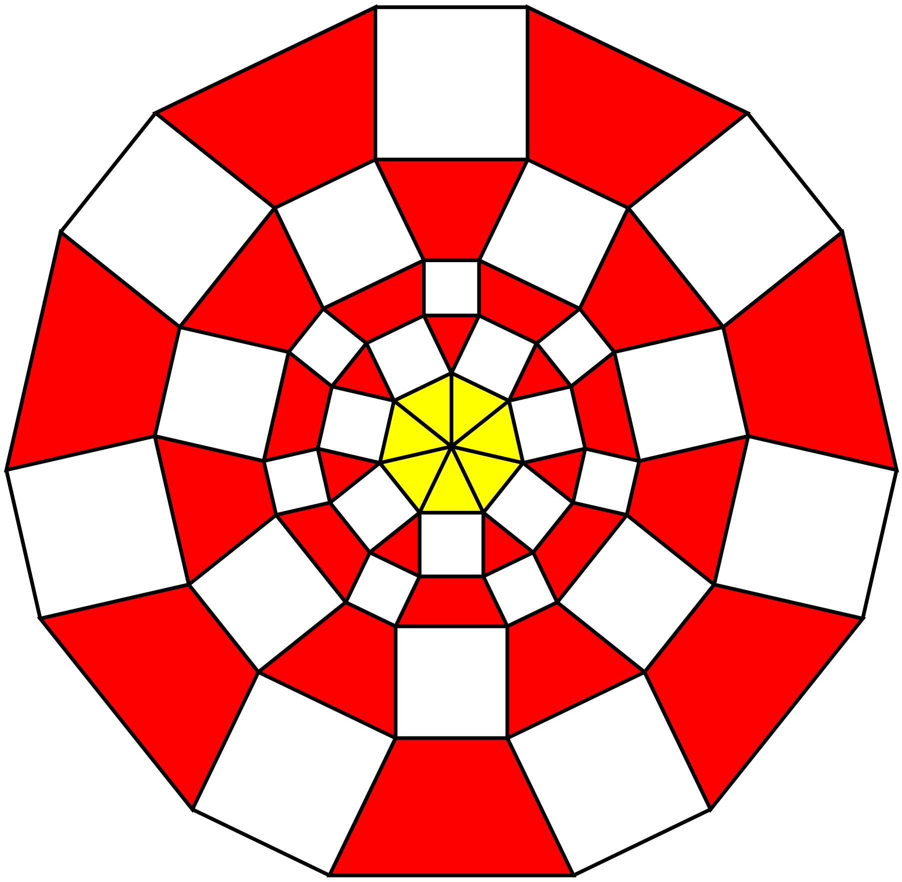
Hans Walser, [20090715b]

Regular *K*-gon and trapezoids

1 Example

Starting by a regular yellow heptagon ($K = 7$) in the unit circle we add squares on every side. Then we proceed as indicated in the following figure.

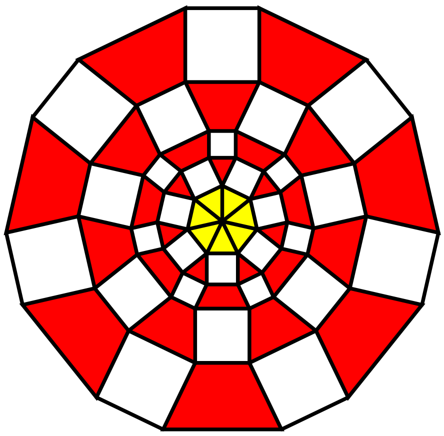
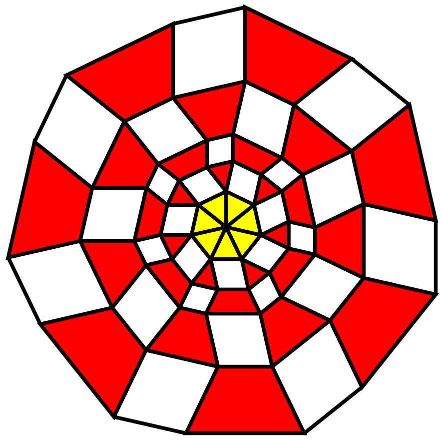
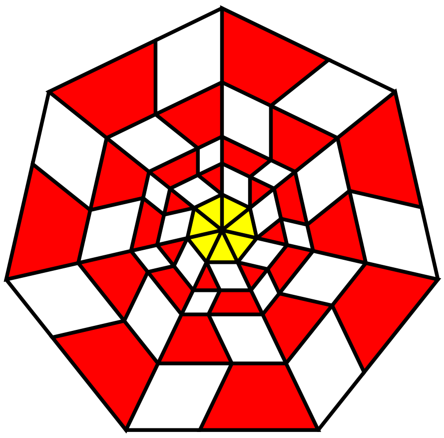
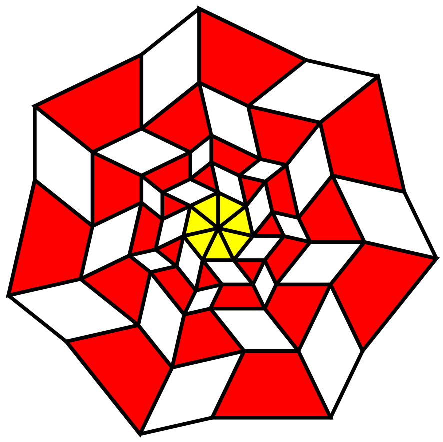


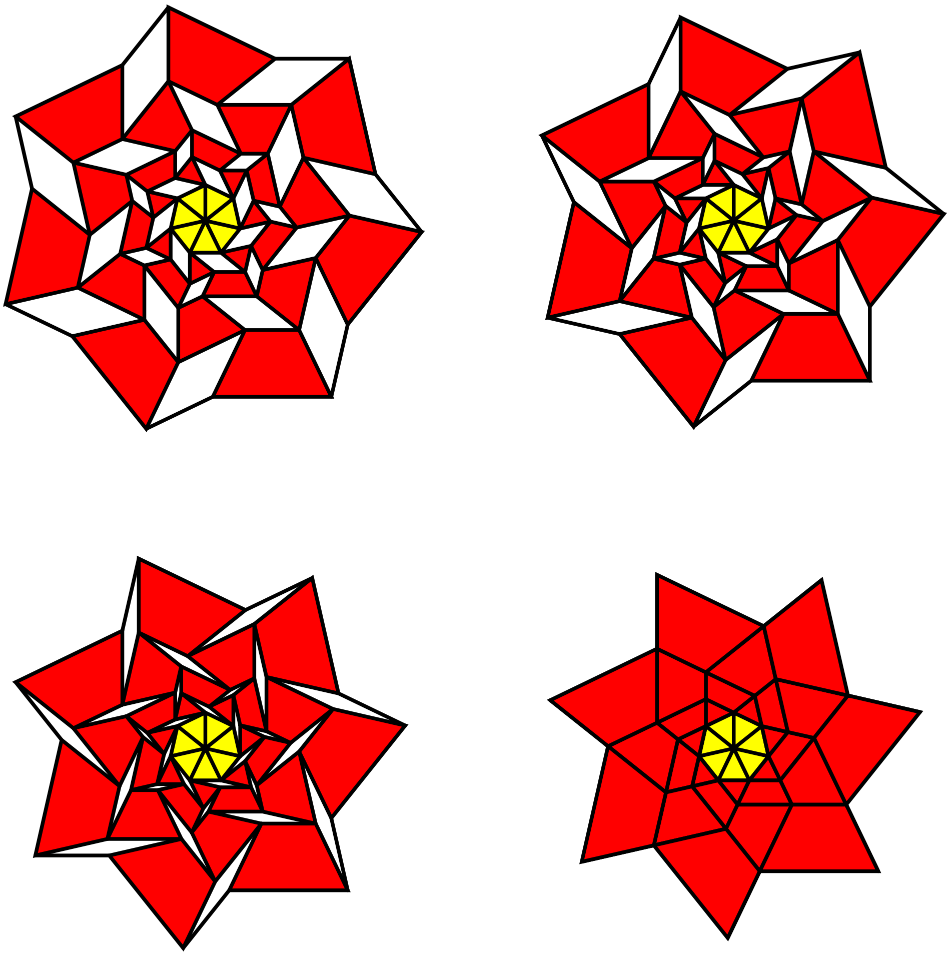
Heptagon and squares

We get red isosceles trapezoids between the white squares. In the first ring we see triangles, bat we count them as special trapezoids with upper side zero. We would like to compare the areas of the trapezoids.

2 Modification

We can modify the figure by transforming the white squares into rhombuses. This does not change shape or size of the red trapezoids.

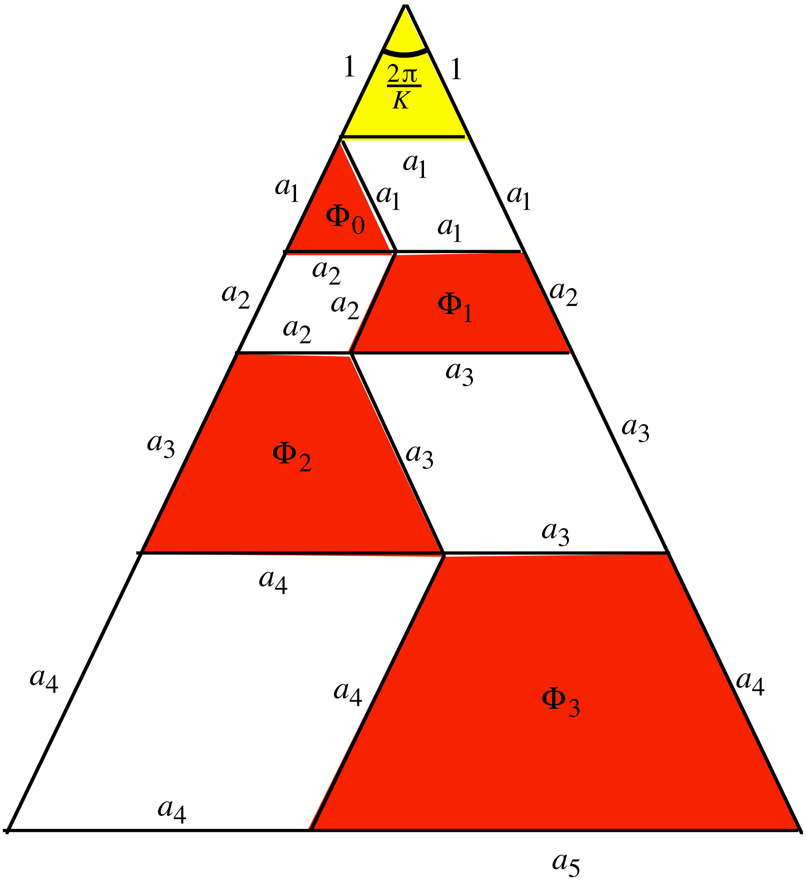







Collapsing the squares

3 Some calculations

We use the notations of the following figure.



Notations

We get:

$$a_1 = 2\sin\left(\frac{\pi}{K}\right)$$

Setting $a_0 = 0$ we can establish the recursion formula:

$$a_{n+2} = 2\sin\left(\frac{\pi}{K}\right)a_{n+1} + a_n$$

For the area Φ_n we get:

$$\Phi_n = \frac{1}{2}\left(a_n + a_{n+2}\right)a_{n+1}\cos\left(\frac{\pi}{K}\right)$$

Since we want to compare the areas of the red polygons, we introduce the relative area:

$$\Psi_n = \frac{\Phi_n}{\Phi_0}$$

4 Special cases

The tables indicate for different *K* the numerical values.

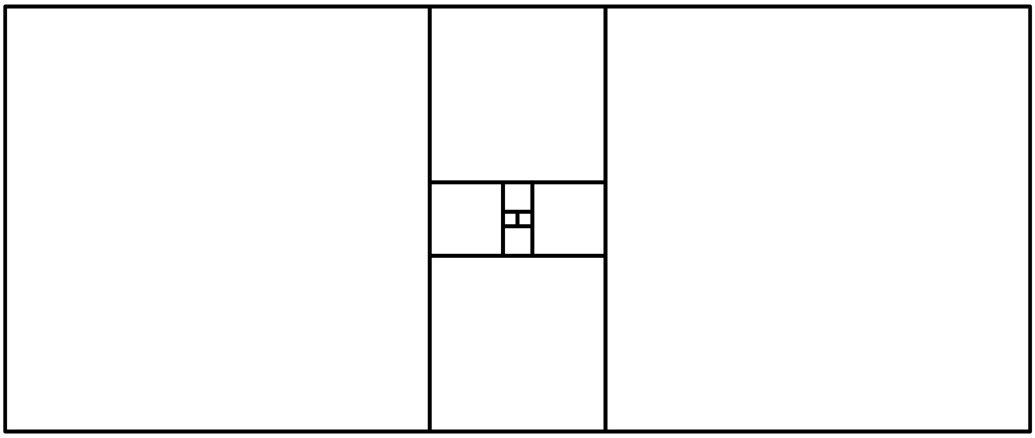
4.1 *K* = 1

We get $a_1 = 2\sin\left(\frac{\pi}{1}\right) = 0$.

4.2 $K = 2$

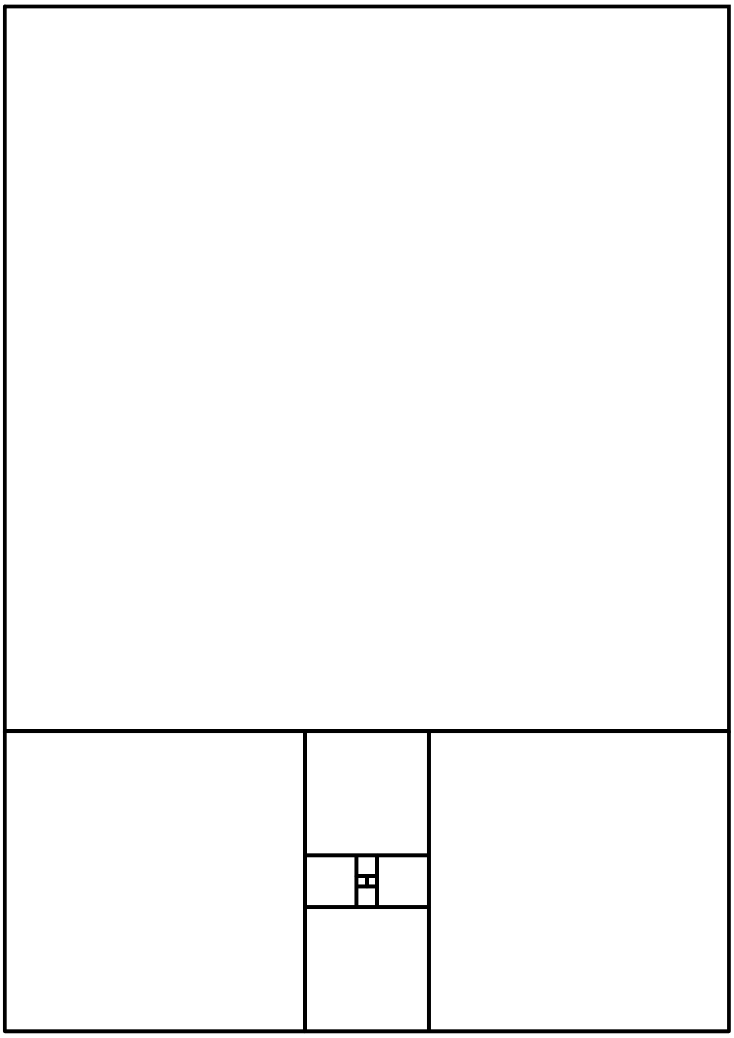
n	a[n]
0	0
1	2.0
2	4.0
3	10.0
4	24.0
5	58.0
6	140.0
7	338.0
8	816.0

We have “flat” trapezoids. But the squares are interesting.



Squares only

If we add only the next square above, we get a rectangle with nearly the shape of the European standard paper shape DIN A.



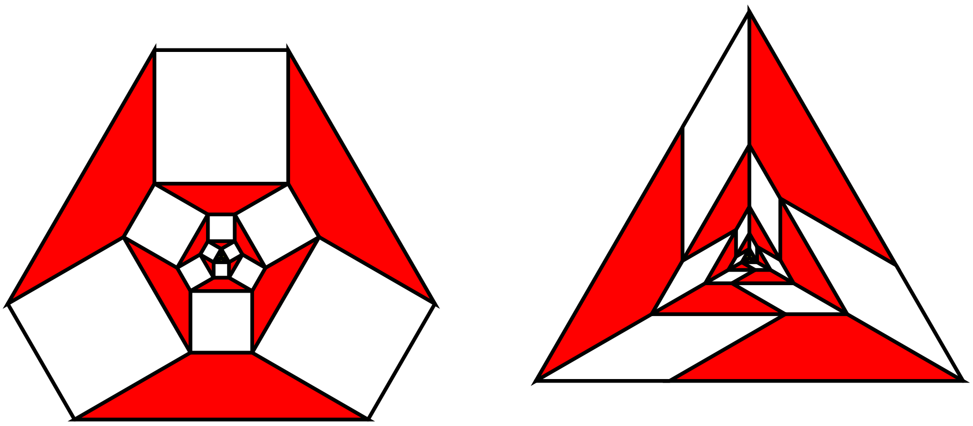
Close to European standard shape DIN A

Indeed we have:

$$\lim_{n\to\infty} \frac{a_n + a_{n+1}}{a_{n+1}} = \sqrt{2}$$

n	q[n]
0	1.0
1	1.5
2	1.4
3	1.4166667
4	1.4137931
5	1.4142857
6	1.4142012
7	1.4142157
8	1.4142132
9	1.4142136
10	1.4142136

4.3 *K* = 3



$K = 3$

n	a[n]	phi[n]	psi[n]
0	0	1.299038106	1.0
1	1.732050808	6.495190528	5.0
2	3.0	31.17691454	24.0
3	6.92820323	149.3893822	115.0
4	15.0	715.7699962	551.0
5	32.90896534	3429.460599	2640.0
6	72.0	16431.533	12649.0
7	157.6166235	78728.20439	60605.0
8	345.0	377209.489	290376.0

We see, that Ψ_n are integer numbers, but not the Φ_n. The case

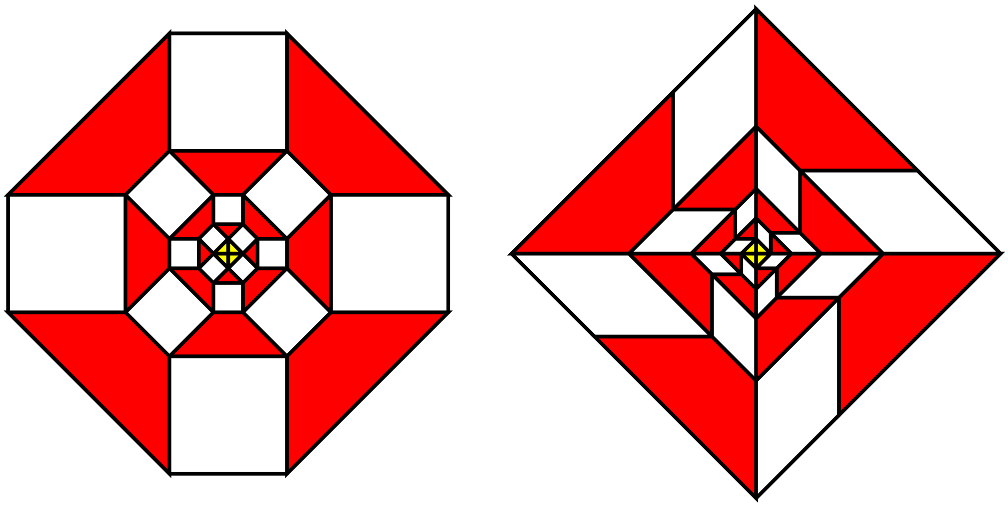
n	a[n]	phi[n]	psi[n]
1	1.732050808	6.495190528	5.0

is subject of [Deshpande 2009].

We have the recursion formula:

$$\Psi_{n+2} = 5\Psi_{n+1} - \Psi_n$$

4.4 *K* = 4



***K* = 4**

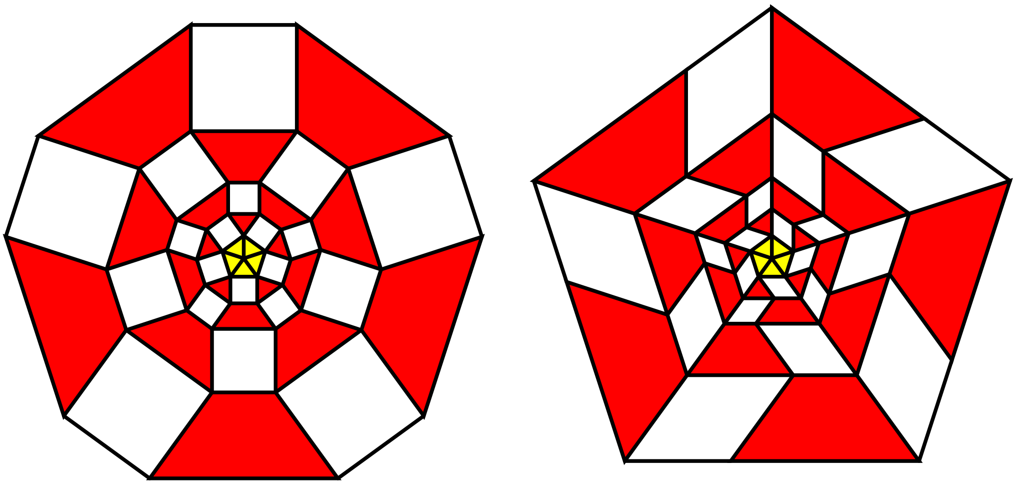
The Figure on the left fits into a square lattice, but not so the figure on the right.

n	a[n]	phi[n]	psi[n]
0	0	1.0	1.0
1	1.414213562	4.0	4.0
2	2.0	15.0	15.0
3	4.242640687	56.0	56.0
4	8.0	209.0	209.0
5	15.55634919	780.0	780.0
6	30.0	2911.0	2911.0
7	57.98275606	10864.0	10864.0
8	112.0	40545.0	40545.0

Both the Φ_n and the Ψ_n are integers. We have the recursion formula:

$$\Psi_{n+2} = 4\Psi_{n+1} - \Psi_n$$

4.5 *K* = 5



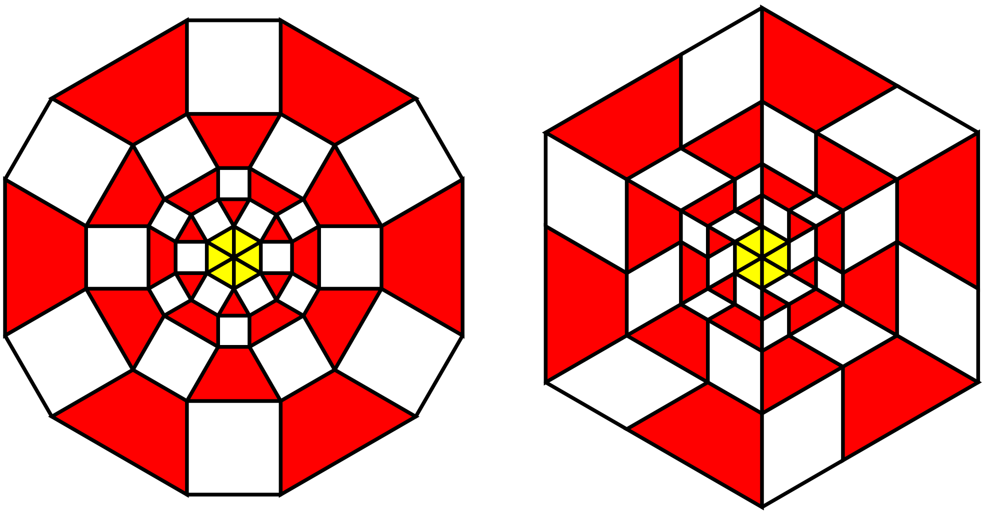
***K* = 5**

n	a[n]	phi[n]	psi[n]
0	0	0.6571638901	1.0
1	1.175570505	2.22250594	3.381966011
2	1.381966011	6.85927566	10.4376941
3	2.800168986	20.9753312	31.91796068
4	4.673762079	64.07858154	97.50776405
5	8.294505831	195.7362536	297.8499832
6	14.42453848	597.8947754	909.8107555
7	25.25156781	1826.323555	2779.099069
8	44.1095368	5578.669413	8489.007836

No integer numbers. Probably there is the golden section in it, but I do not see it. But there is a very interesting recursion formula:

$$\Psi_{n+2} = \Psi_1 \Psi_{n+1} - \Psi_n$$

4.6 *K* = 6



***K* = 6**

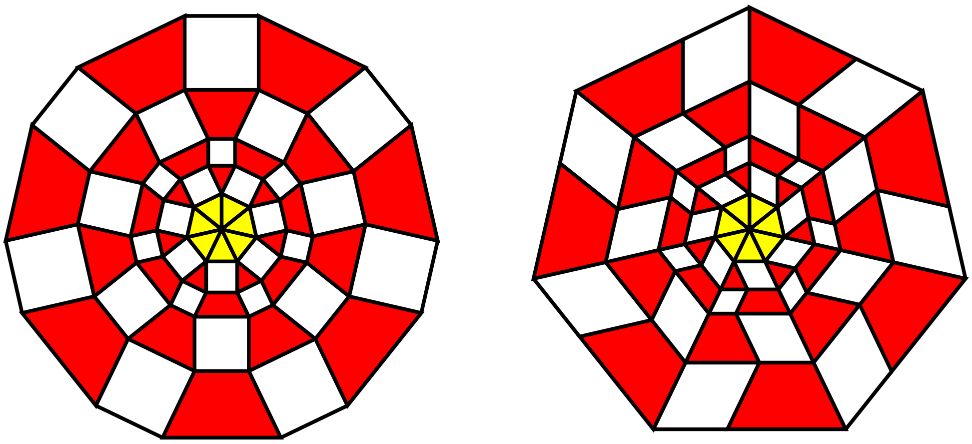
The figure on the right fits into a regular triangular lattice. The figure on the left not, since squares and regular triangles don't like each other.

n	a[n]	phi[n]	psi[n]
0	0	0.4330127019	1.0
1	1.0	1.299038106	3.0
2	1.0	3.464101615	8.0
3	2.0	9.09326674	21.0
4	3.0	23.8156986	55.0
5	5.0	62.35382907	144.0
6	8.0	163.2457886	377.0
7	13.0	427.3835368	987.0
8	21.0	1118.904822	2584.0

The a_n are the Fibonacci numbers ($a_n = f_n$), and the Ψ_n are every second Fibonacci number ($\Psi_n = f_{2n+2}$).We have the recursion formula:

$$\Psi_{n+2} = 3\Psi_{n+1} - \Psi_n$$

4.7 *K* = 7



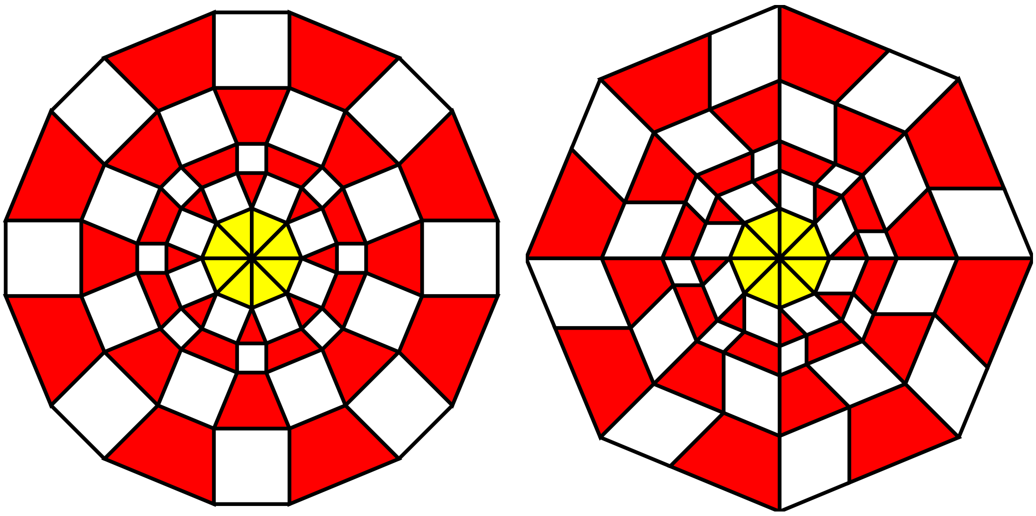
K **= 7**

n	a[n]	phi[n]	psi[n]
0	0	0.2943675264	1.0
1	0.8677674782	0.8103998041	2.753020396
2	0.7530203963	1.936679664	6.579121302
3	1.521214089	4.521318811	15.35943474
4	2.07308051	10.51060324	35.70571581
5	3.320165935	24.41458629	82.93912915
6	4.95421253	56.70325077	192.6273984
7	7.619270449	131.6906196	447.3680275
8	11.56596763	305.8437111	1038.985906

Awful numbers. We have again the interesting recursion formula:

$$\Psi_{n+2} = \Psi_1 \Psi_{n+1} - \Psi_n$$

4.8 *K* = 8



K **= 8**

n	a[n]	phi[n]	psi[n]
0	0	0.2071067812	1.0
1	0.7653668647	0.5355339059	2.585786438
2	0.5857864376	1.17766953	5.686291501
3	1.213708394	2.509667992	12.11774901
4	1.514718626	5.311795927	25.64761953
5	2.373023839	11.22550187	54.20151774
6	3.330952442	23.71495458	114.5059299
7	4.922424466	50.09630604	241.8863629
8	7.098413022	105.8233942	510.9605468

We have again the interesting recursion formula:

$$\Psi_{n+2} = \Psi_1 \Psi_{n+1} - \Psi_n$$

5 Summary

Cases with integer numbers only for $K = 2, 3, 4, 6$.

In the general case there is the conjecture:

$$\Psi_{n+2} = \Psi_1 \Psi_{n+1} - \Psi_n$$

If anybody has time to prove it, I would be glad to hear about.

References

[Deshpande 2009] Deshpande, M. N. : Proof Without Words: Beyond Extriangles. MATHEMATICS MAGAZINE. Vol. 82, No. 3, June 2009, p. 208.