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Ford Circles and an Amusing Function

1 What about?

Link between the Ford circles and the functions described by Steinerberger (2018), Fibonacci numbers, and the Golden ratio.

Just visualizations. Proofs are left to the reader.

2 Ford Circles

Figure 1 gives a collection of Ford circles.

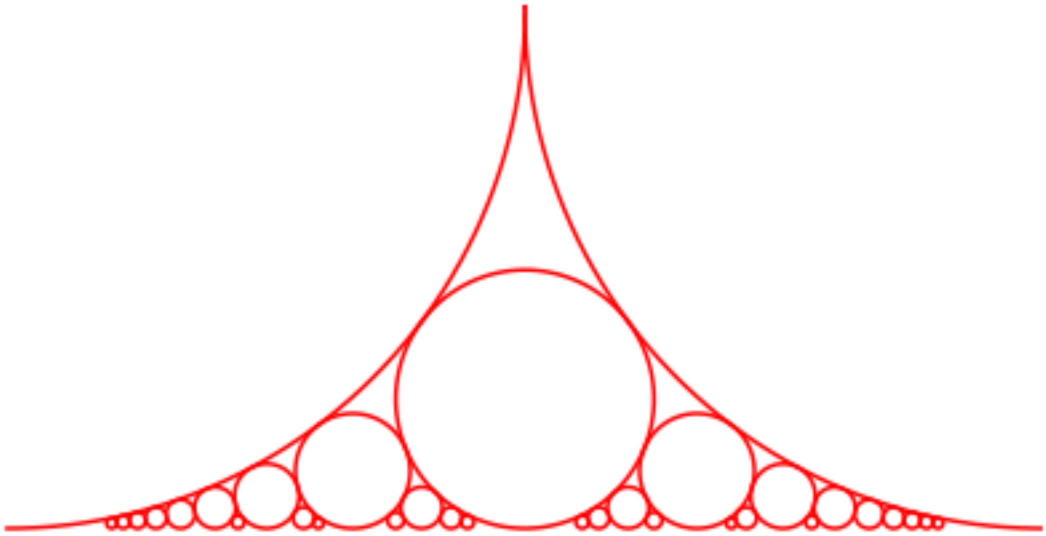

Fig. 1: Ford circles

3 An amusing function

We use the function:

$$f_n(x) = -\frac{1}{5}\sum_{k=1}^{n}\frac{\left|\sin(k\pi x)\right|}{k} \tag{1}$$

Apart from the insignificant factor $-\frac{1}{5}$, these are the functions described by Steinerberger (2018).

Figure 2 gives the diagram of f_{10}. Striking are the tip-shaped local maxima.

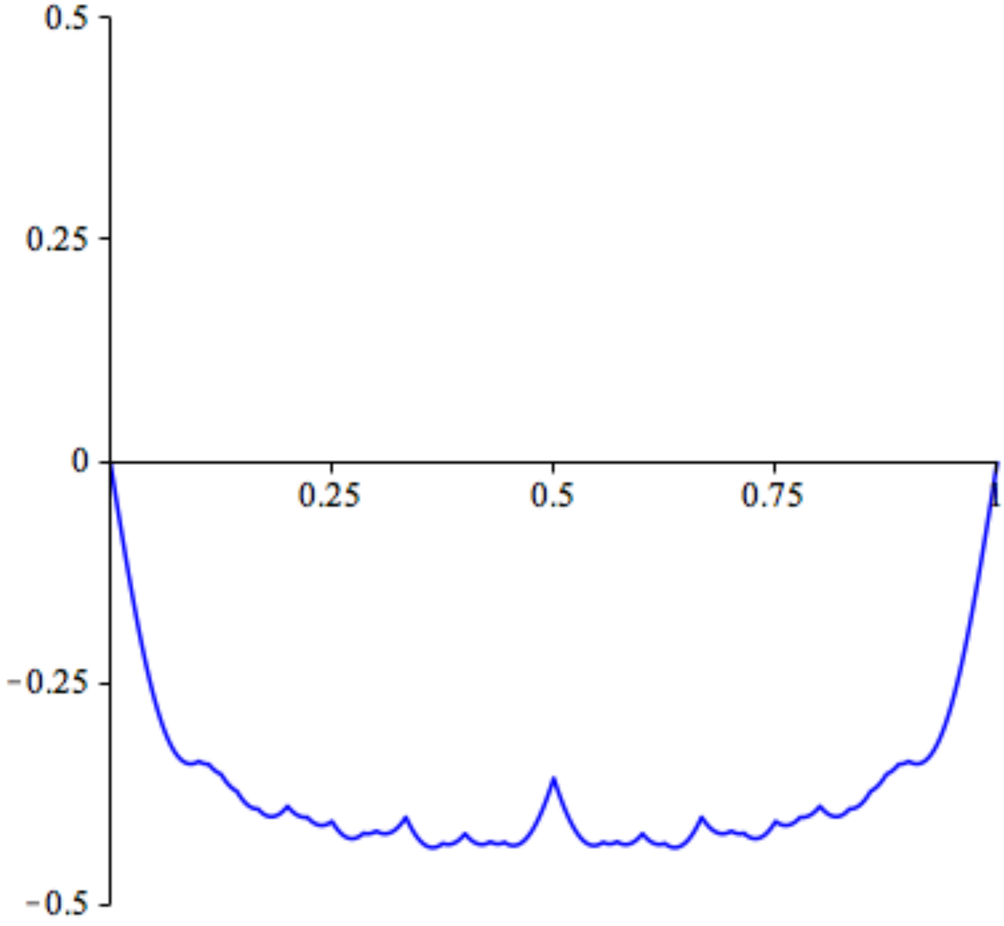

Fig. 2: Diagram

4 Link

Figure 3 gives the link between the Figures 1 and 2.

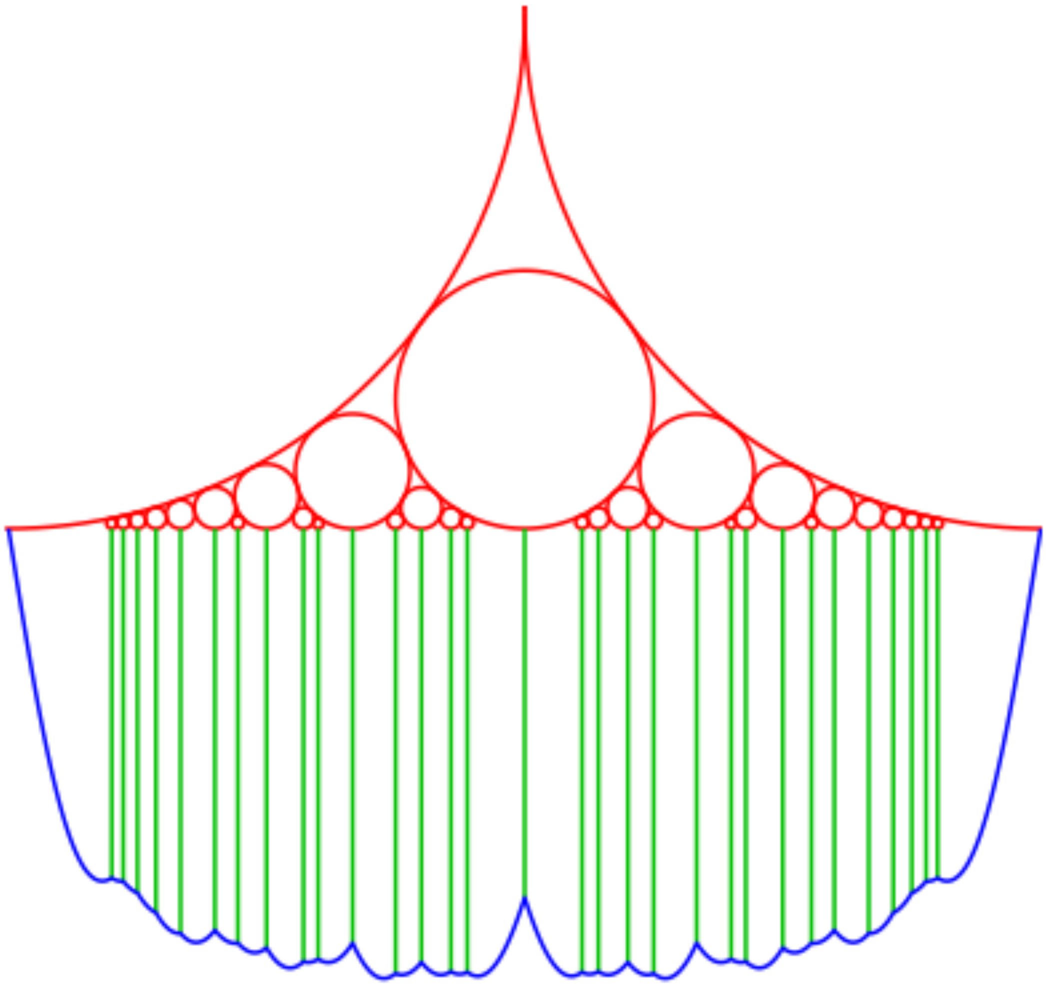

Fig. 3: Link

5 Further examples

Just for fun some more examples.

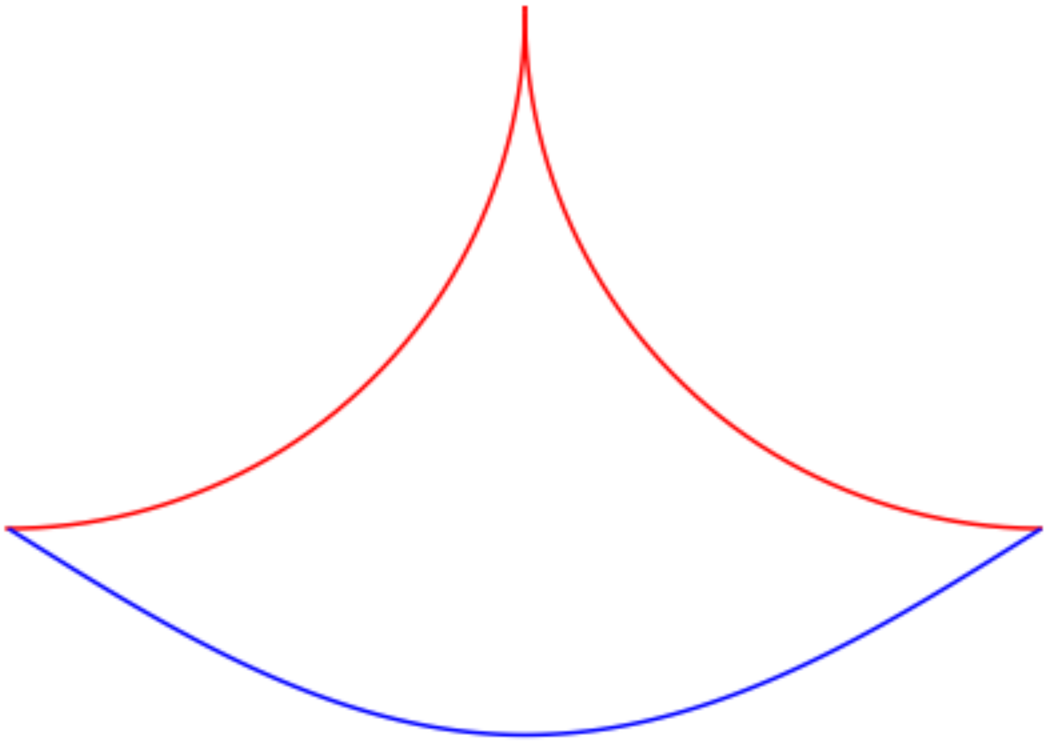

Fig. 4.1: *n* = 1

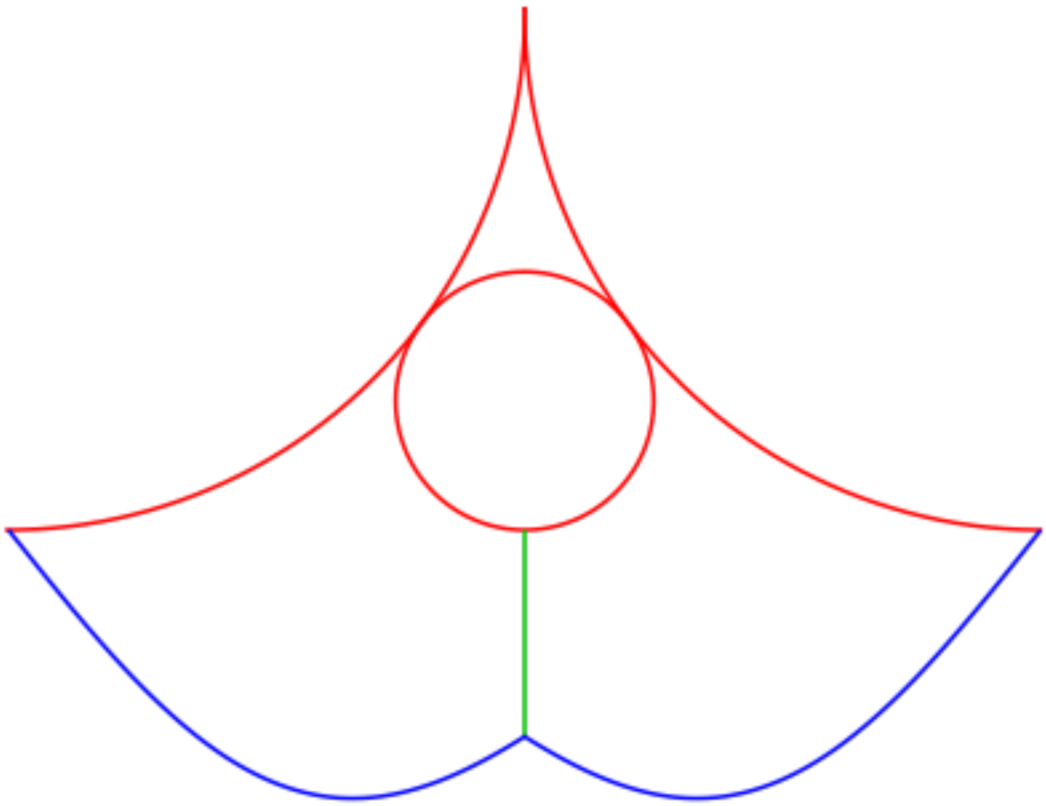

Fig. 4.2: *n* = 2

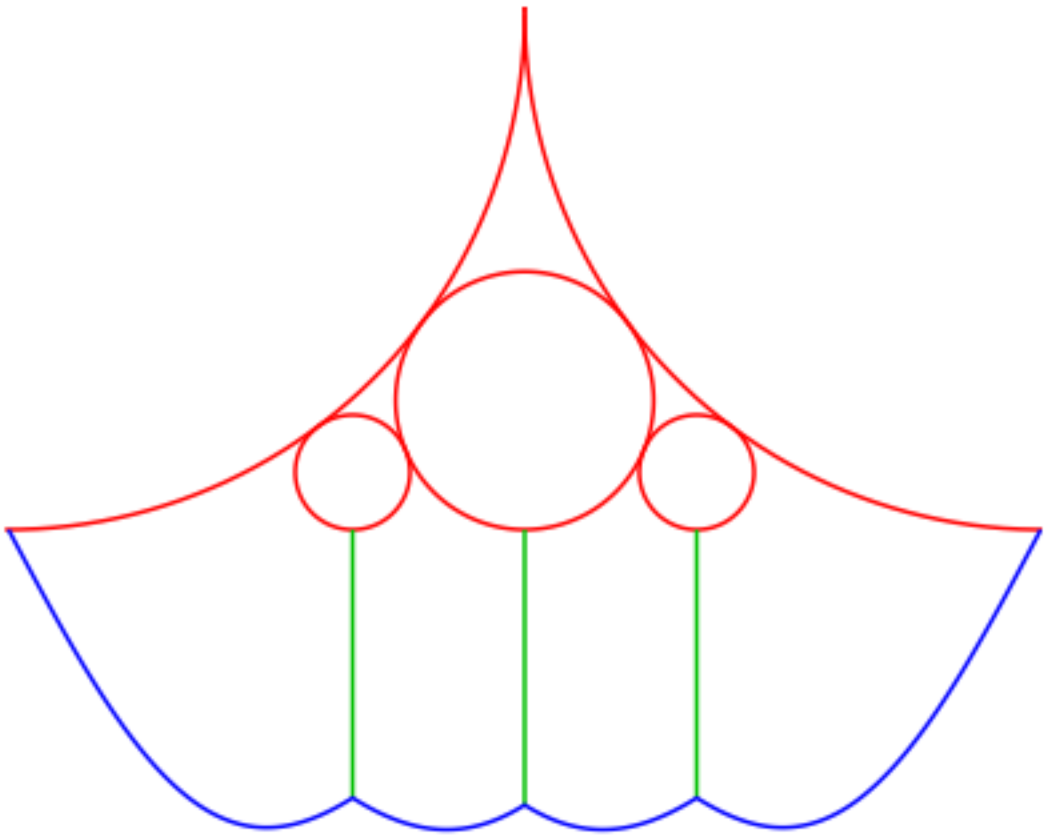

Fig. 4.3: *n* = 3

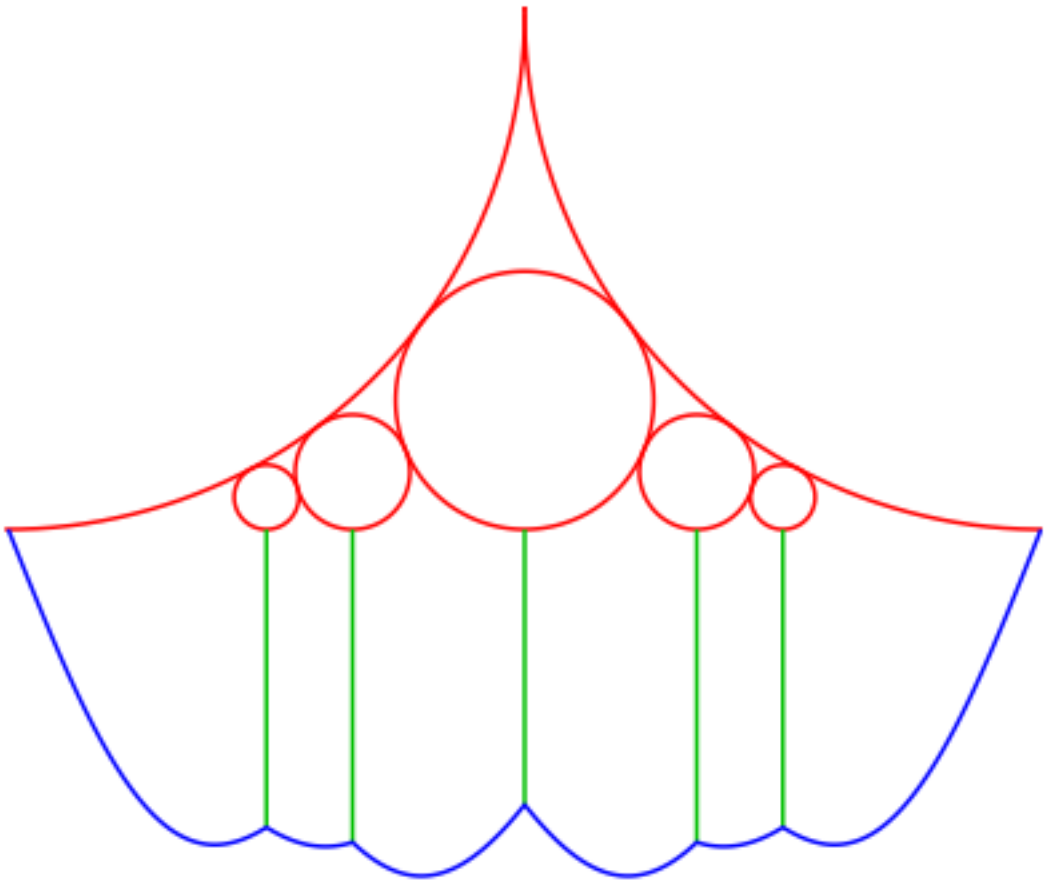

Fig. 4.4: *n* = 4

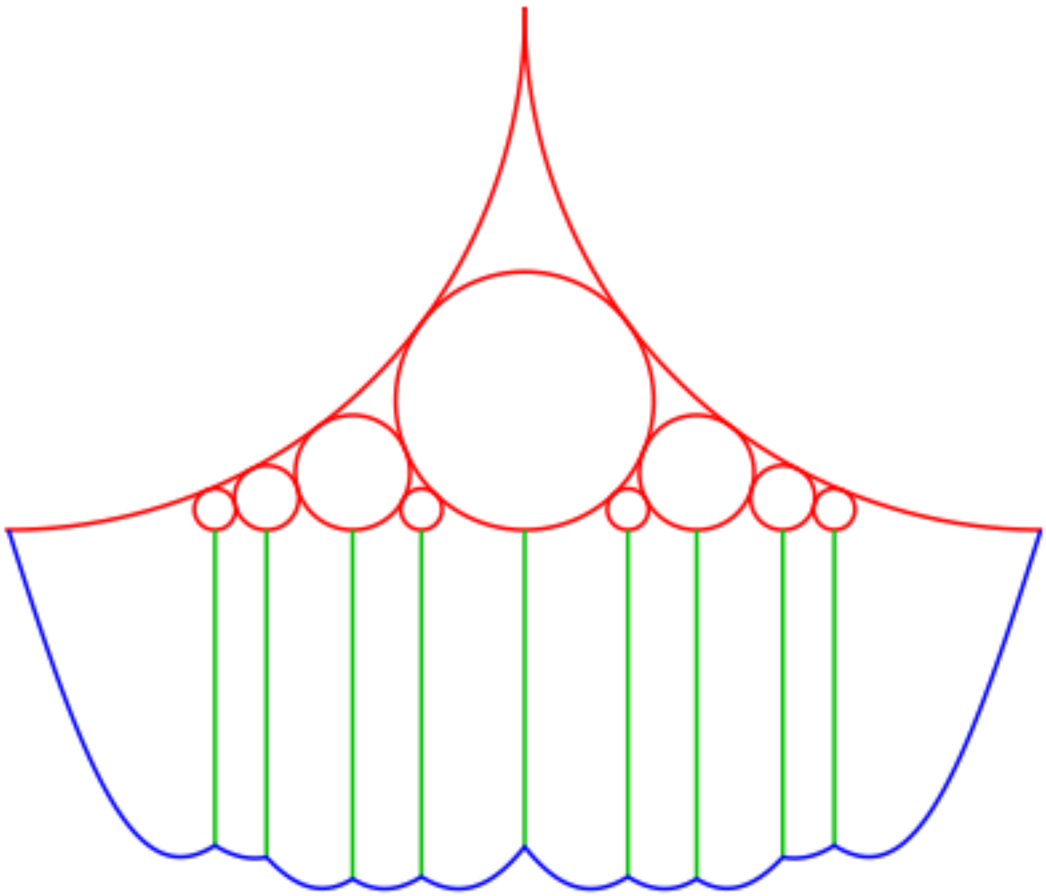

Fig. 4.5: *n* = 5

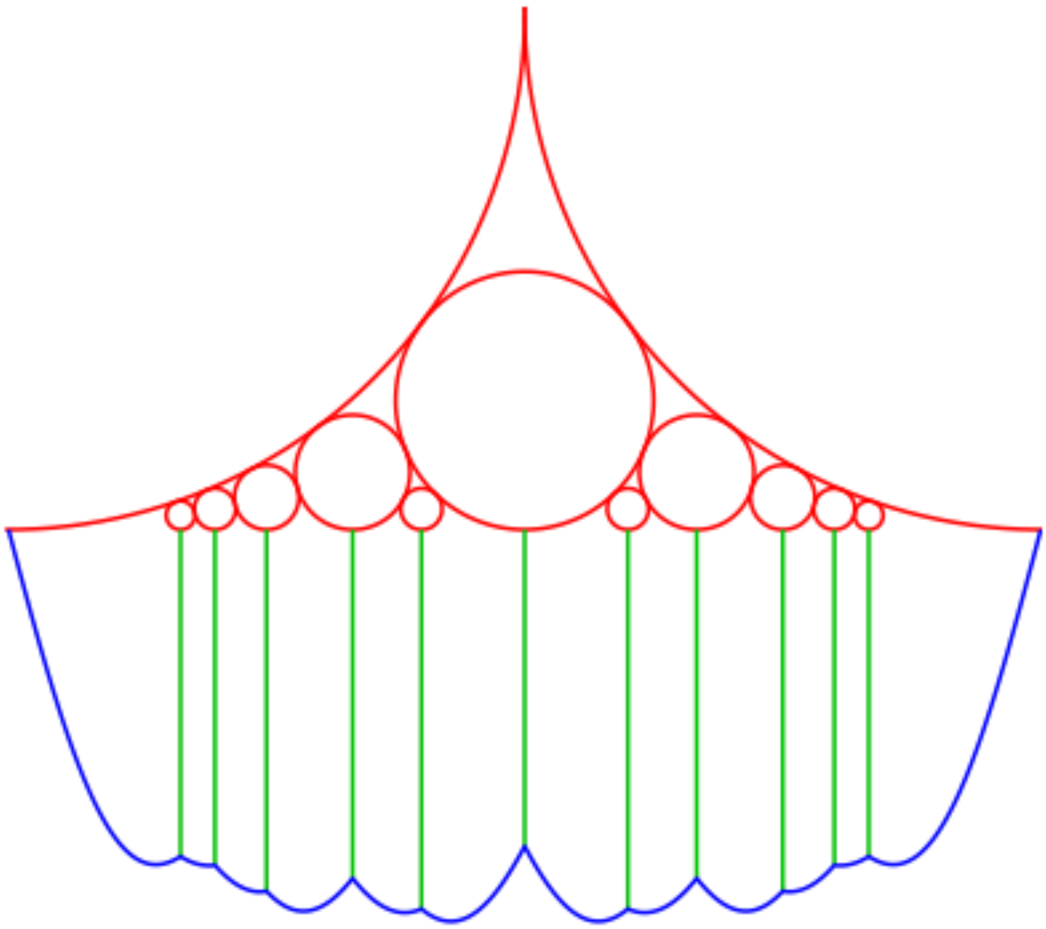

Fig. 4.6: *n* = 6

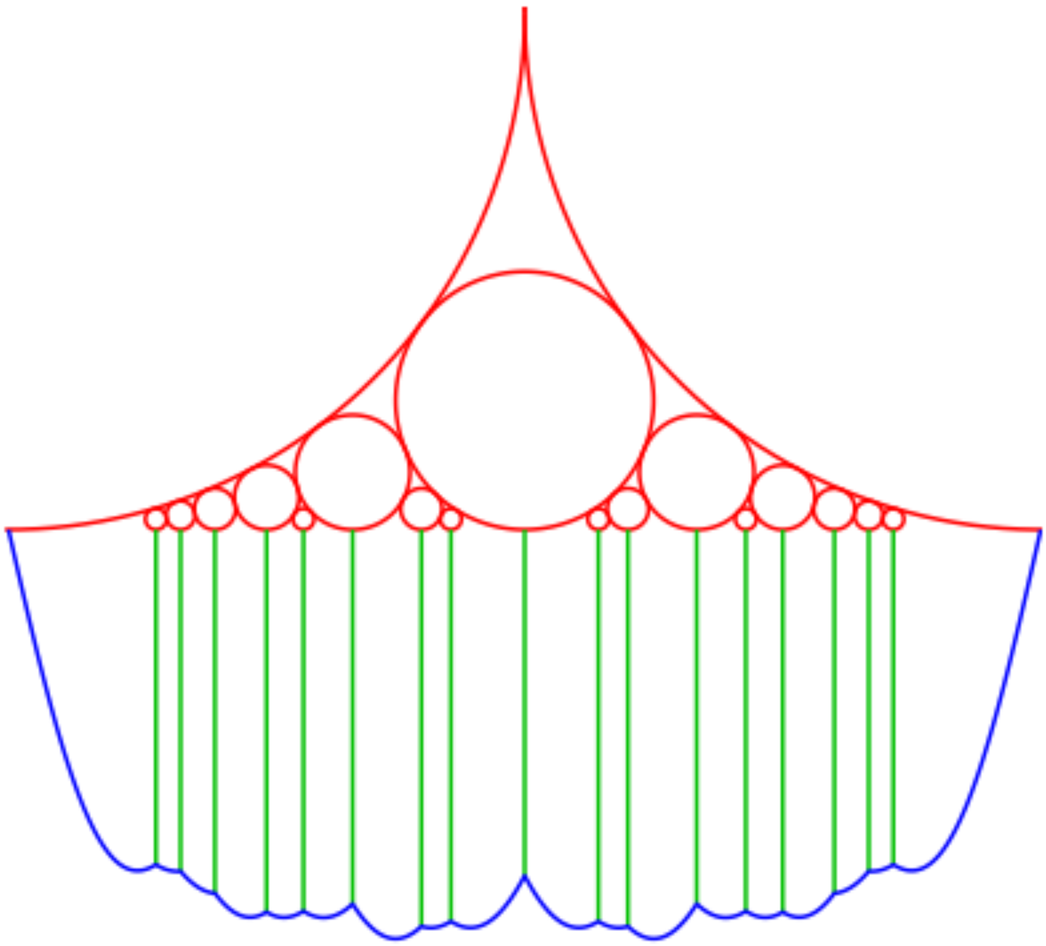

Fig. 4.7: *n* = 7

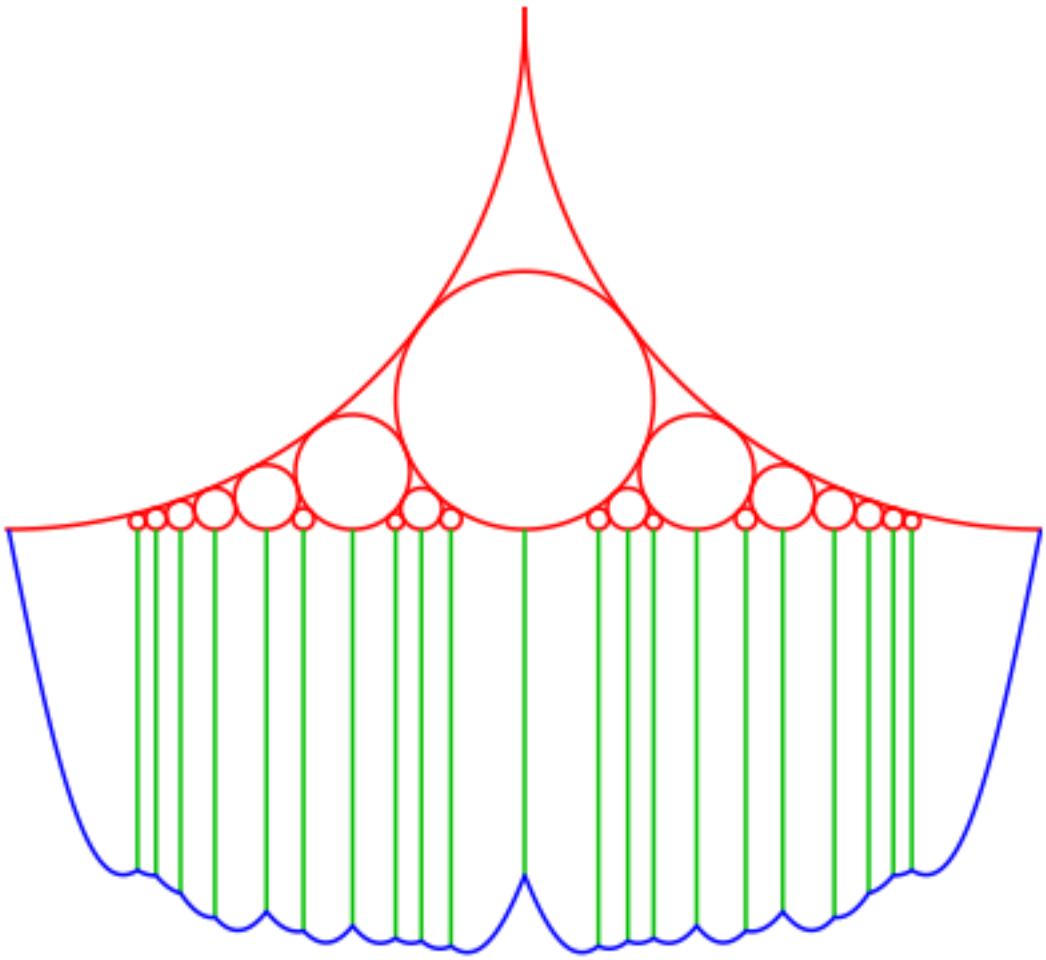

Fig. 4.8: *n* = 8

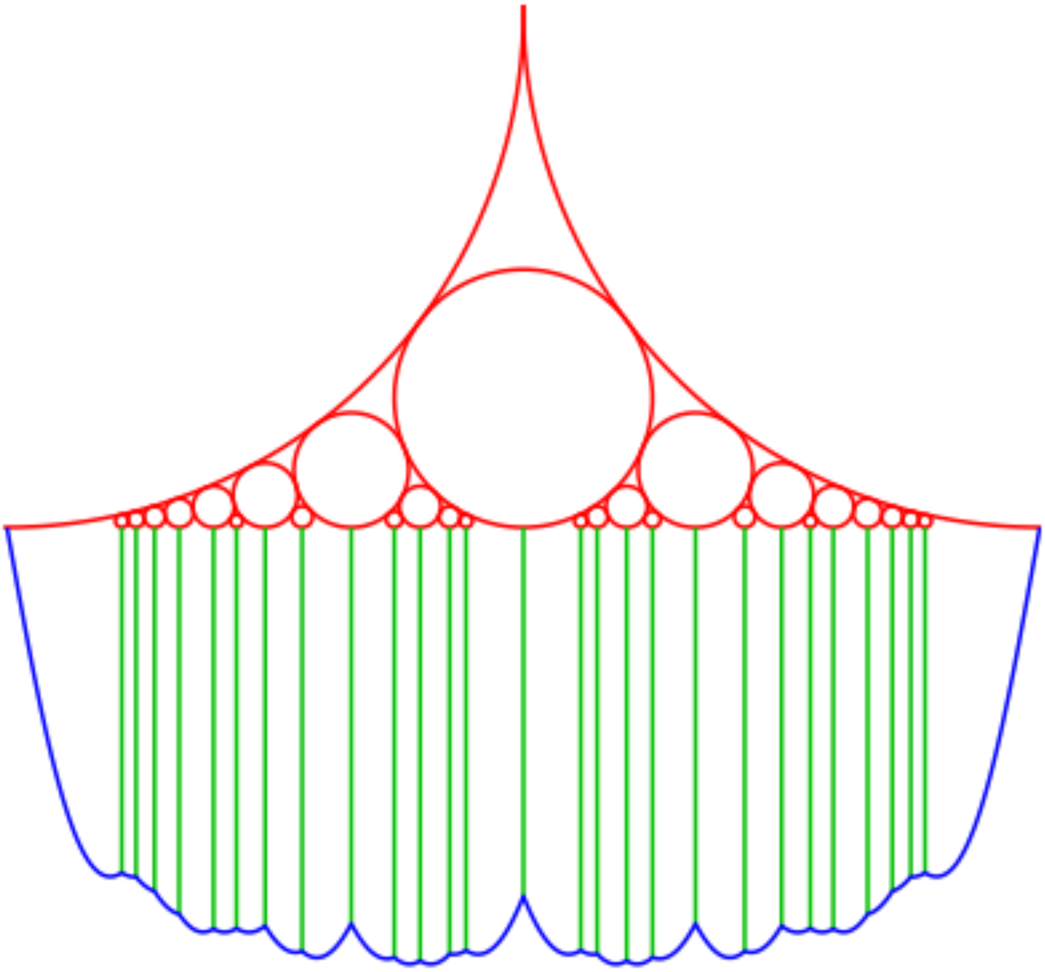

Fig. 4.9: *n* = 9

6 Zigzag lines, Fibonacci numbers, and the Golden ratio

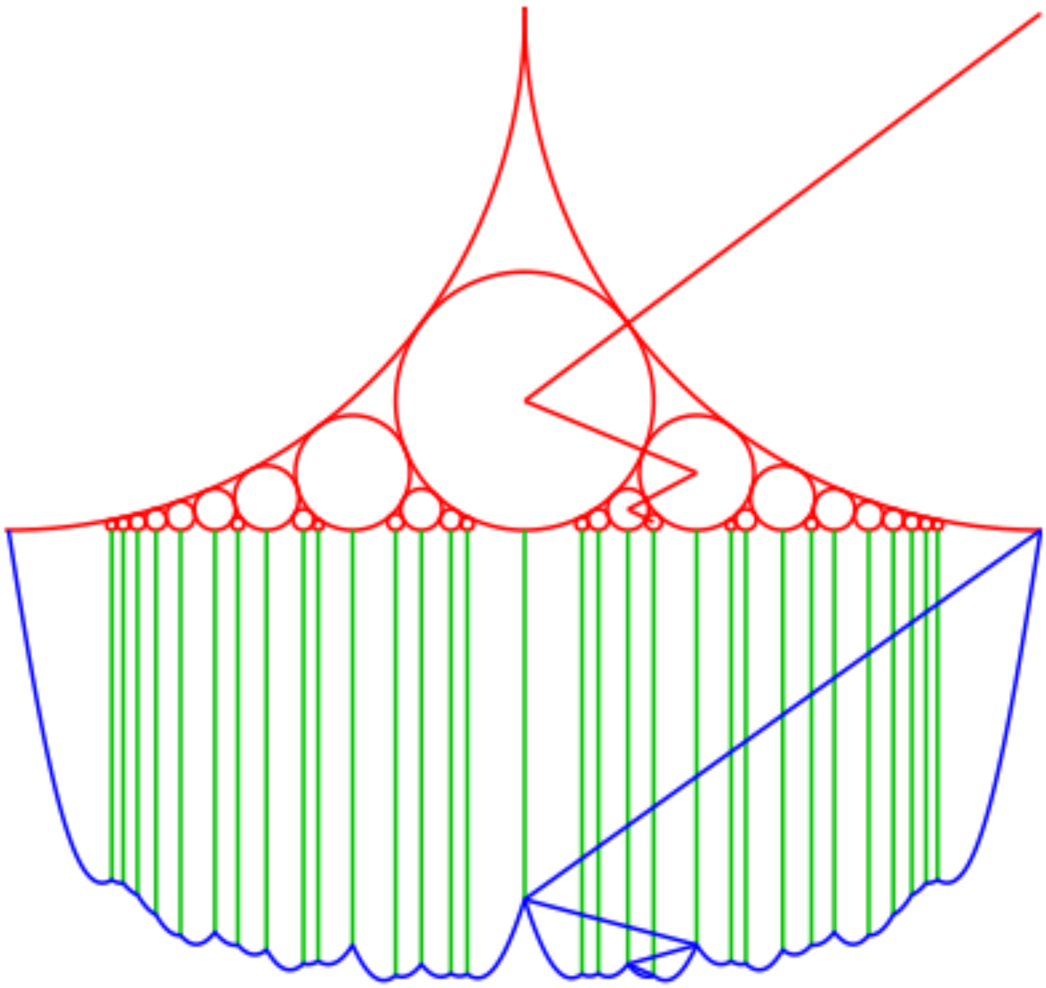

Fig. 5: Zigzag lines

Figure 5 depicts two zigzag lines. The vertices of the red zigzag line are the centers of adjoining Ford circles, the vertices of the blue zigzag line peaks of the function diagram.

The x-coordinates of the vertices of both zigzag lines are:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \ldots \quad (2)$$

We recognize in the numerators and denominators of (2) the Fibonacci numbers (Walser 2012). Hence the limit of these x-coordinates is the Golden ratio (Walser 2001, Walser 2013):

$$\frac{-1+\sqrt{5}}{2} \approx 0.618 \tag{3}$$

Figure 6 shows the situation.

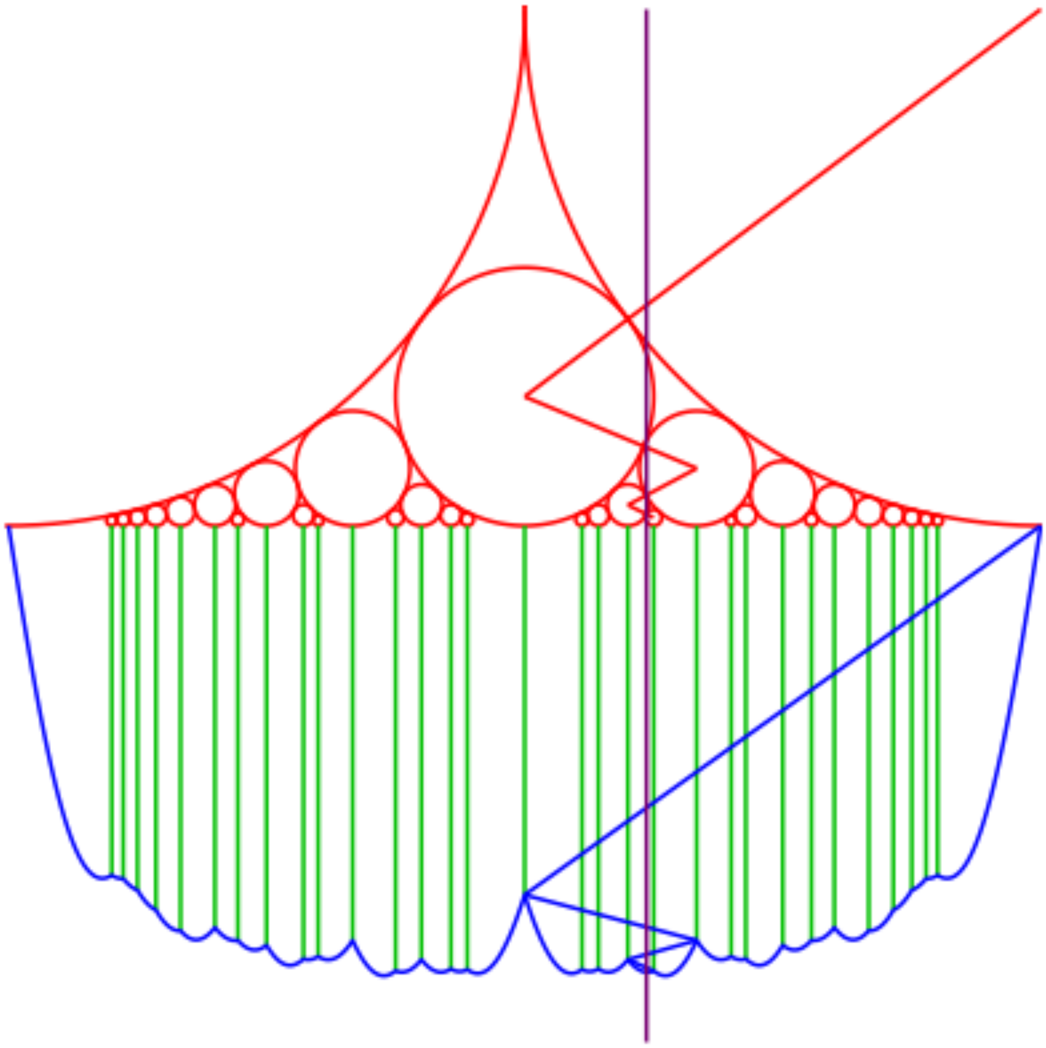

Fig. 6: Golden ratio

7 Pythagorean triangles

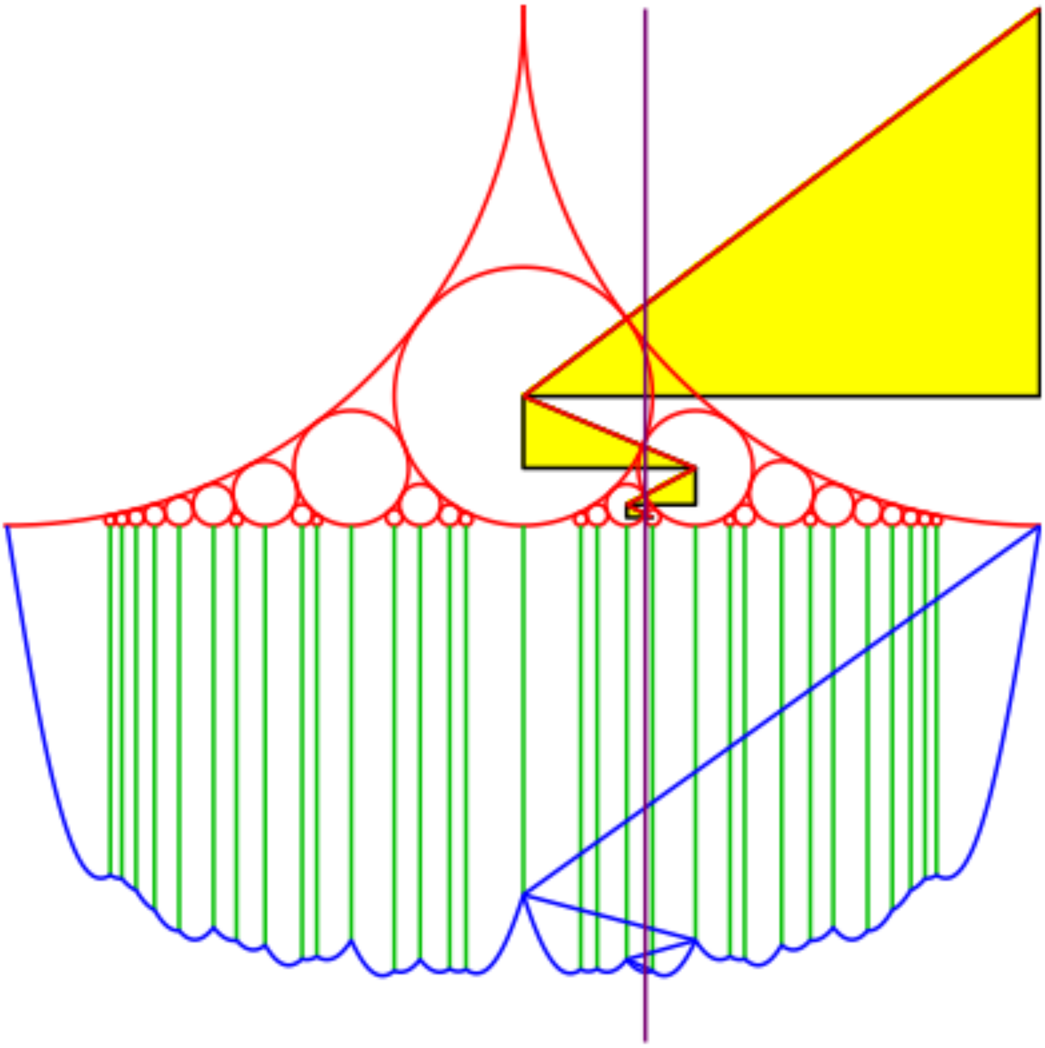

Fig. 7: Pythagorean triangles

The yellow triangles in figure 7 have the sides:

k	a_k	b_k	c_k	$a_k : b_k : c_k$
1	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	3 : 4 : 5
2	$\frac{5}{72}$	$\frac{1}{6}$	$\frac{13}{72}$	5 : 12 : 13
3	$\frac{8}{225}$	$\frac{1}{15}$	$\frac{17}{225}$	8 : 15 : 17
4	$\frac{39}{3200}$	$\frac{1}{40}$	$\frac{89}{3200}$	39 : 80 : 89
5	$\frac{105}{21632}$	$\frac{1}{104}$	$\frac{233}{21632}$	105 : 208 : 233

Tab. 1: Sides and ratios

The triangles are Pythagorean triangles with:

$$\lim_{k \to \infty} \frac{b_k}{a_k} = 2 \quad \text{and} \quad \lim_{k \to \infty} \frac{c_k}{a_k} = \sqrt{5} \qquad (4)$$

References

Steinerberger, Stefan (2018): An Amusing Sequence of Functions. Mathematics Magazine, Vol. 91, No. 4, October 2018, p 262-266.

Walser, Hans (2001): The Golden Section. Translated by Peter Hilton and Jean Pedersen. The Mathematical Association of America 2001. ISBN 0-88385-534-8.

Walser, Hans (2012): Fibonacci. Zahlen und Figuren. Leipzig, EAGLE, Edition am Gutenbergplatz. ISBN 978-3-937219-60-8.

Walser, Hans (2013): Der Goldene Schnitt. 6., bearbeitete und erweiterte Auflage. Edition am Gutenbergplatz, Leipzig. ISBN 978-3-937219-85-1.

Links

Francis Bonahon: Funny Fractions and Ford Circles (30.11.2018):
https://www.youtube.com/watch?v=0hlvhQZIOQw&t=604s

Hans Walser: Ford-Kreise (30.11.2018):
http://www.walser-h-m.ch/hans/Miniaturen/F/Ford-Kreise/Ford-Kreise.htm