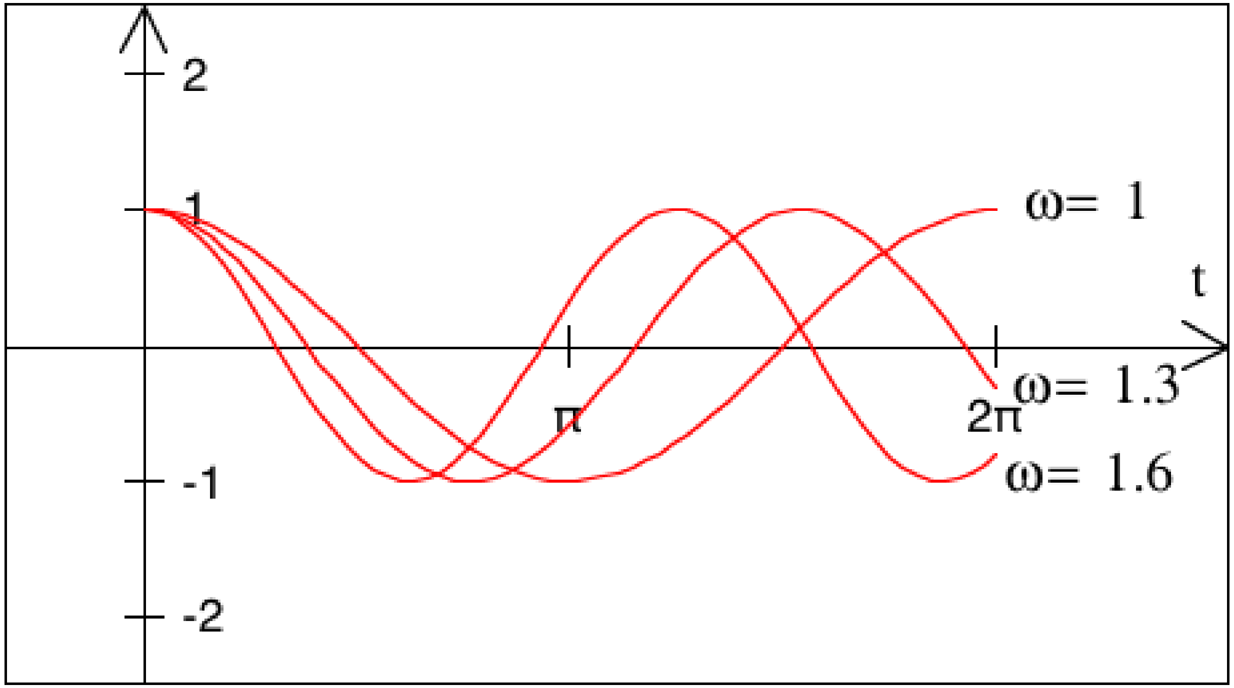
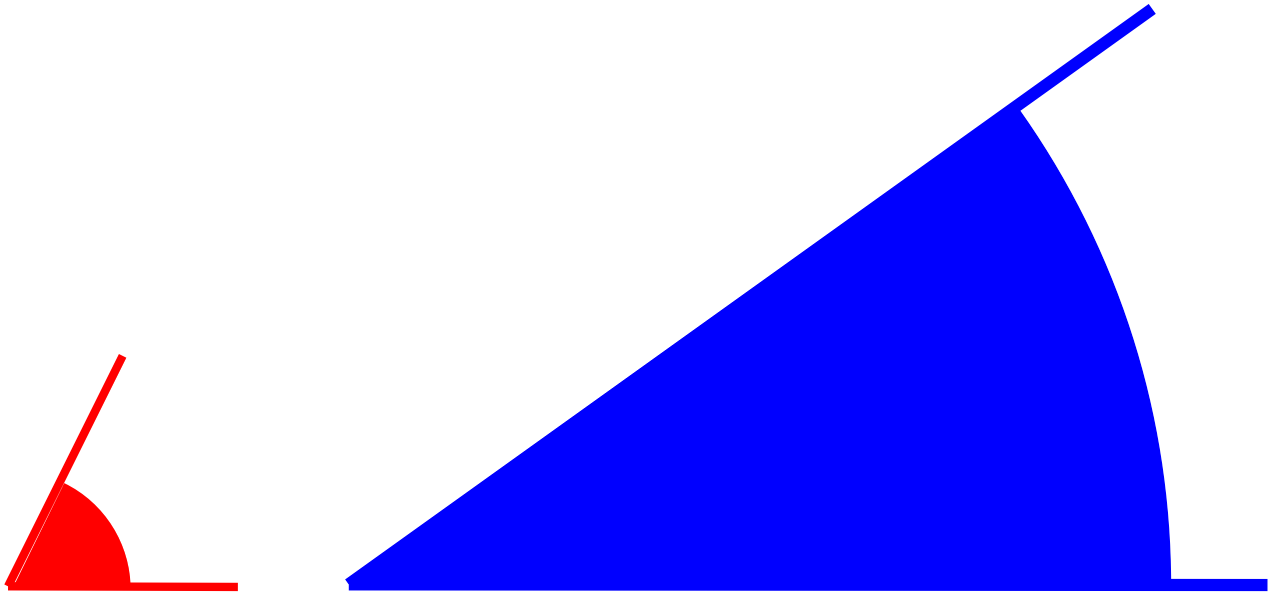


Modul 102: Funktionen, Folgen, Grenzwerte

Winkelfunktionen



Welches ist der größere Winkel?

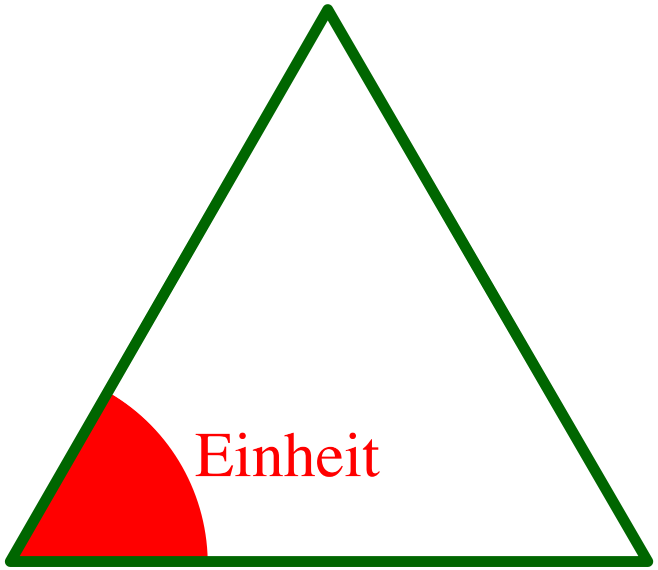
Positiver Drehsinn: Gegenuhrzeigersinn

Positiver Drehsinn

Winkelmaße

	Grad Degree	Gon Grad	Radiant Radians
Vollwinkel	360°	400 gon	2π
Gestreckter Winkel	180°	200 gon	π
Rechter Winkel	90°	100 gon	$\frac{\pi}{2}$
Winkel im gleichseitigen Dreieck	60°	66.66667 gon	$\frac{\pi}{3}$
Halber rechter Winkel	45°	50 gon	$\frac{\pi}{4}$

Babylon, Schule



Babylonier

Unterteilung nach dem
Sexagesimalsystem (Basis 60)

60°
1° = 60'
1' = 60''

Winkelmaße

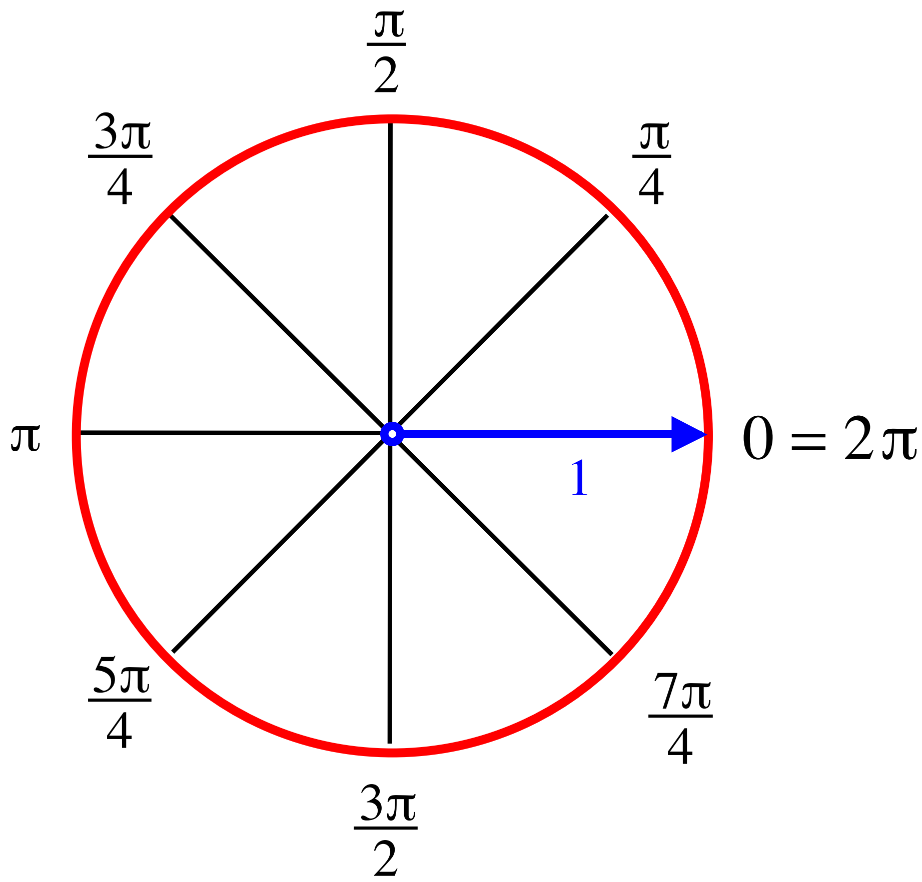
	Grad Degree	Gon Grad	Radiant Radians
Vollwinkel	360°	400 gon	2π
Gestreckter Winkel	180°	200 gon	π
Rechter Winkel	90°	100 gon	$\frac{\pi}{2}$
Winkel im gleichseitigen Dreieck	60°	66.66667 gon	$\frac{\pi}{3}$
Halber rechter Winkel	45°	50 gon	$\frac{\pi}{4}$

Technik, Vermessung

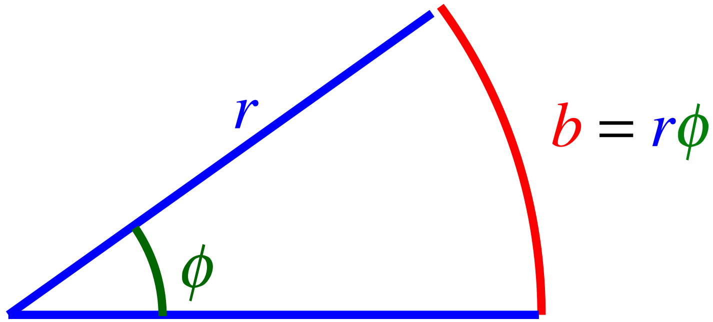
Winkelmaße

	Grad Degree	Gon Grad	Radiant Radians
Vollwinkel	360°	400 gon	2π
Gestreckter Winkel	180°	200 gon	π
Rechter Winkel	90°	100 gon	$\frac{\pi}{2}$
Winkel im gleichseitigen Dreieck	60°	66.66667 gon	$\frac{\pi}{3}$
Halber rechter Winkel	45°	50 gon	$\frac{\pi}{4}$

Bogenmaß, Wissenschaft

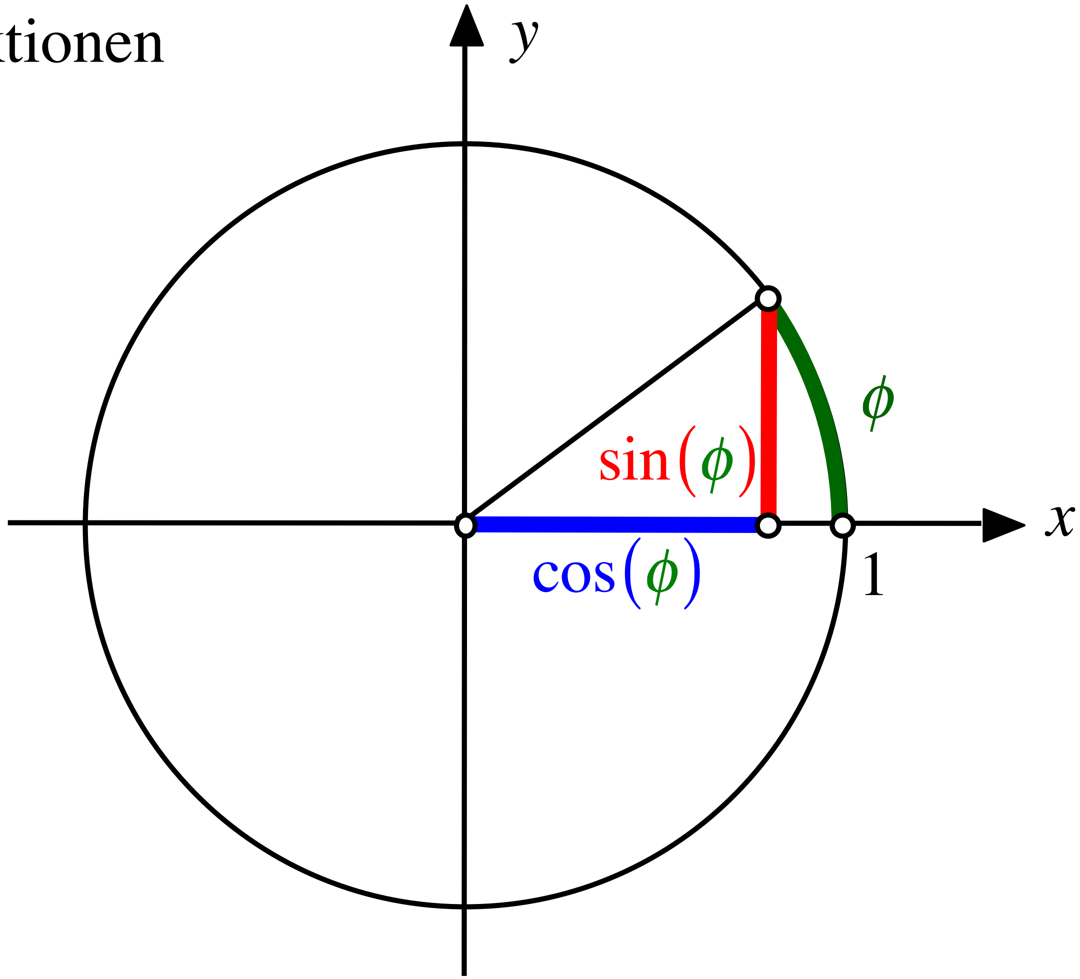


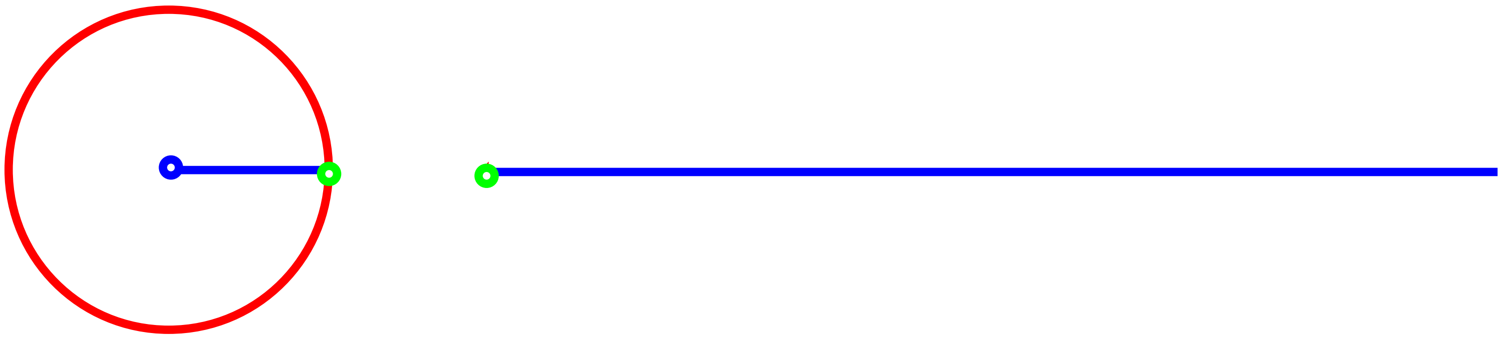
Bogenmaß: Bogenlänge auf dem Einheitskreis ($r = 1$)

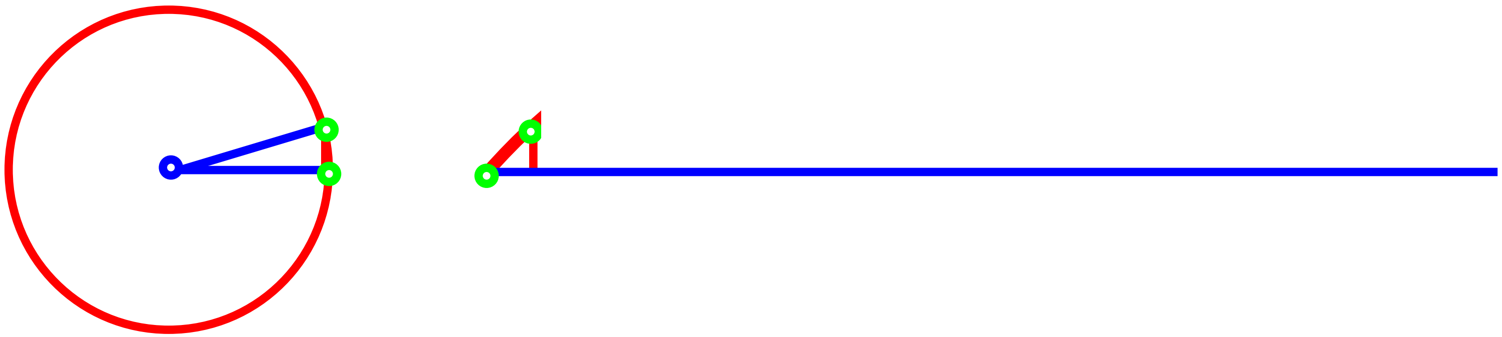


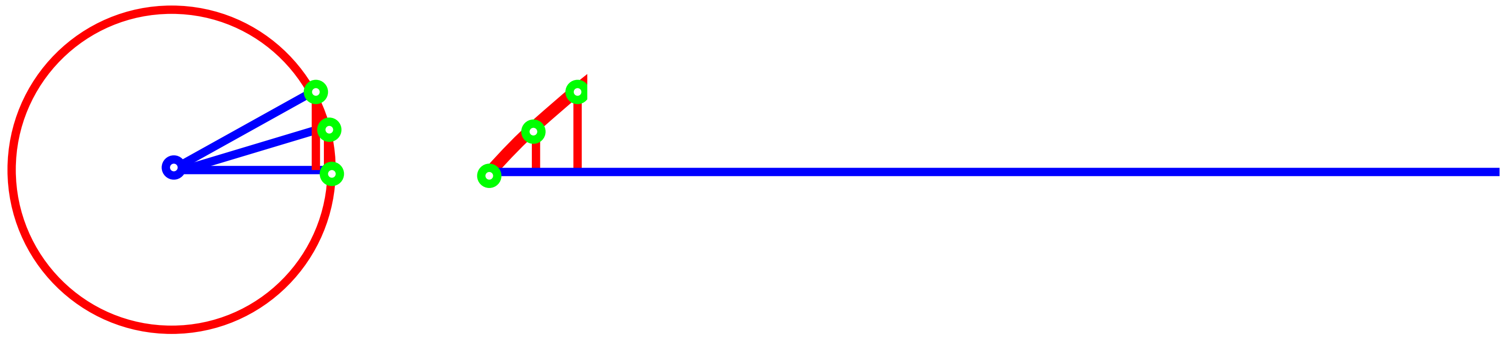
Bogenlänge b

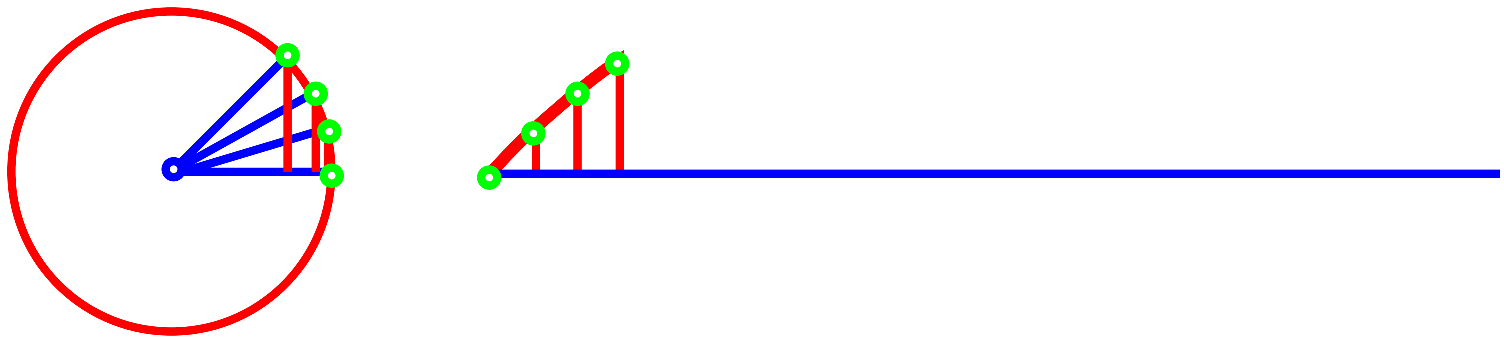
Winkelfunktionen

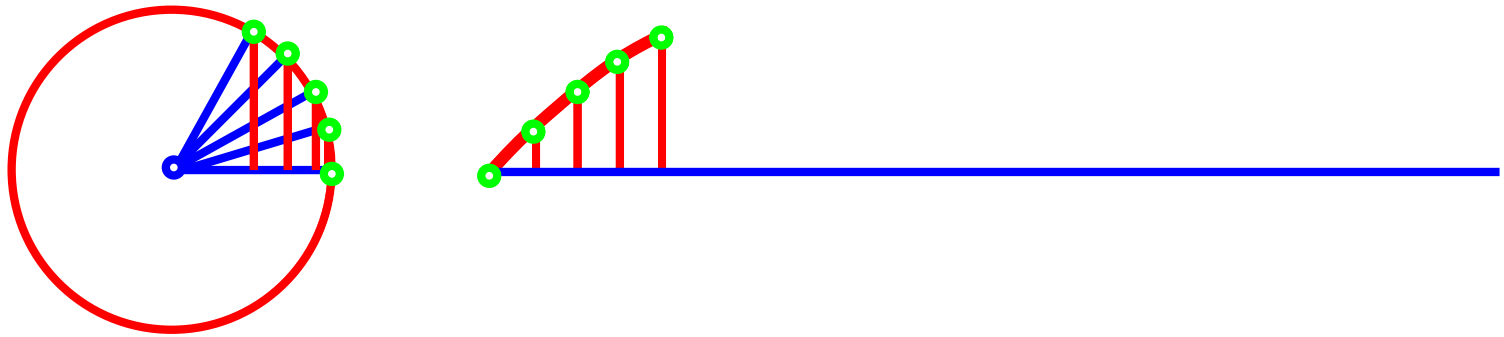


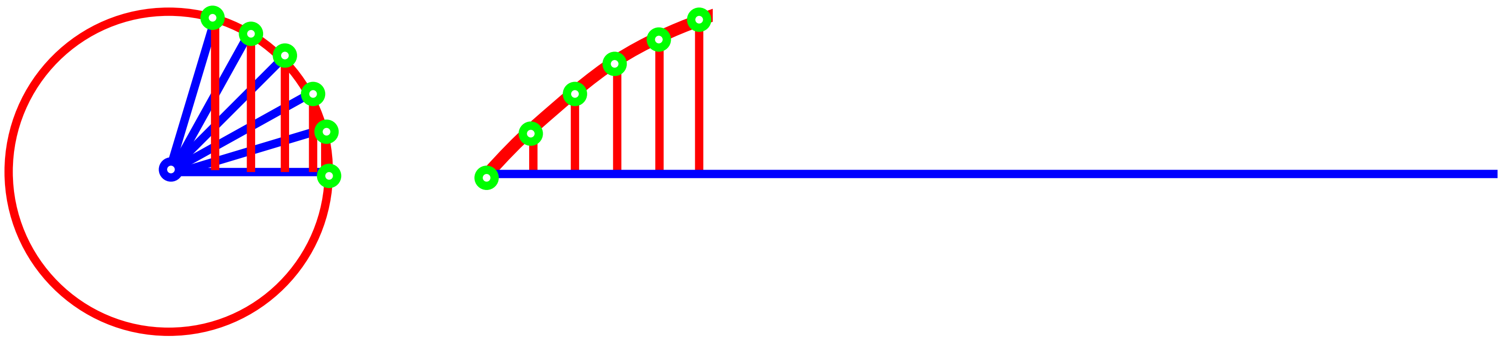


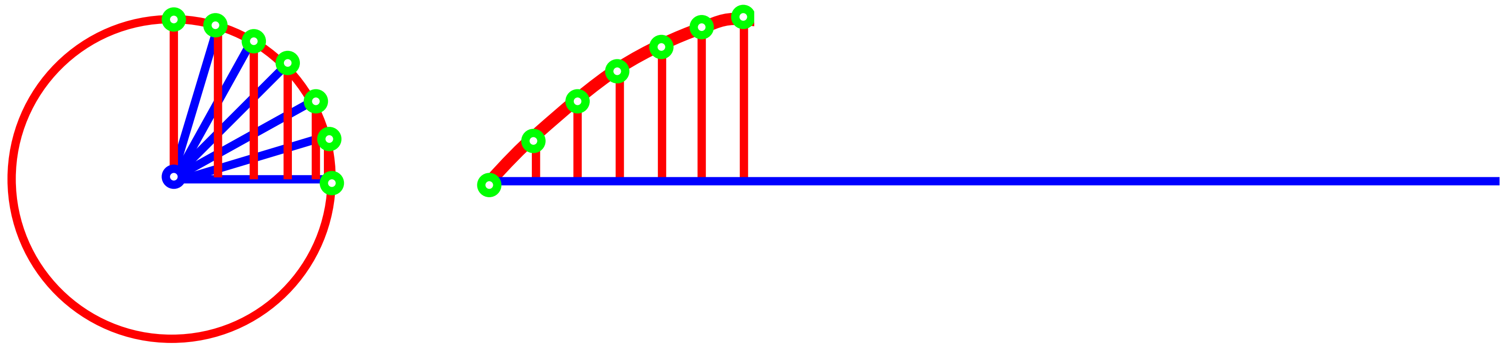


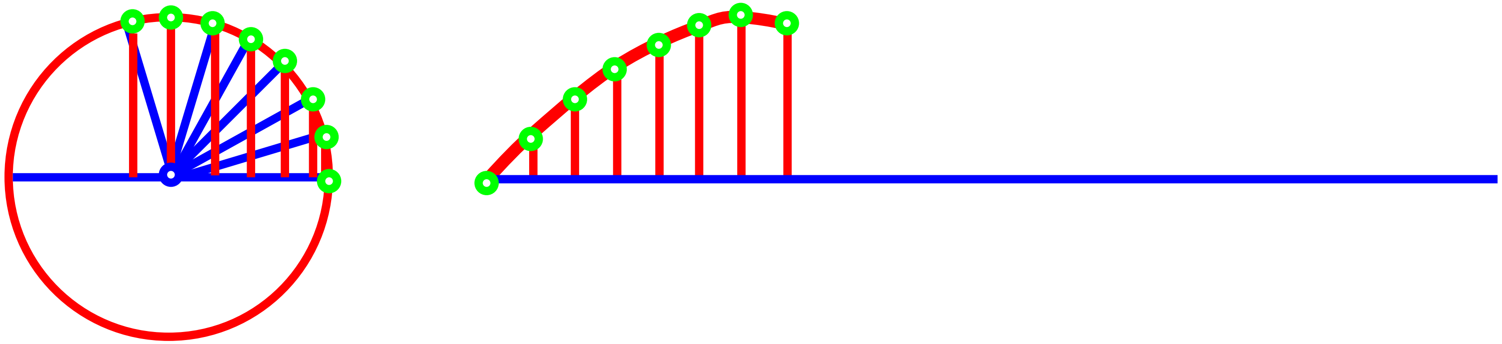


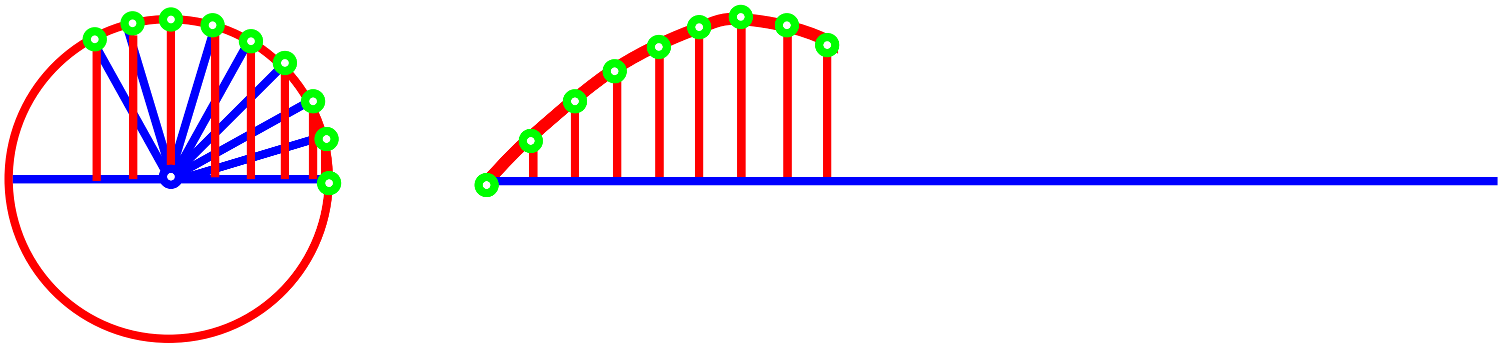


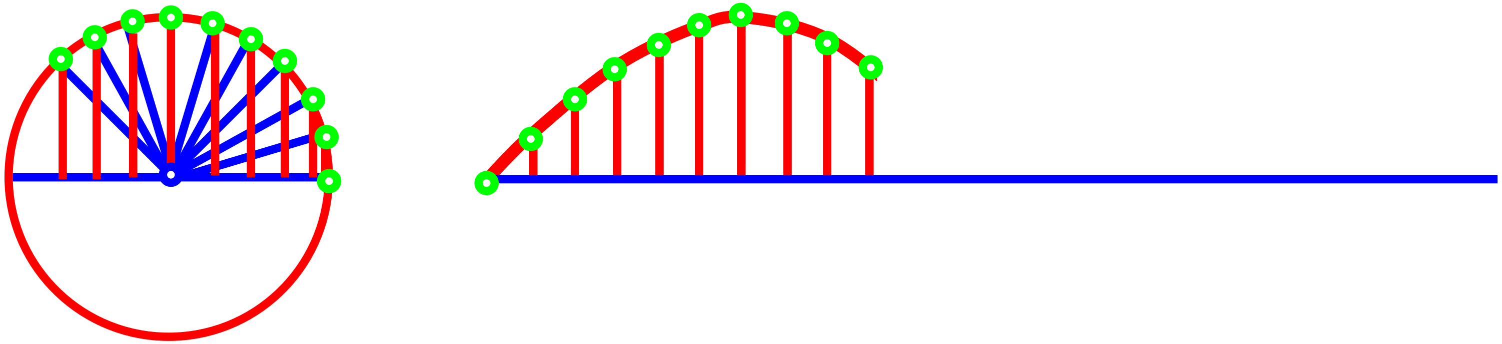


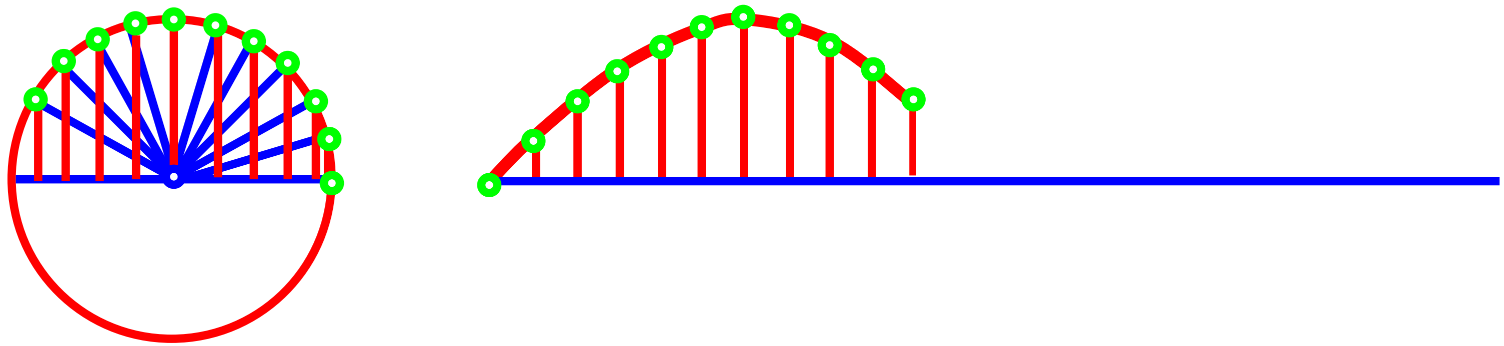


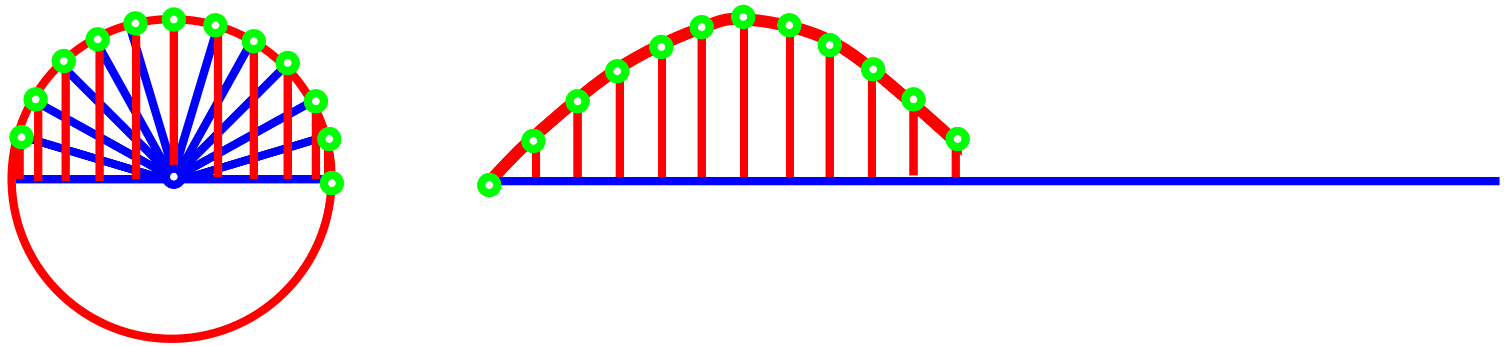


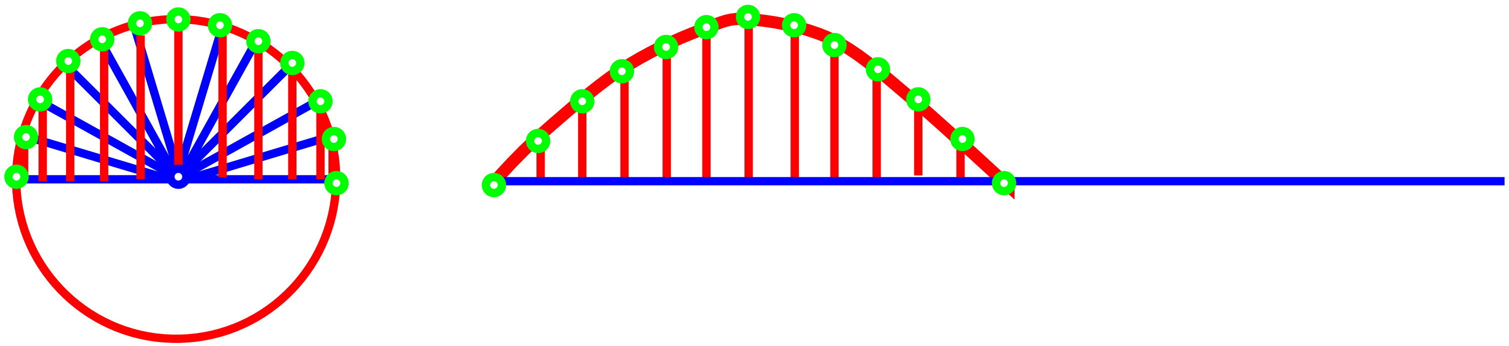


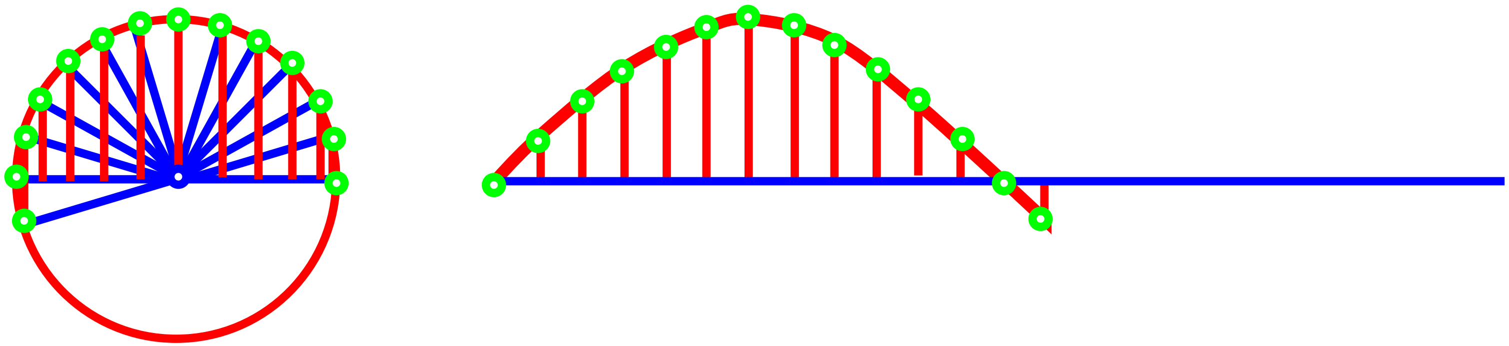


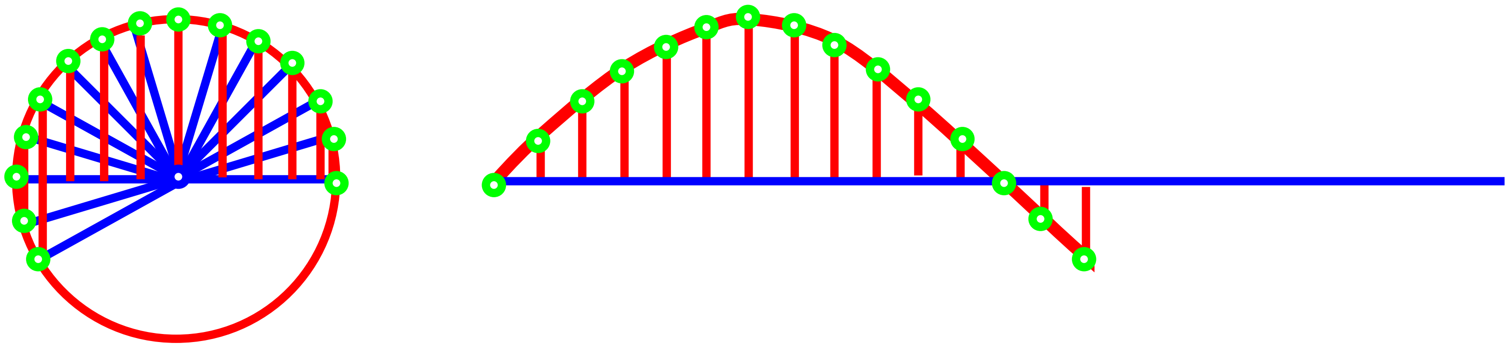


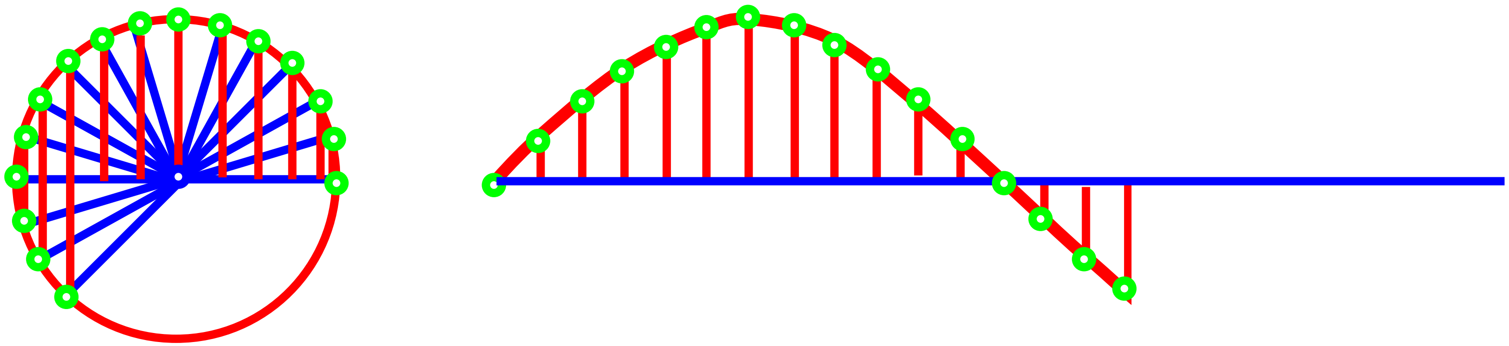


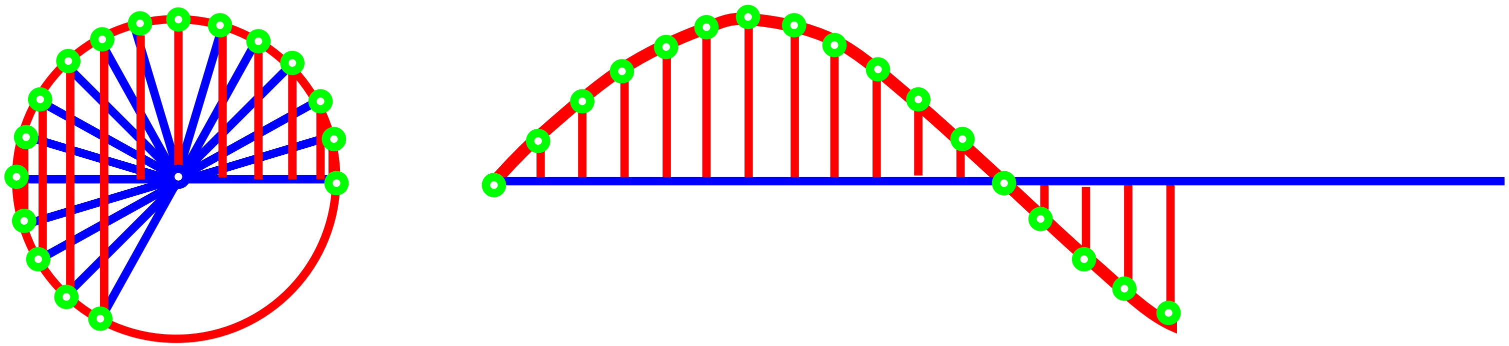


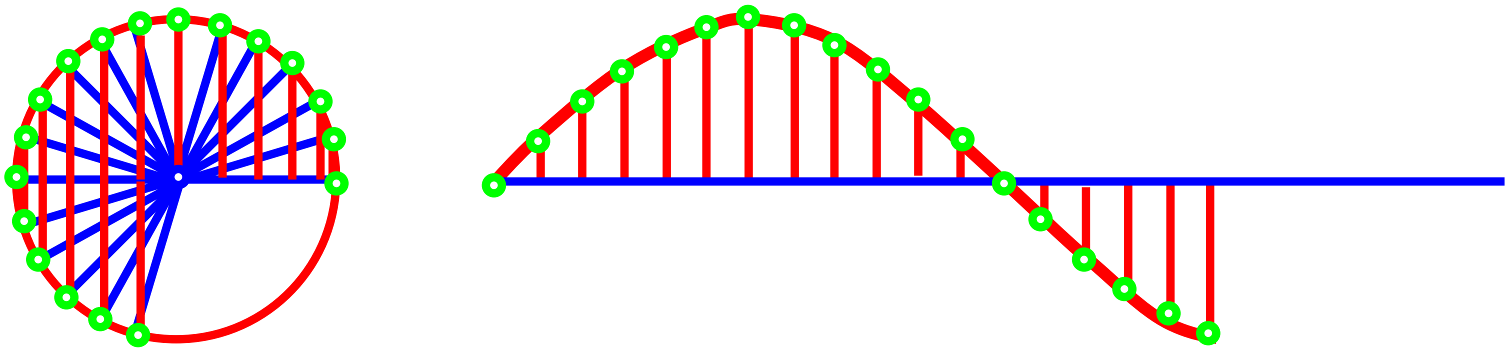


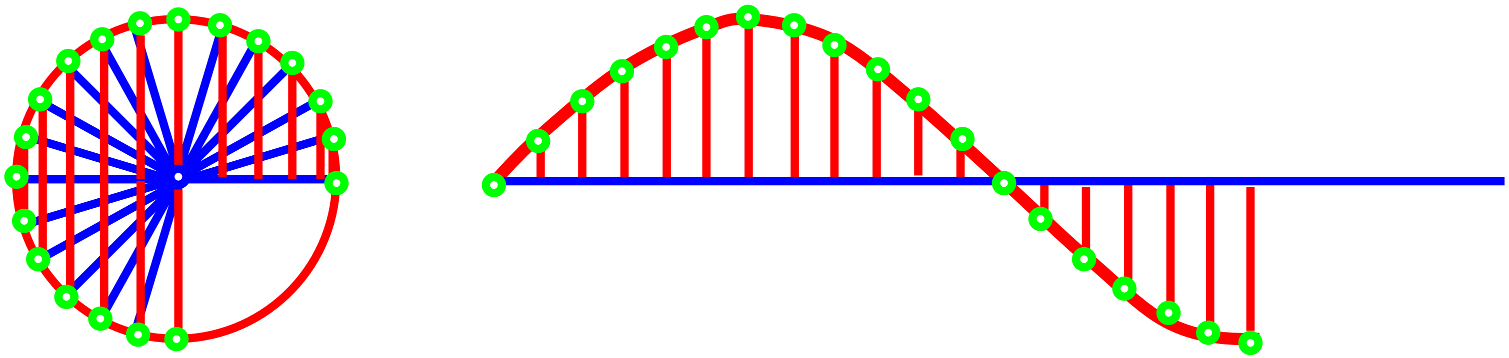


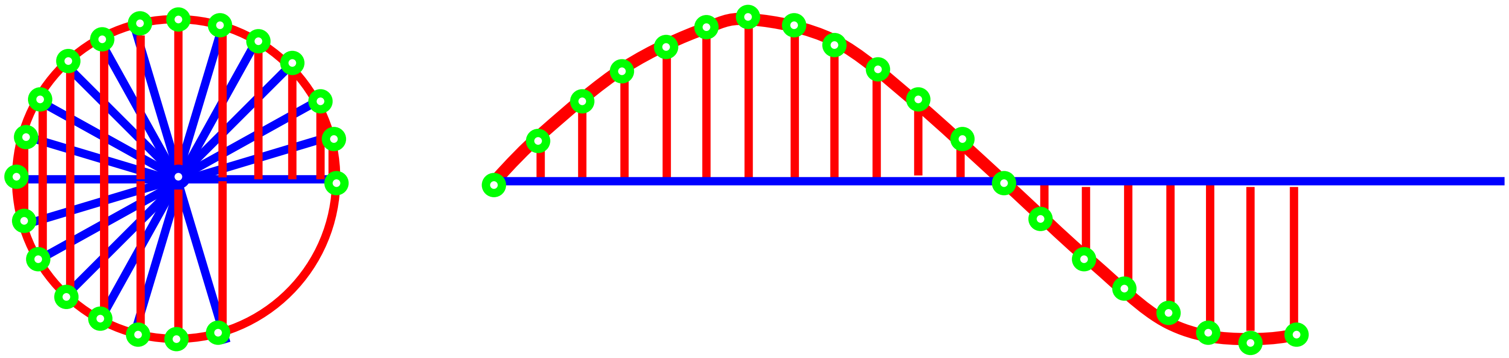


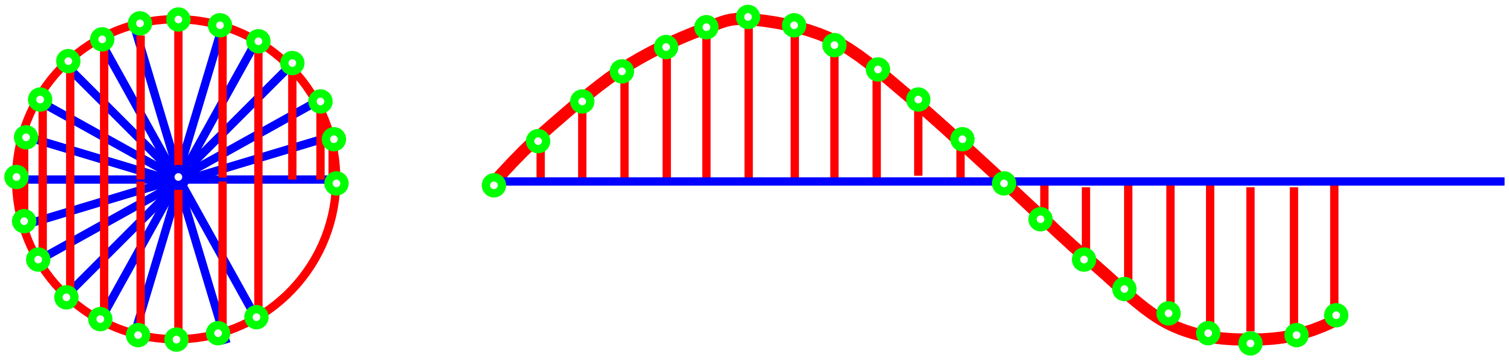


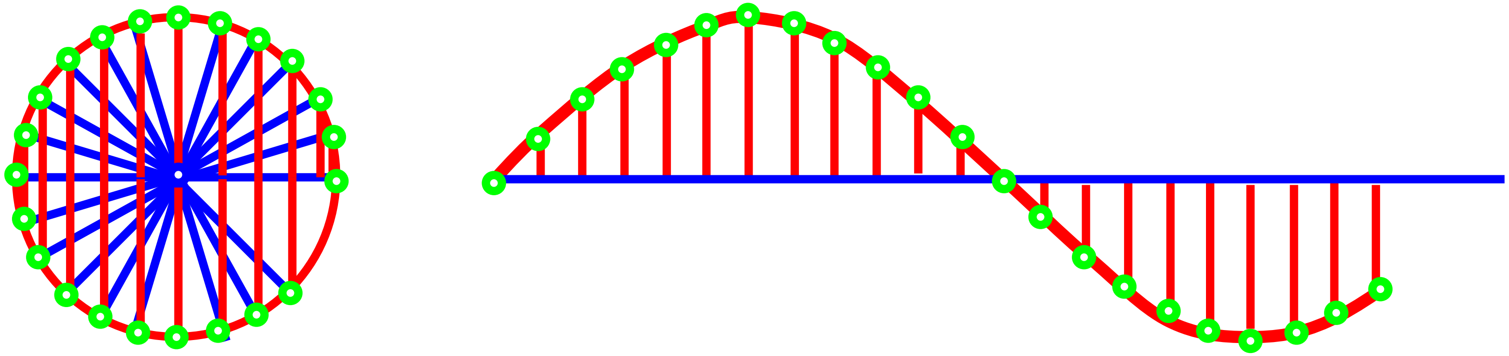


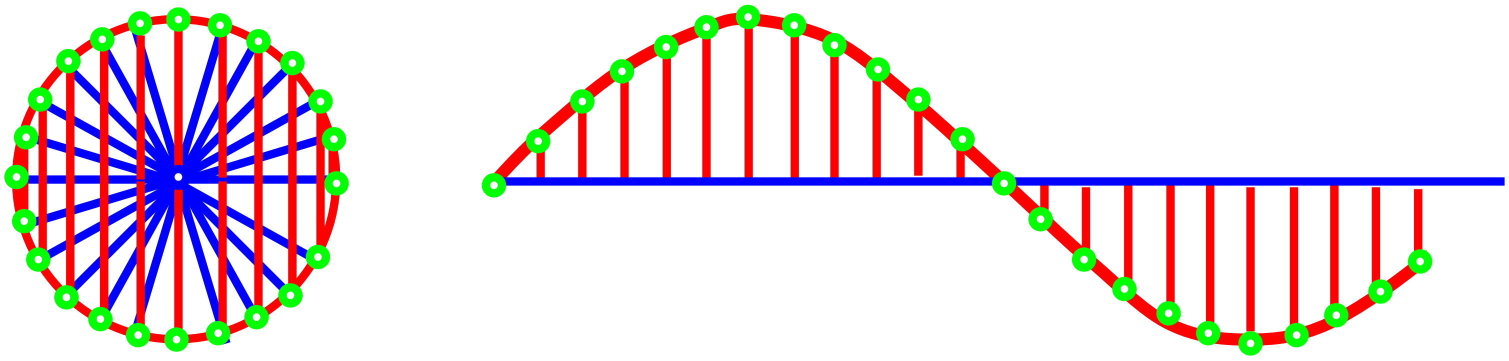


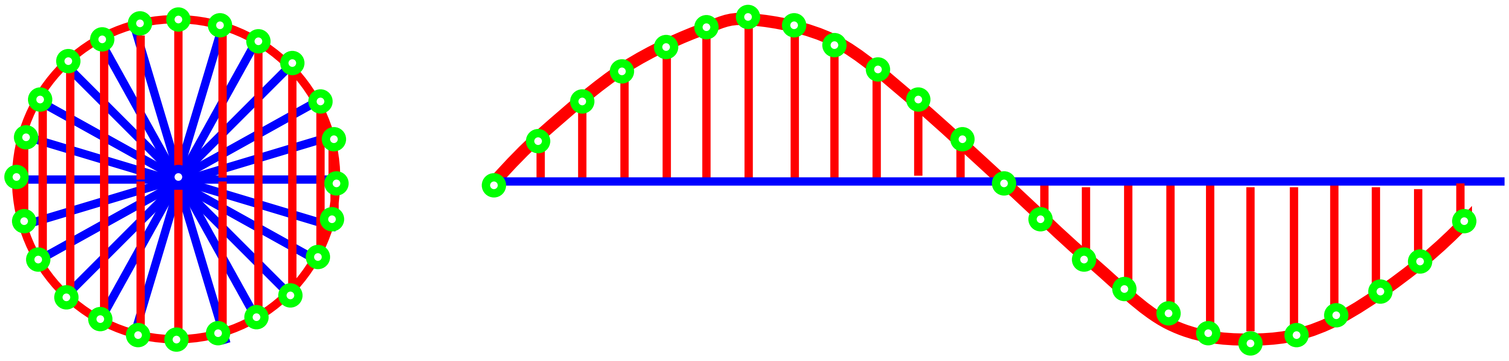


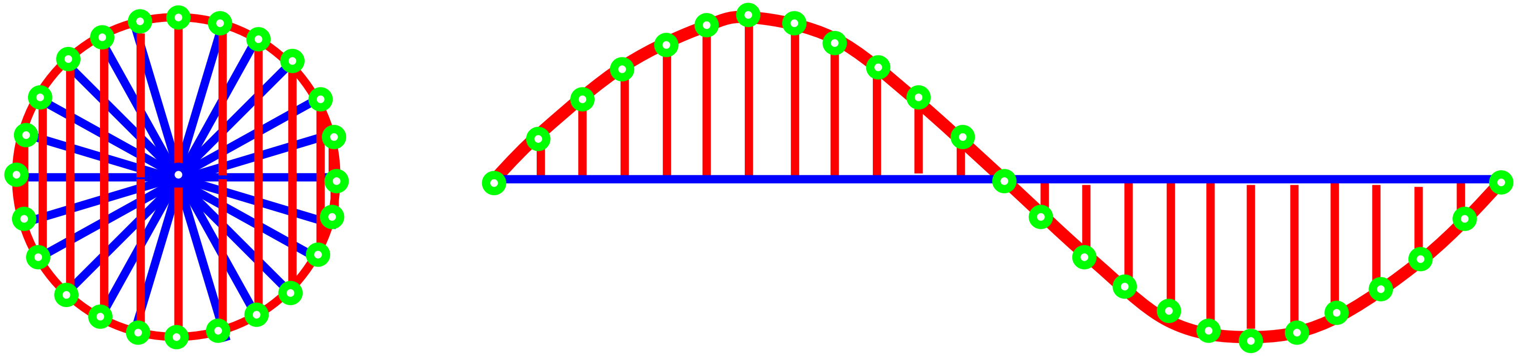




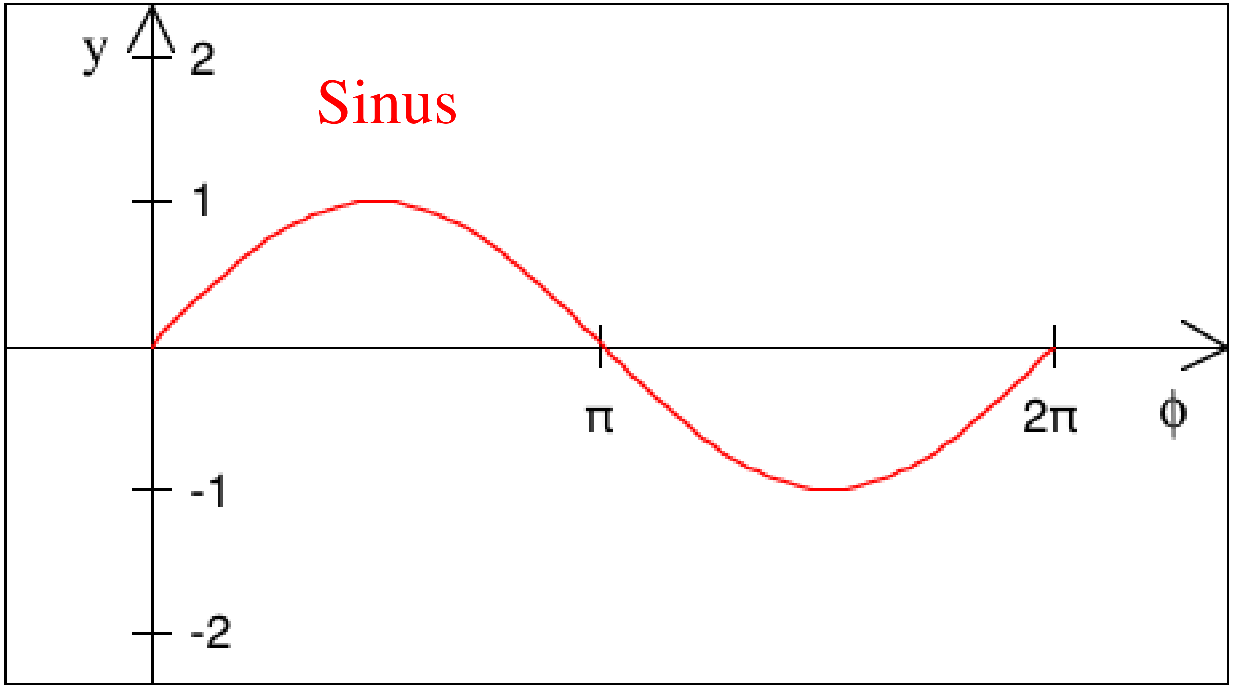




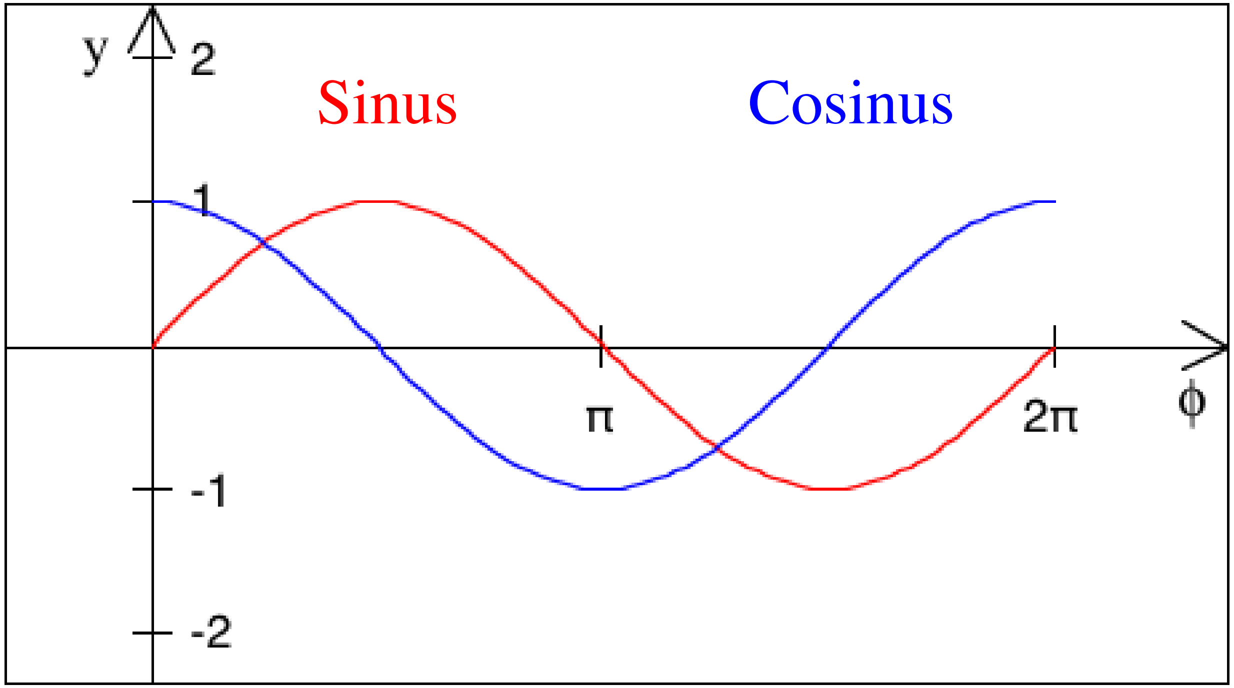




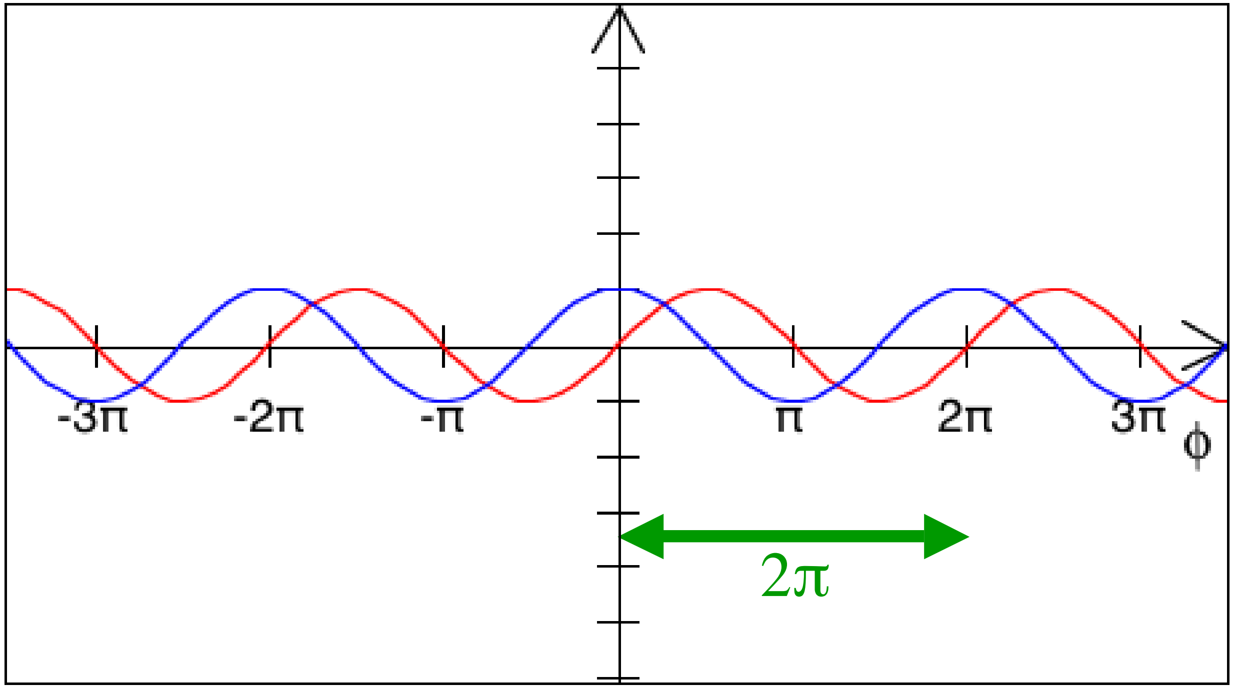
Winkelfunktionen



Winkelfunktionen

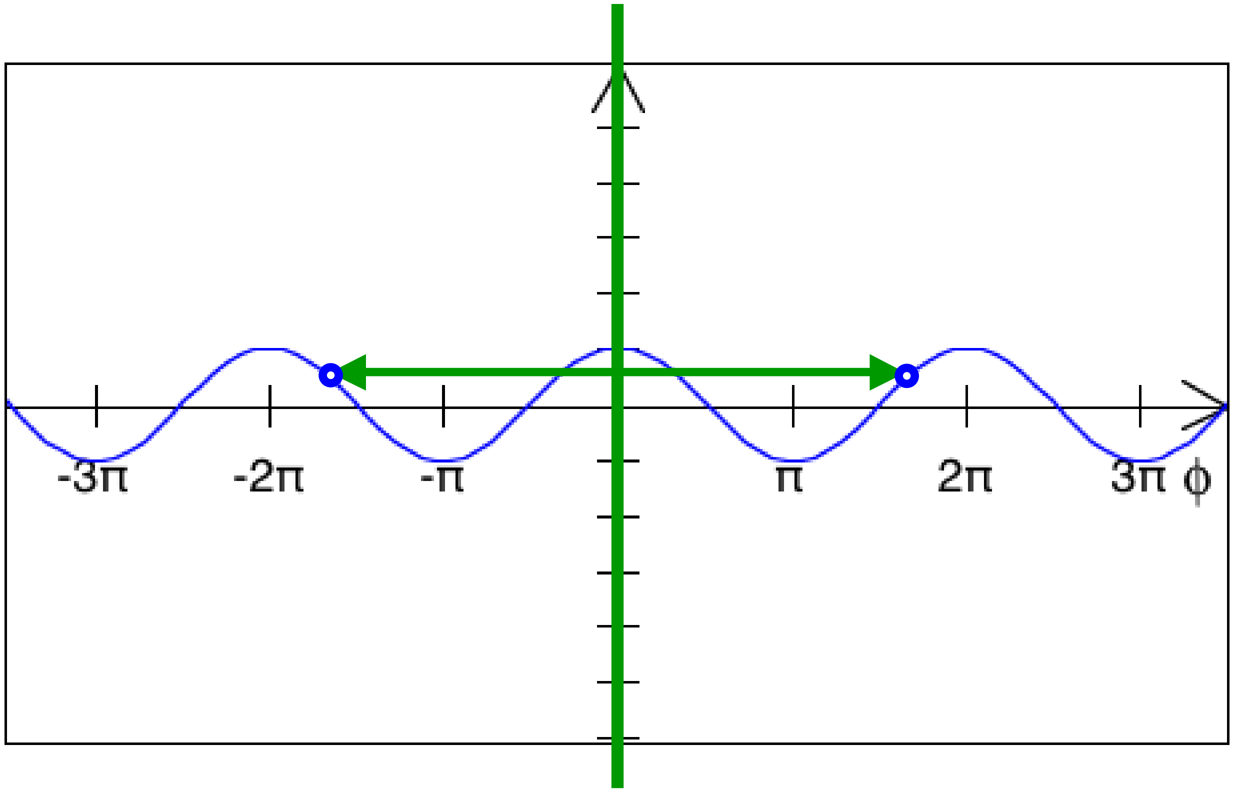


Winkelfunktionen



Periodenlänge 2π

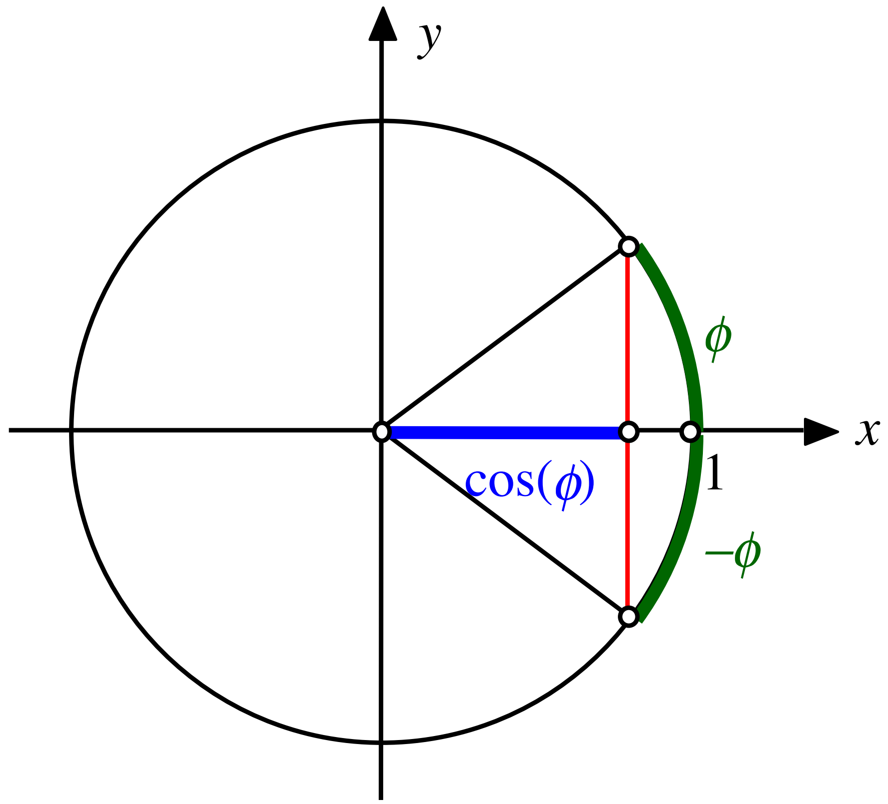
Winkelfunktionen



$$\cos(-\phi) = \cos(\phi)$$

Gerade Funktion, *y*-Achse Symmetrieachse

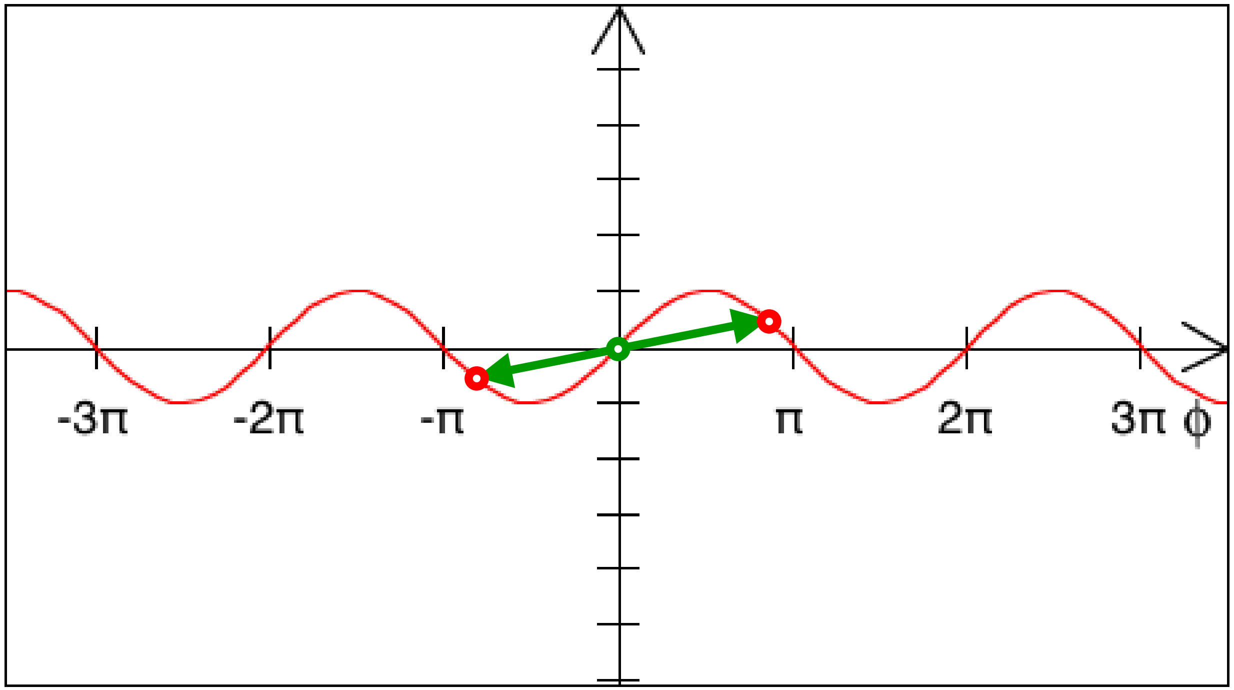
Winkelfunktionen



$$\cos(-\phi) = \cos(\phi)$$

Gerade Funktion

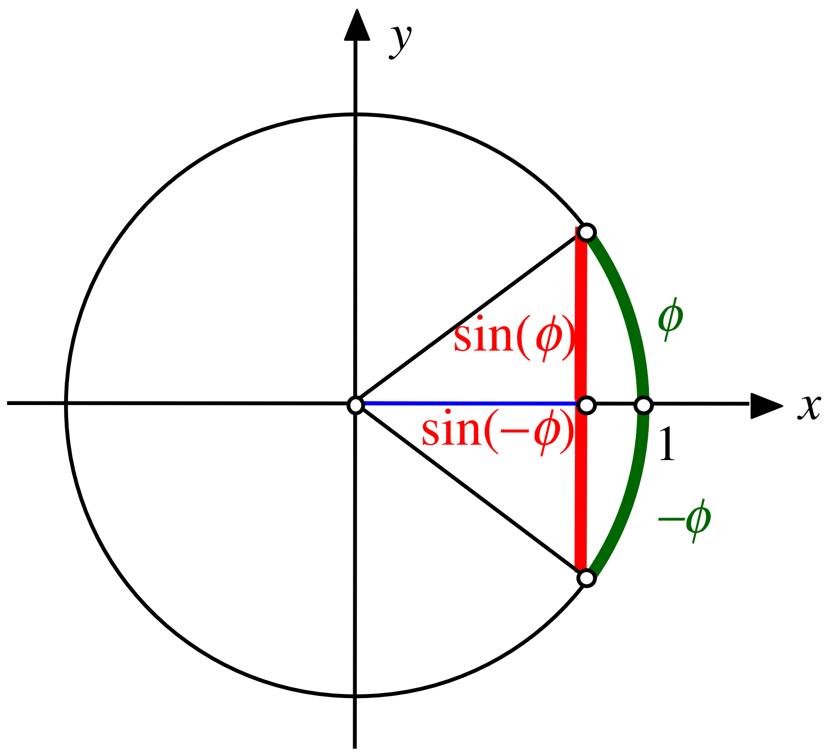
Winkelfunktionen



$$\sin(-\phi) = -\sin(\phi)$$

Ungerade Funktion, Ursprung Symmetriezentrum

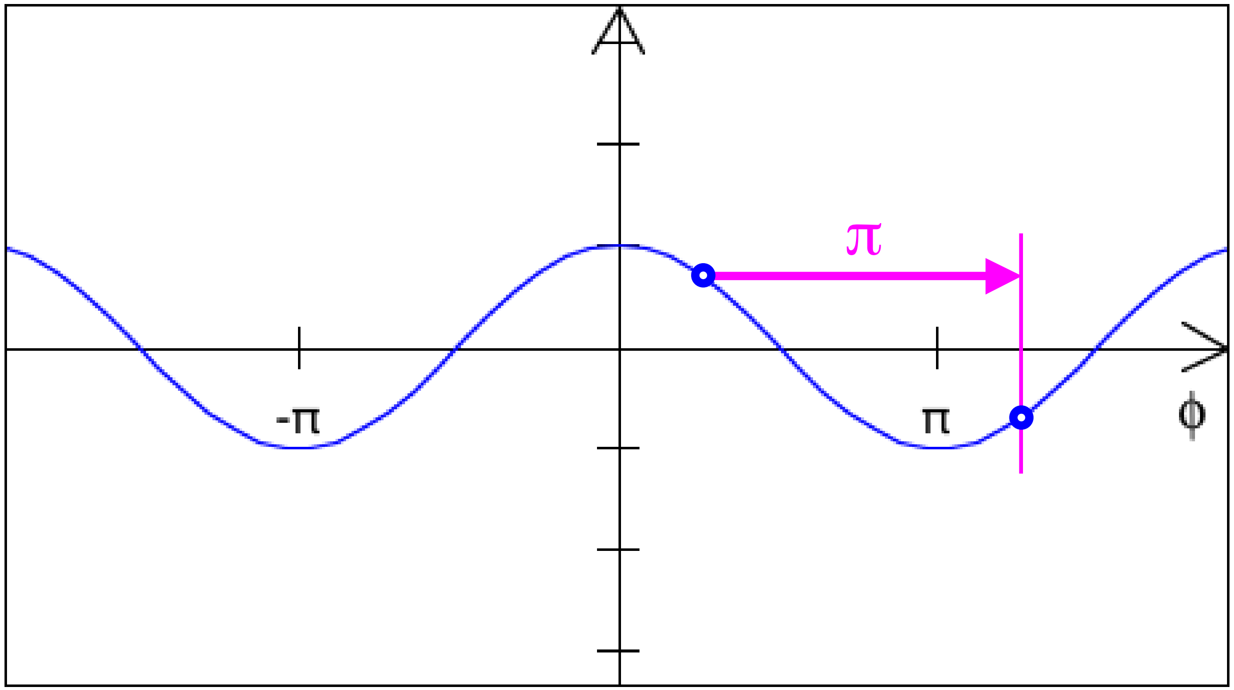
Winkelfunktionen



$$\sin(-\phi) = -\sin(\phi)$$

Ungerade Funktion

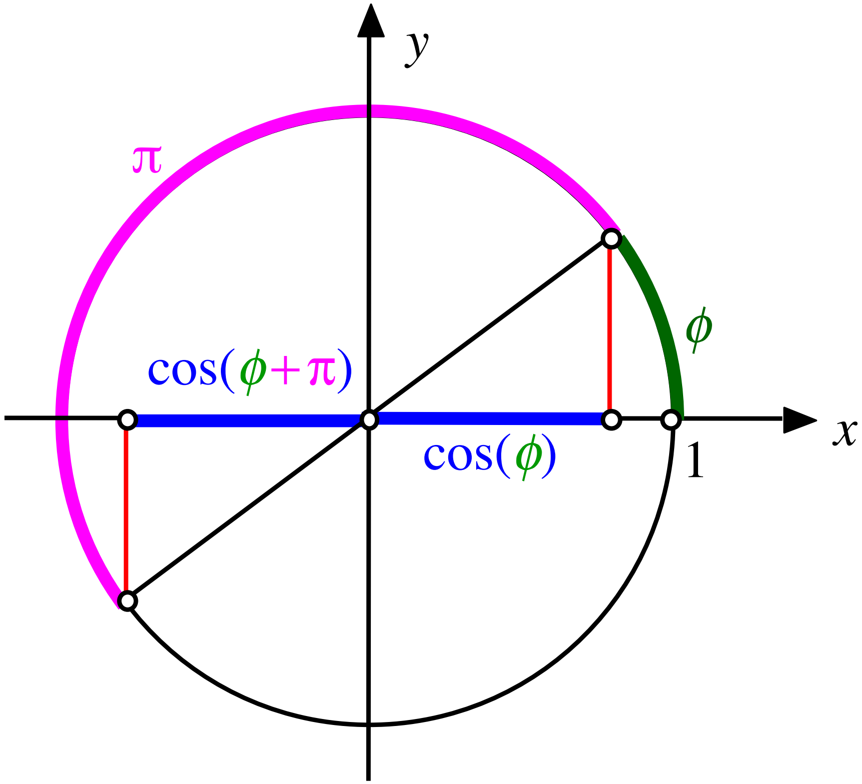
Winkelfunktionen



$$\cos(\phi+\pi) = -\cos(\phi)$$

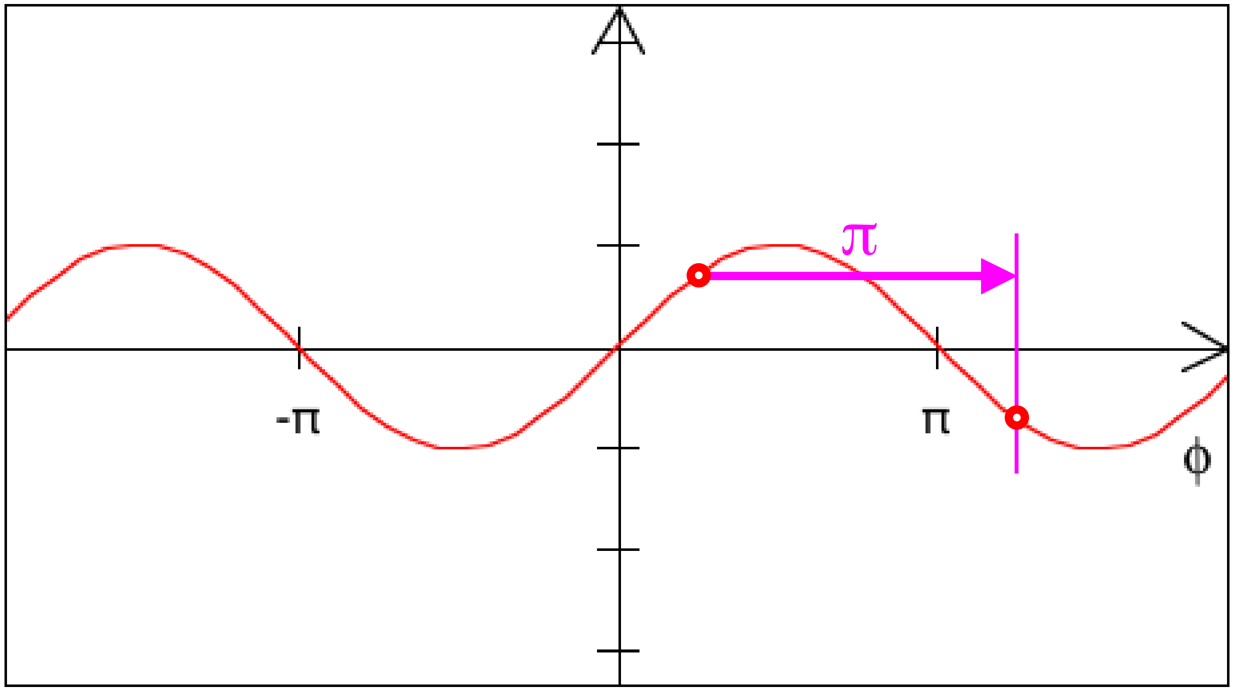
Schubspiegelsymmetrie

Winkelfunktionen



$$\cos(\phi+\pi) = -\cos(\phi)$$

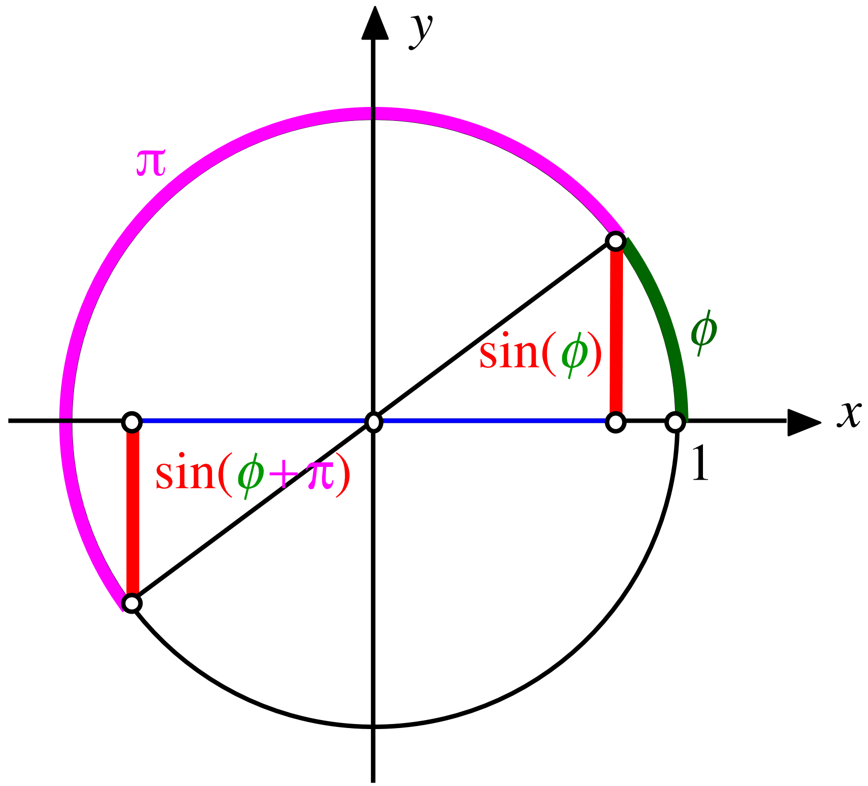
Winkelfunktionen



$$\sin(\phi+\pi) = -\sin(\phi)$$

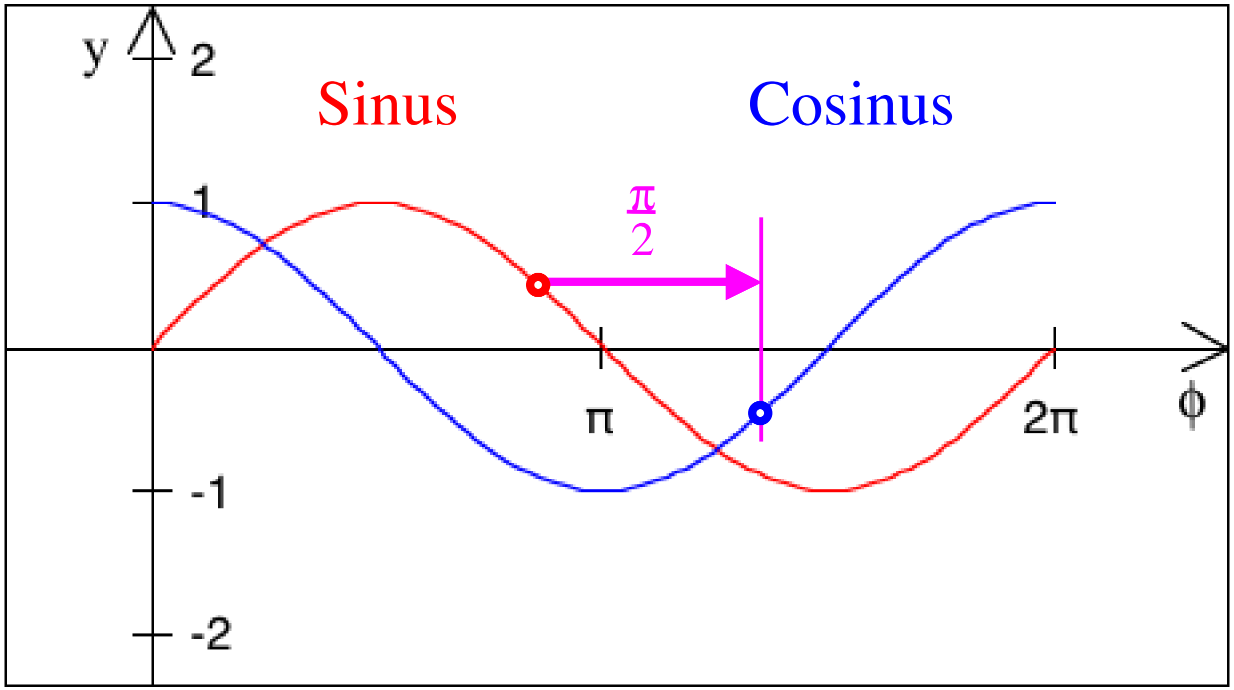
Schubspiegelsymmetrie

Winkelfunktionen



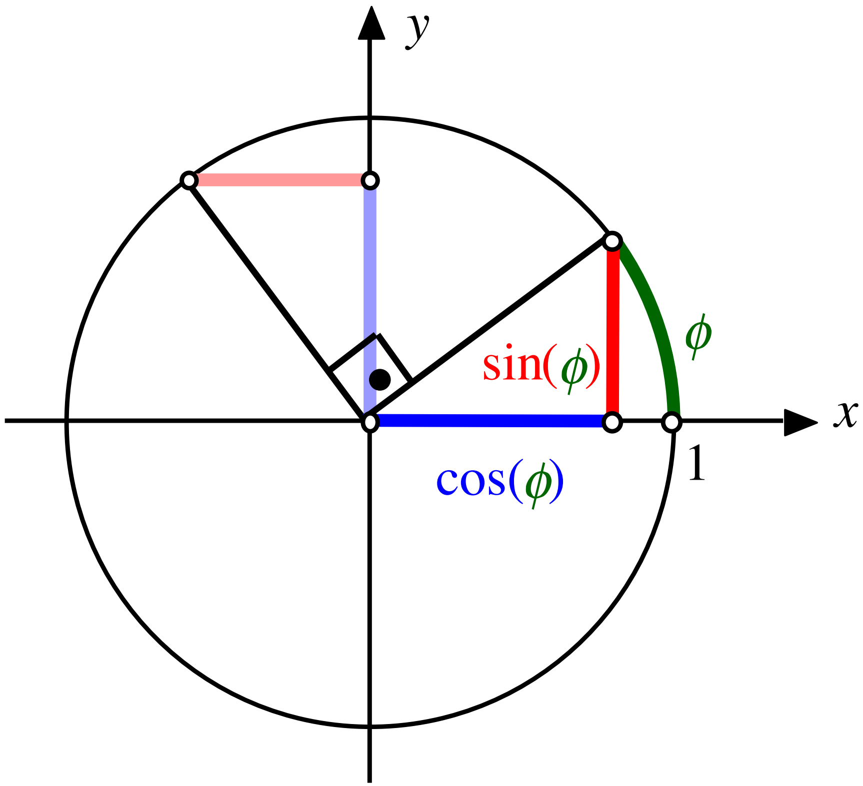
$$\sin(\phi+\pi) = -\sin(\phi)$$

Winkelfunktionen



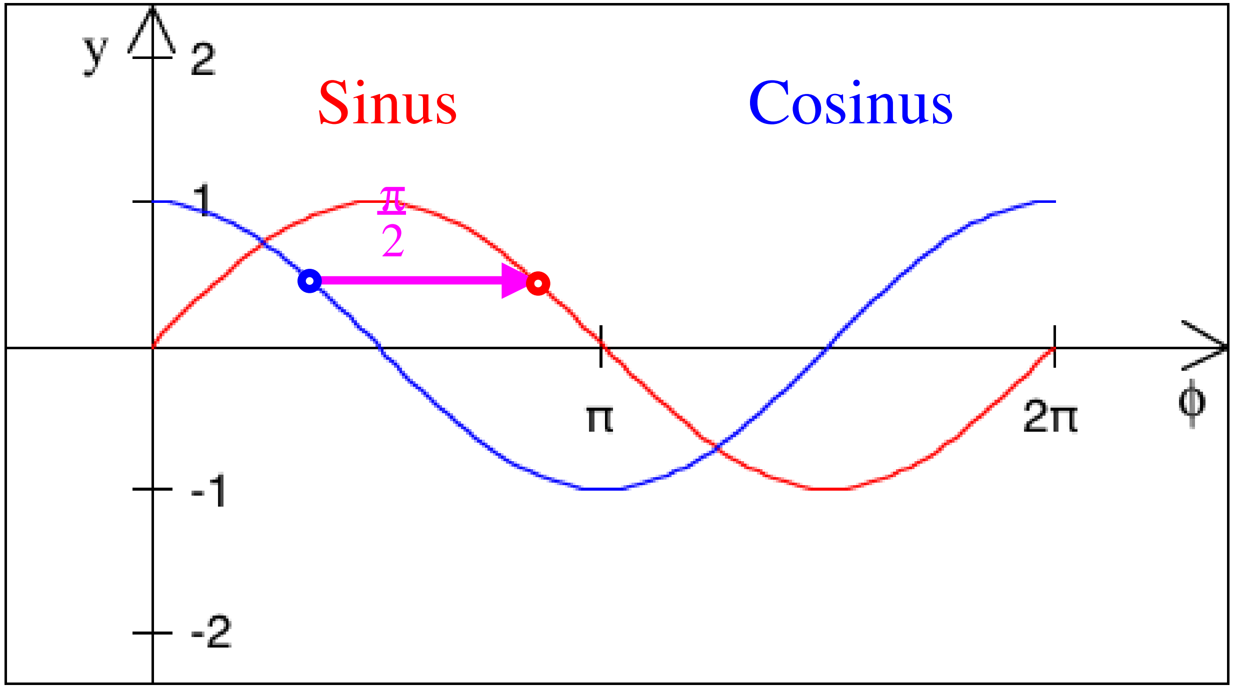
$$\cos\left(\phi + \frac{\pi}{2}\right) = -\sin(\phi)$$

Winkelfunktionen



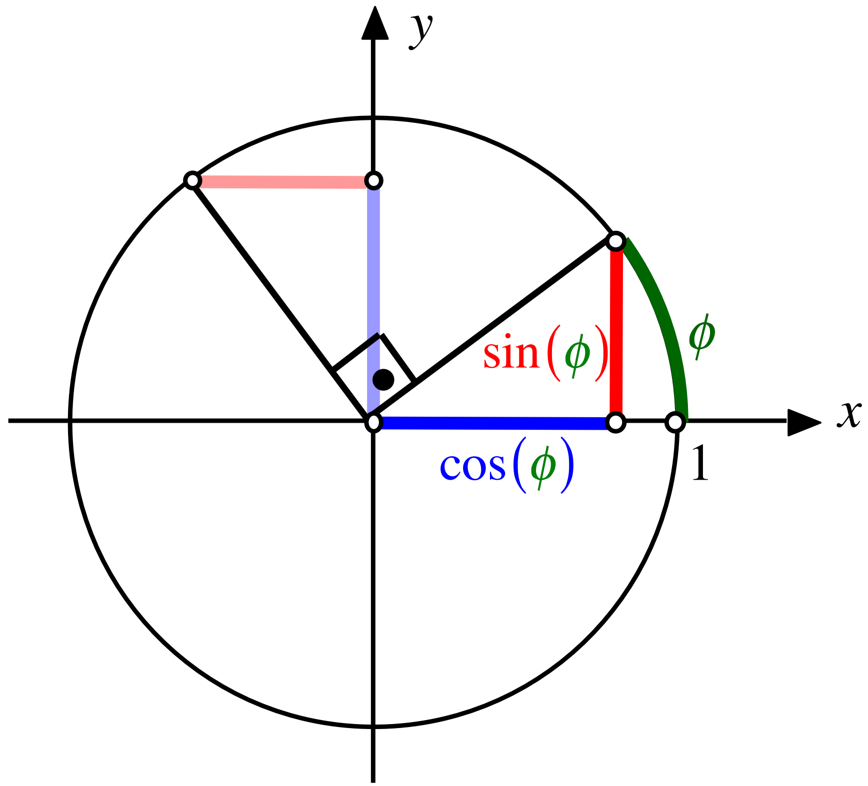
$$\cos\left(\phi + \frac{\pi}{2}\right) = -\sin(\phi)$$

Winkelfunktionen



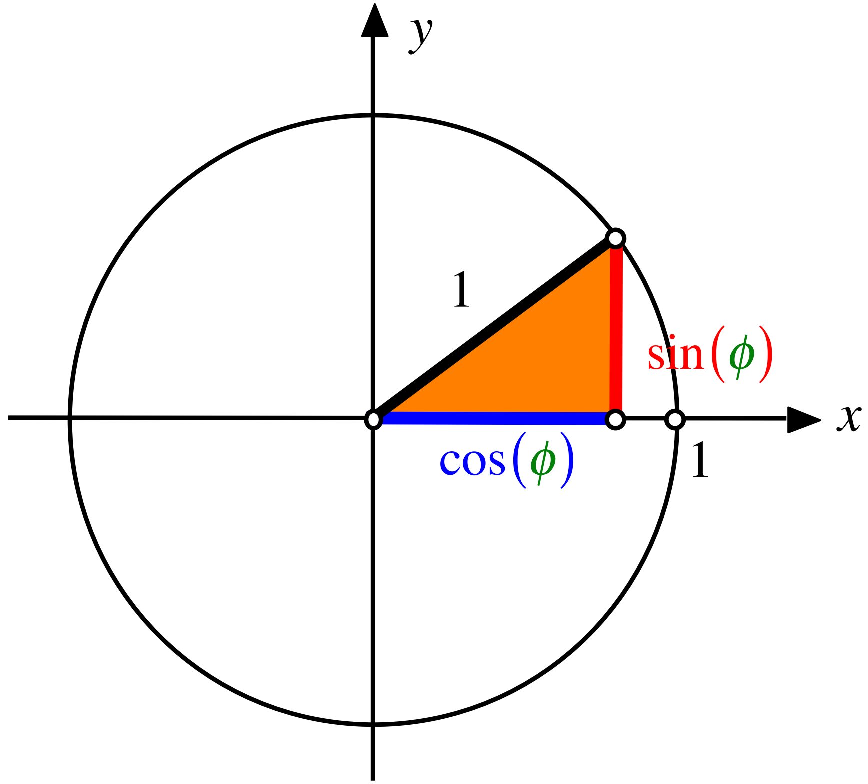
$$\sin\left(\phi + \frac{\pi}{2}\right) = \cos(\phi)$$

Winkelfunktionen



$$\sin\left(\phi + \frac{\pi}{2}\right) = \cos(\phi)$$

Winkelfunktionen



$$\left(\cos(\phi)\right)^2 + \left(\sin(\phi)\right)^2 = 1$$

Grüße von Pythagoras

Winkelfunktionen

Schreibweisen beim Quadrat

$$\sin^2\phi = \sin^2(\phi) \quad = \quad (\sin(\phi))^2$$

Kurzschreibweisen

logisch korrekt

Computerverständlich

$$\cos^2\phi + \sin^2\phi = 1$$

$$\cos^2(\phi) + \sin^2(\phi) = 1 \qquad (\cos(\phi))^2 + (\sin(\phi))^2 = 1$$

Grüße von Pythagoras

Winkelfunktionen

Additionstheoreme
Doppelwinkelformeln

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

Sonderfall: $\cos(2\alpha)=(\cos(\alpha))^2-(\sin(\alpha))^2$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

Sonderfall: $\sin(2\alpha)=2\sin(\alpha)\cos(\alpha)$

Siehe Formelsammlung

Winkelfunktionen

Additionstheoreme
Doppelwinkelformeln

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

Sonderfall: $\cos(2\alpha)=(\cos(\alpha))^2-(\sin(\alpha))^2$

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

Sonderfall: $\sin(2\alpha)=2\sin(\alpha)\cos(\alpha)$

Haben Sie es in der Formelsammlung gesehen?

Winkelfunktionen Arcus, im Prinzip die Umkehrung

Beispiele $\arcsin(\sin(1.1)) = 1.1$

Dasselbe

Winkelfunktionen Arcus, im Prinzip die Umkehrung

Beispiele $\arcsin(\sin(1.1)) = 1.1$

$$\arcsin(\sin(6.5)) = 0.2168$$

NICHT dasselbe

Haben Sie Bogenmaß (radians) eingestellt?

Winkelfunktionen Arcus, im Prinzip die Umkehrung

Beispiele

$$\arcsin(\sin(1.1)) = 1.1$$

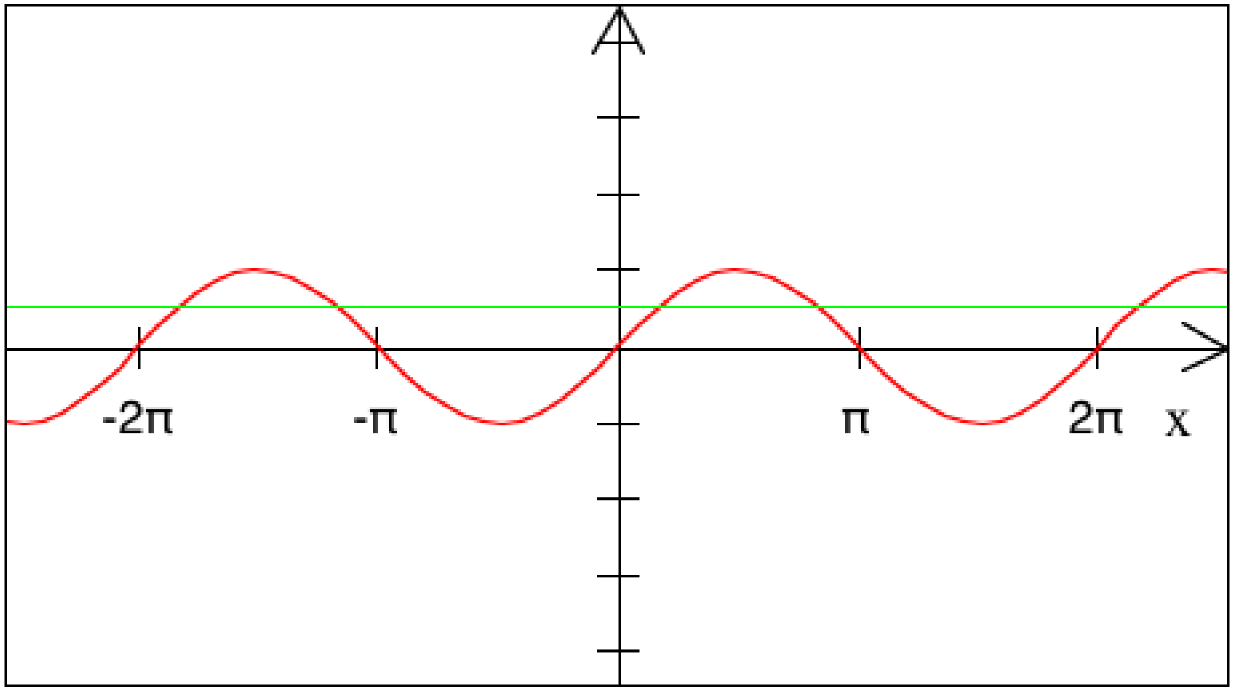
$$\arcsin(\sin(6.5)) = 0.2168$$

Fehlbetrag:

$$6.5 - 0.2168 = 6.2832 = 2\pi$$

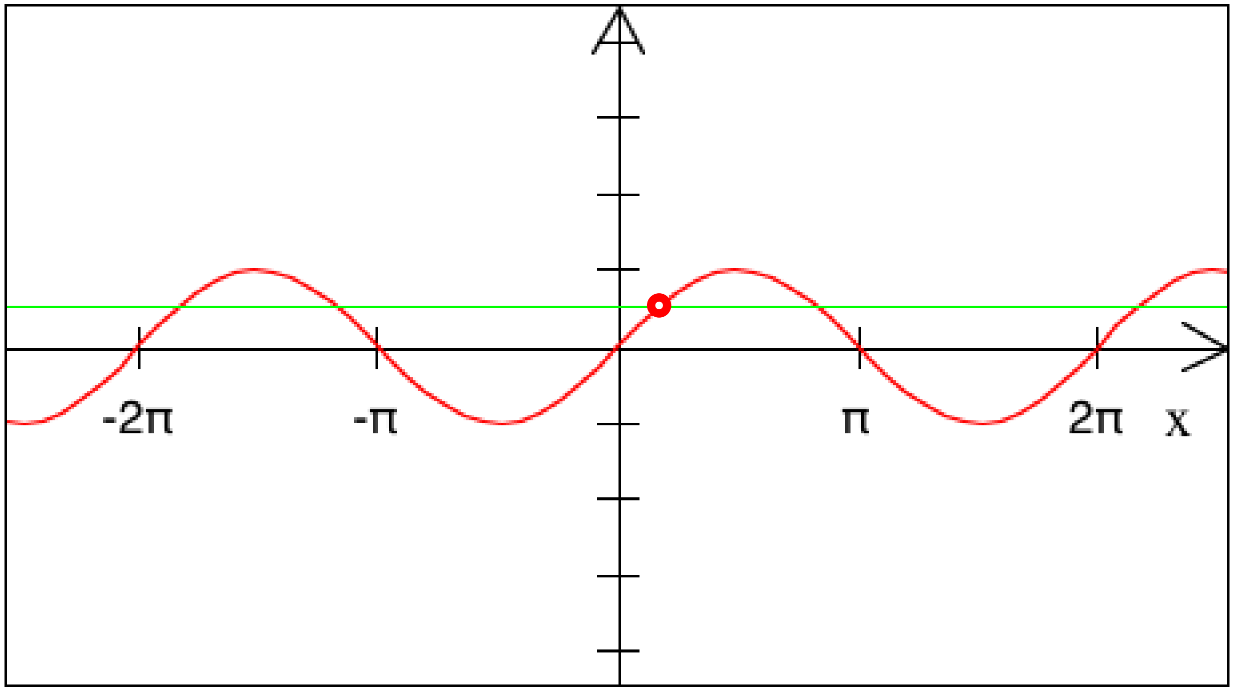
Die Arcusfunktionen beschränken sich auf einen Standardbereich. Warum?

Winkelfunktionen Arcus, im Prinzip die Umkehrung



$$\sin(x) = 0.5 \qquad x = ?$$

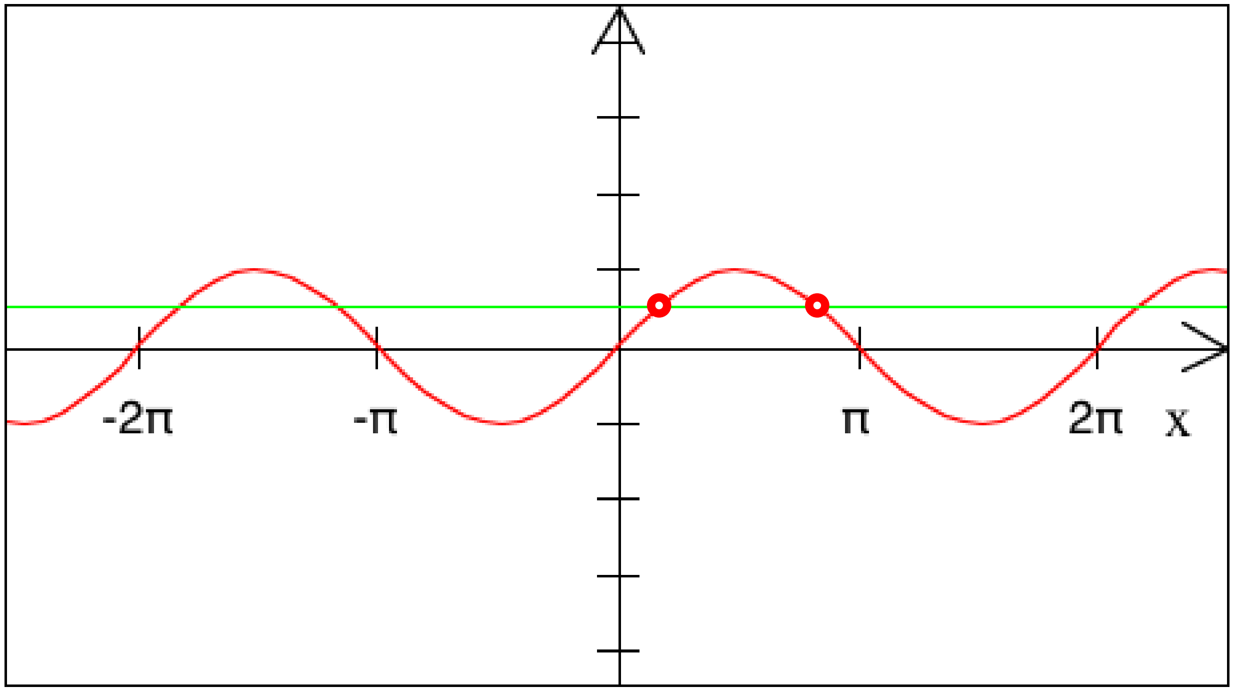
Winkelfunktionen Arcus, im Prinzip die Umkehrung



$$\sin(x) = 0.5 \qquad x = ?$$

$$x \in \left\{\frac{\pi}{6}\right.$$

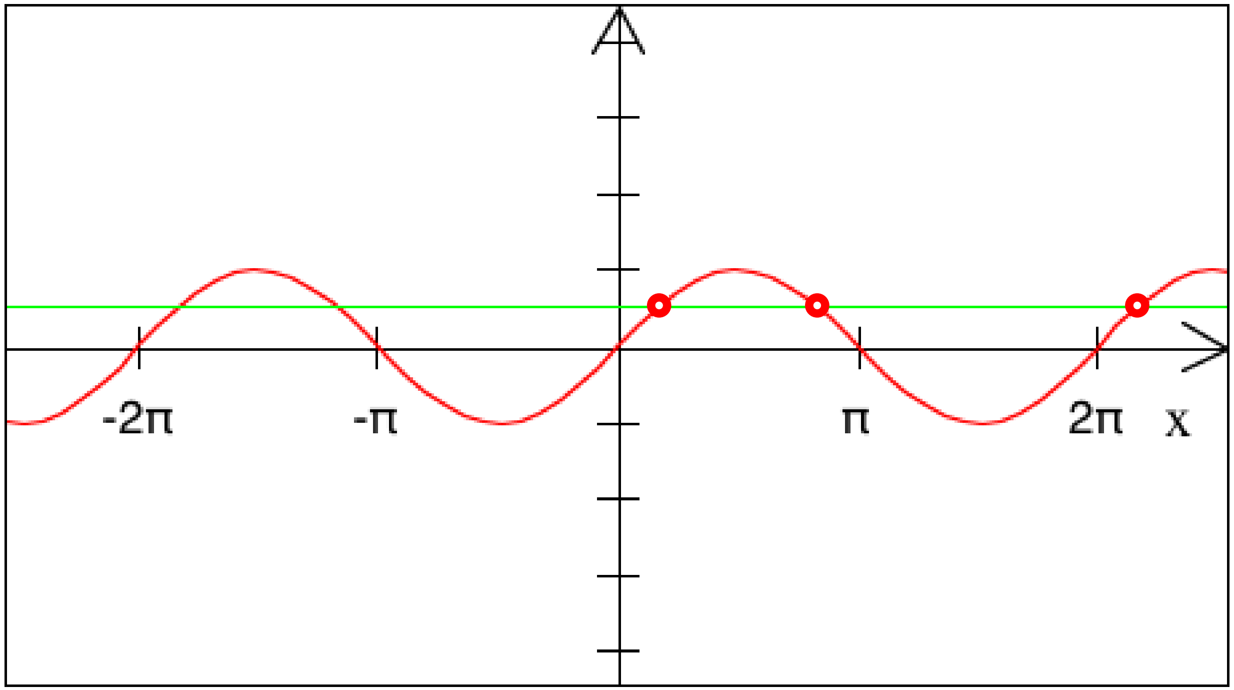
Winkelfunktionen Arcus, im Prinzip die Umkehrung



$$\sin(x) = 0.5 \qquad x = ?$$

$$x \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right.$$

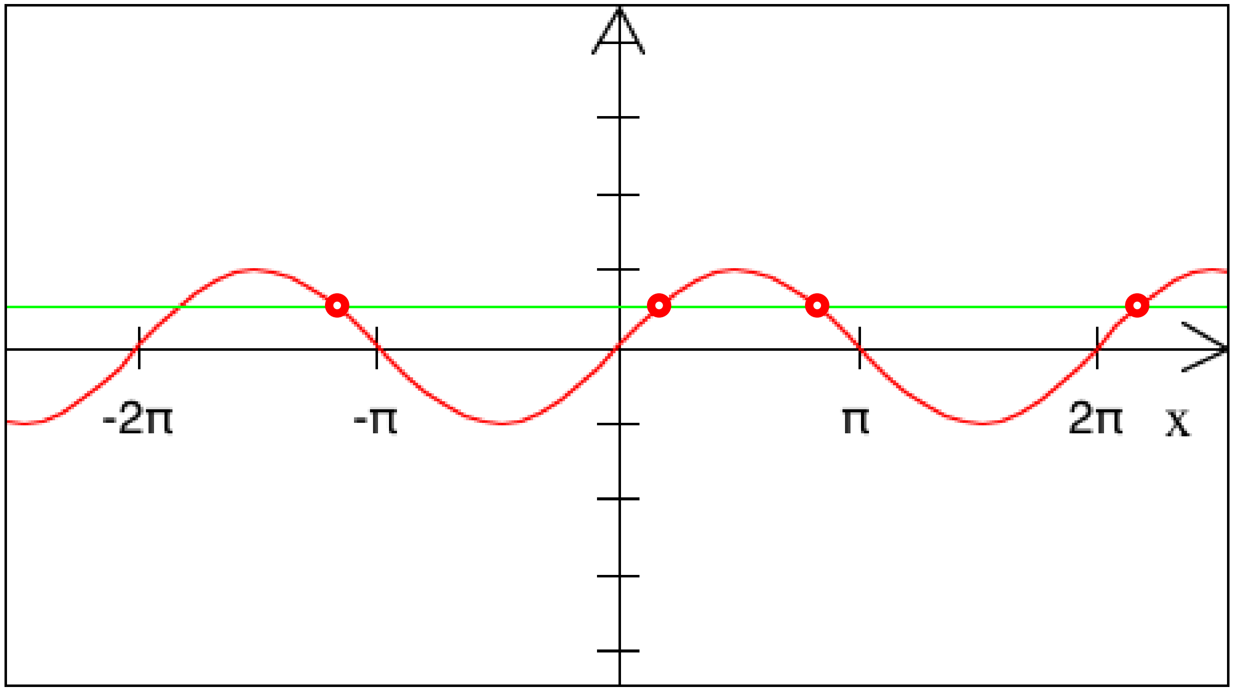
Winkelfunktionen Arcus, im Prinzip die Umkehrung



$$\sin(x) = 0.5 \qquad x = ?$$

$$x \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6} \right.$$

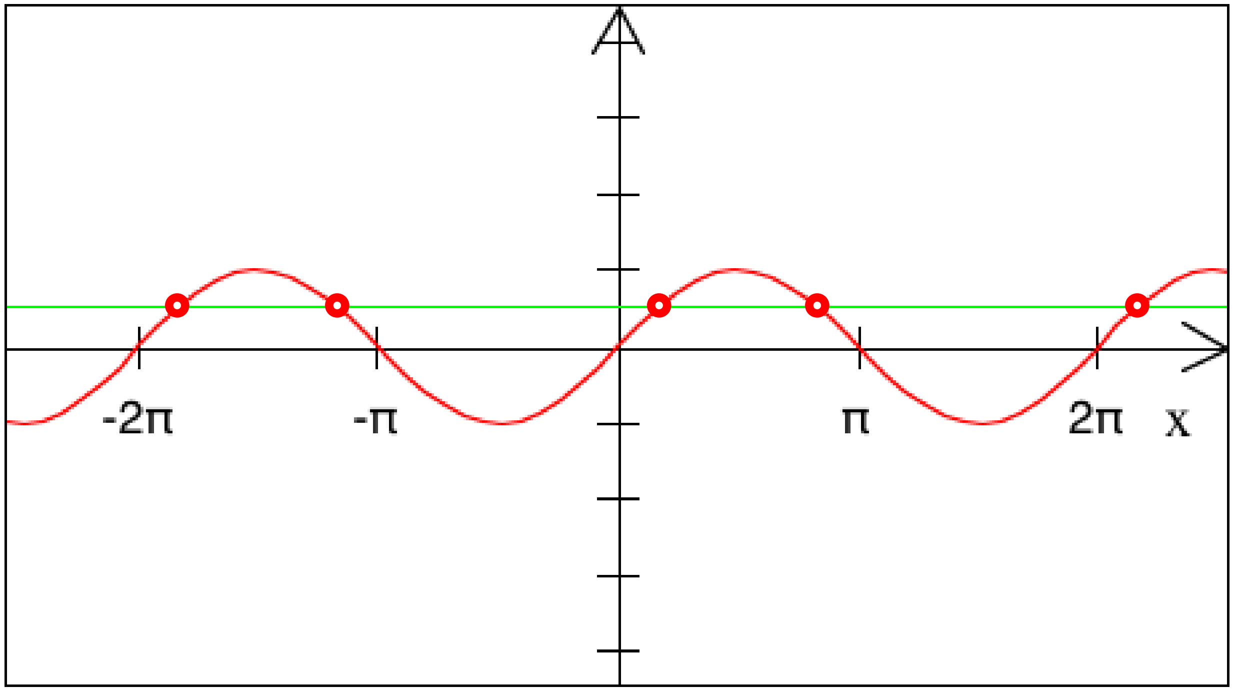
Winkelfunktionen Arcus, im Prinzip die Umkehrung



$$\sin(x) = 0.5 \qquad x = ?$$

$$x \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots, \frac{-7\pi}{6} \right.$$

Winkelfunktionen Arcus, im Prinzip die Umkehrung

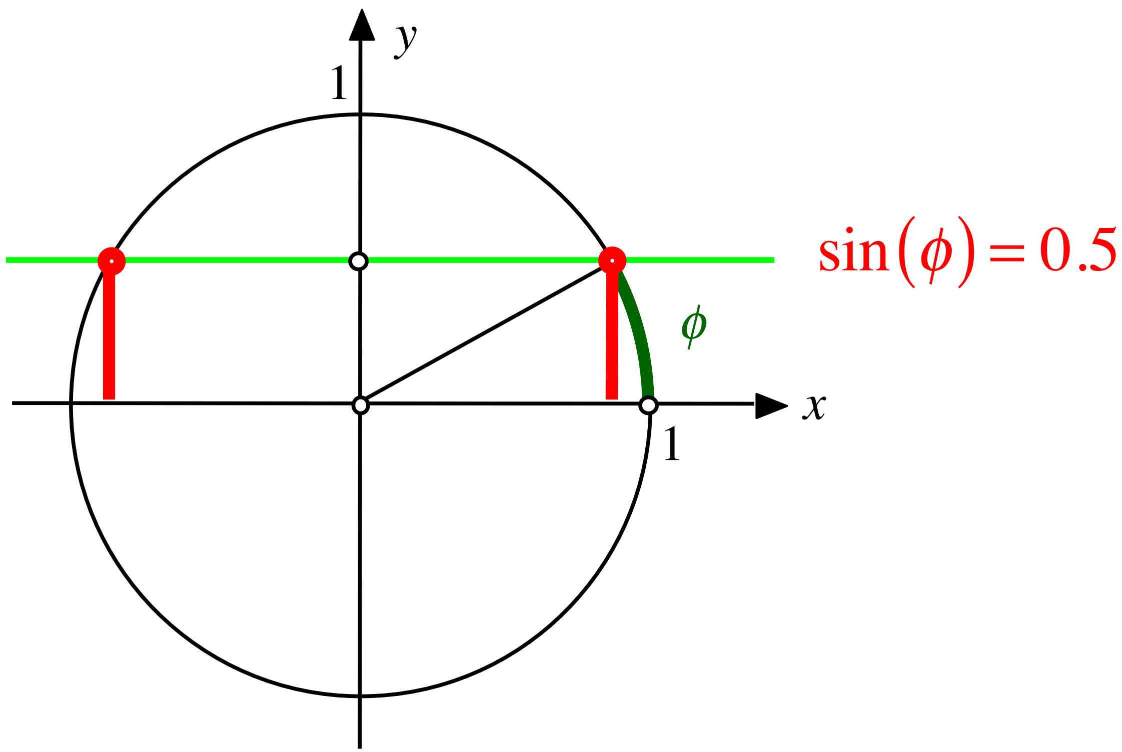


$$\sin(x) = 0.5 \qquad x = ?$$

$$x \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots, \frac{-7\pi}{6}, \frac{-11\pi}{6}, \ldots\right\}$$

Winkelfunktionen

Arcus, im Prinzip die Umkehrung

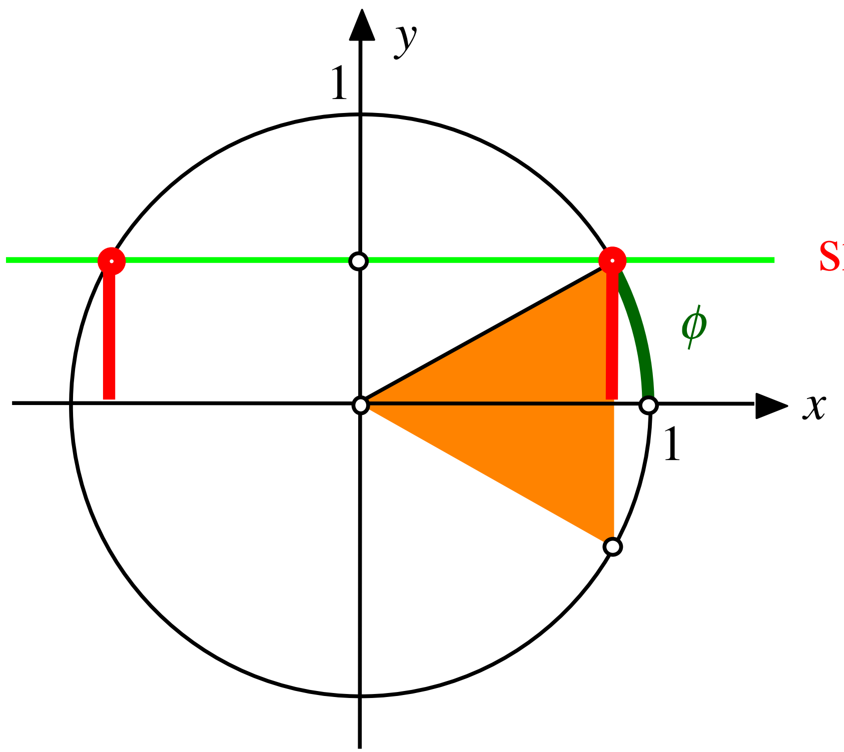


$$\sin(\phi) = 0.5 \qquad \phi = ?$$

$$\phi \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots, \frac{-7\pi}{6}, \frac{-11\pi}{6}, \ldots \right\}$$

Winkelfunktionen

Arcus, im Prinzip die Umkehrung

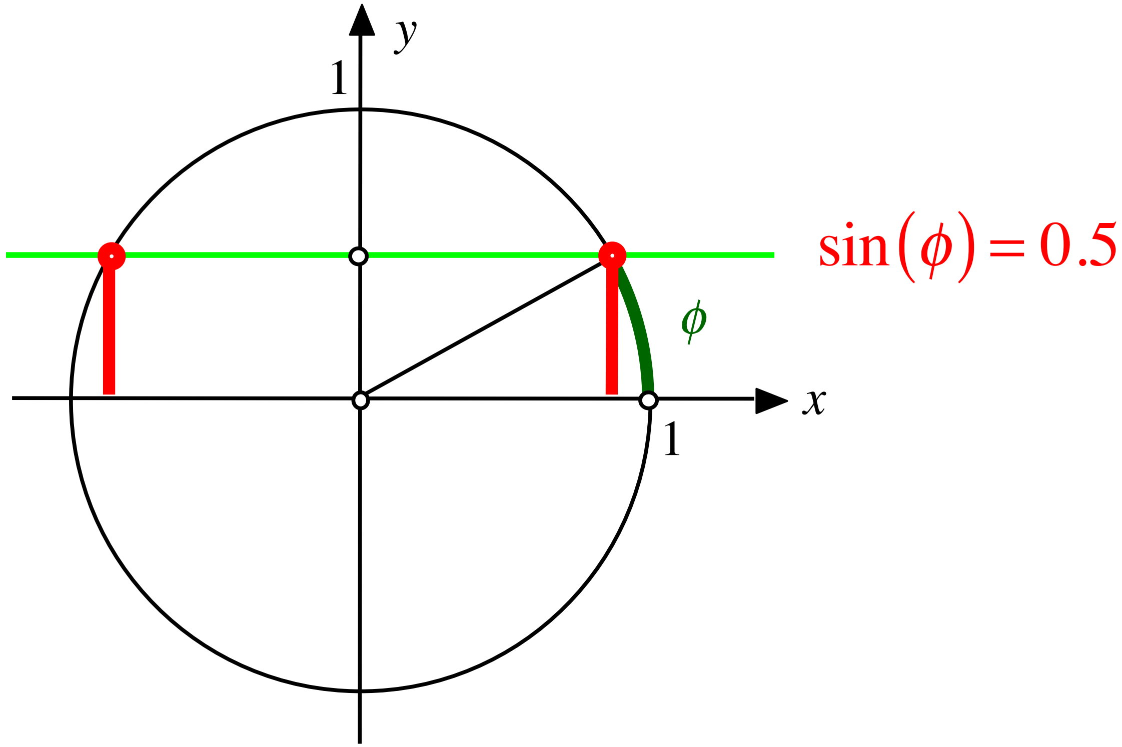


$$\sin(\phi) = 0.5 \qquad \phi = ?$$

$$\phi \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots, \frac{-7\pi}{6}, \frac{-11\pi}{6}, \ldots \right\}$$

Winkelfunktionen

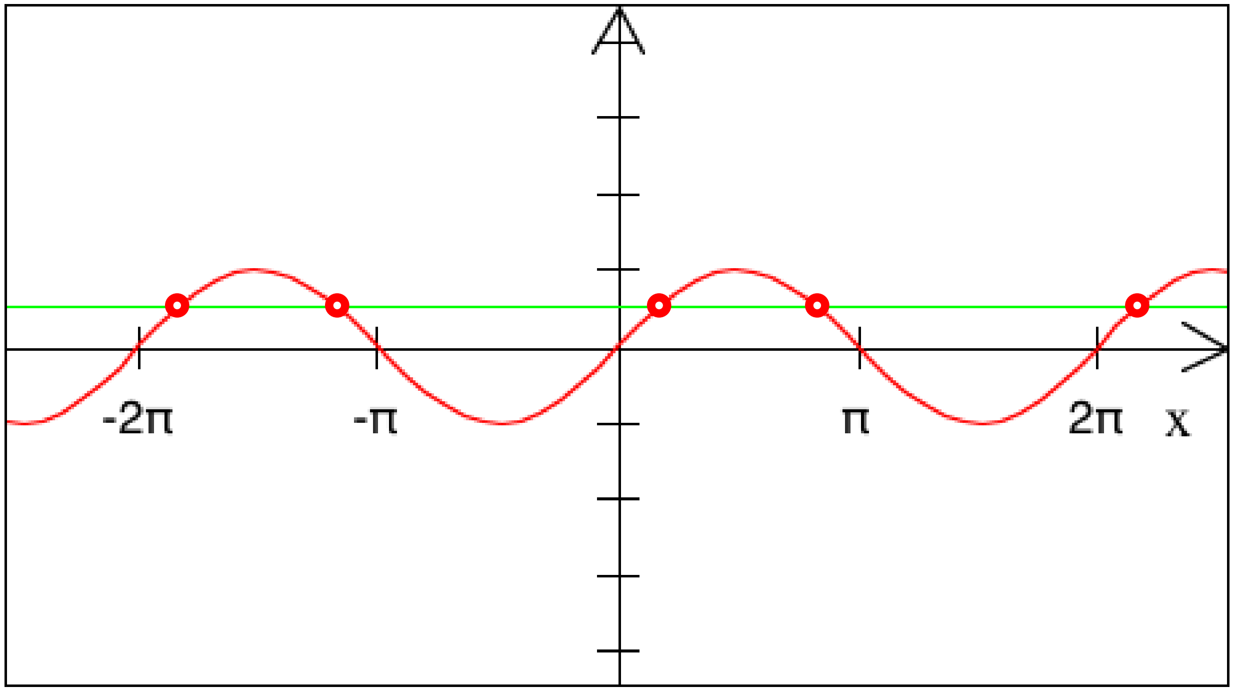
Arcus, im Prinzip die Umkehrung



$$\sin(\phi) = 0.5 \qquad \phi = ?$$

$$\phi \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots, \frac{-7\pi}{6}, \frac{-11\pi}{6}, \ldots \right\}$$

Winkelfunktionen Arcus, im Prinzip die Umkehrung



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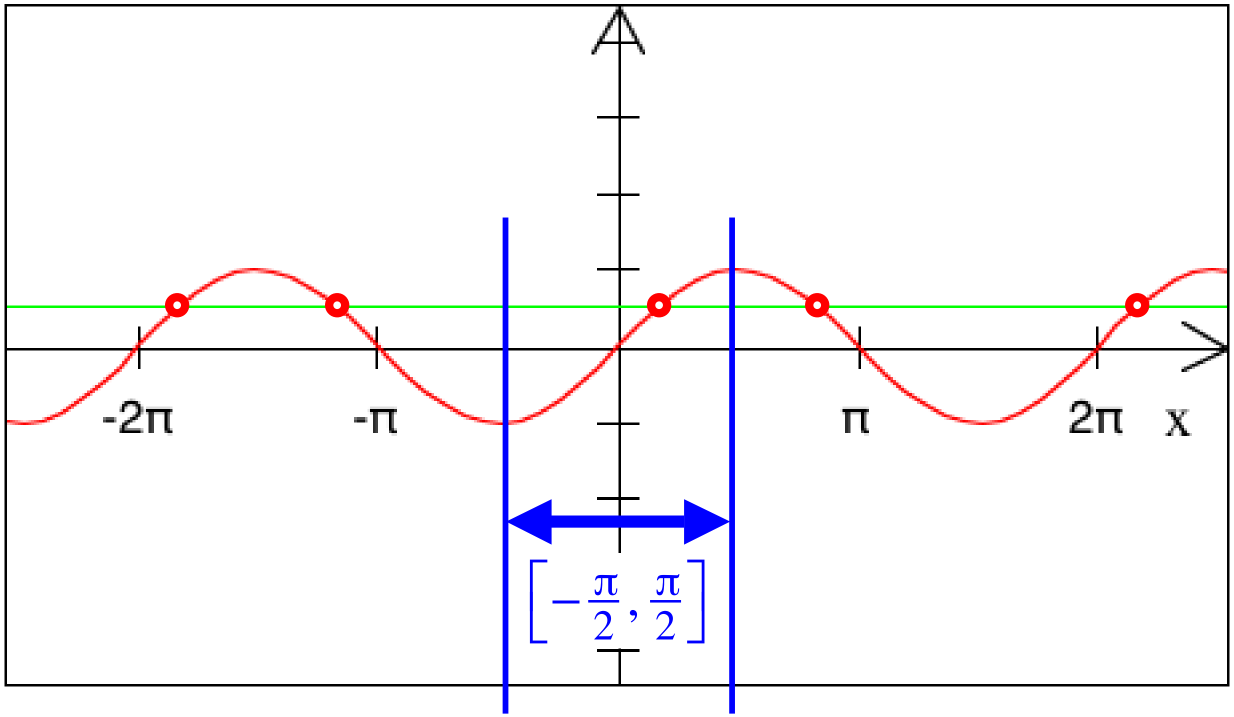
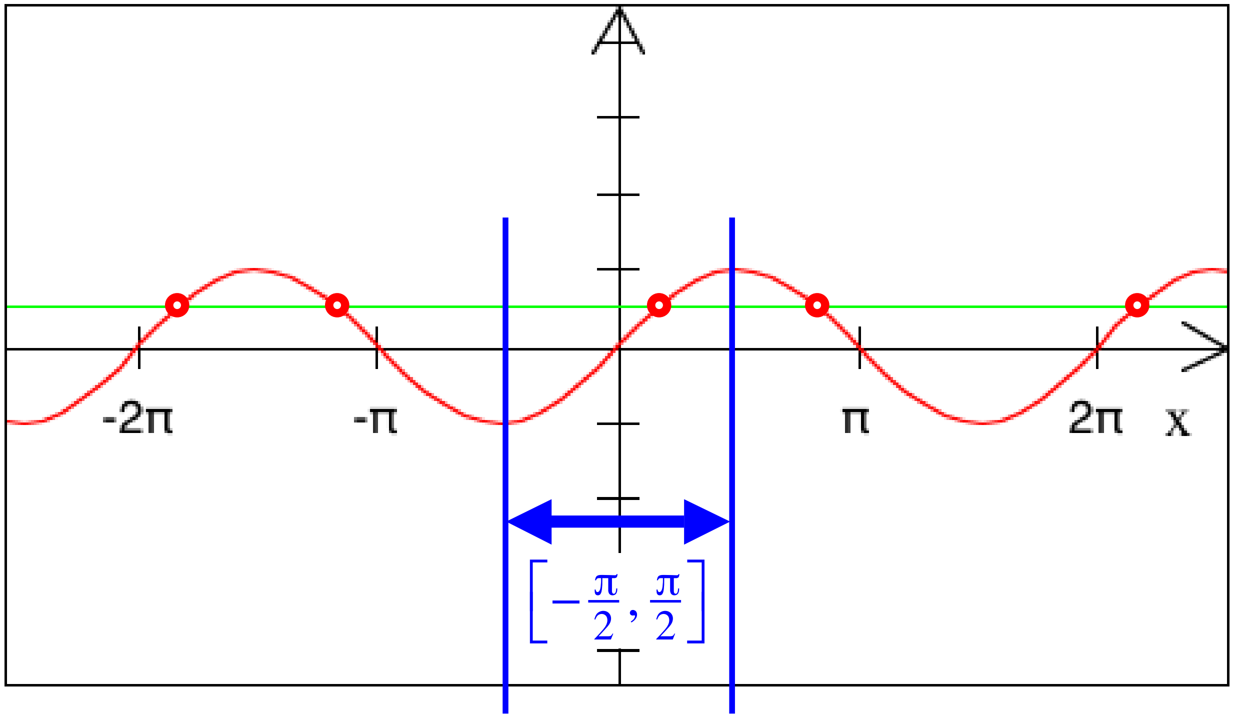
Winkelfunktionen

Arcus, im Prinzip die Umkehrung



Arcussinus gibt die Lösung im Bereich $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Winkelfunktionen Arcus, im Prinzip die Umkehrung

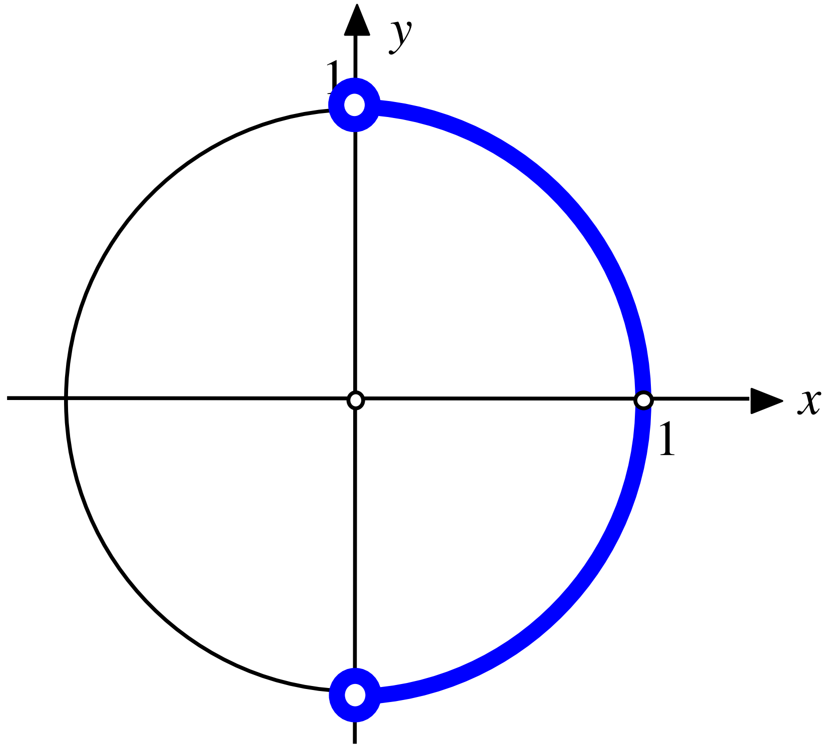


Arcussinus gibt die Lösung im Bereich $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Eingeschränkte Sinus-Funktion

Winkelfunktionen

Arcus, im Prinzip die Umkehrung



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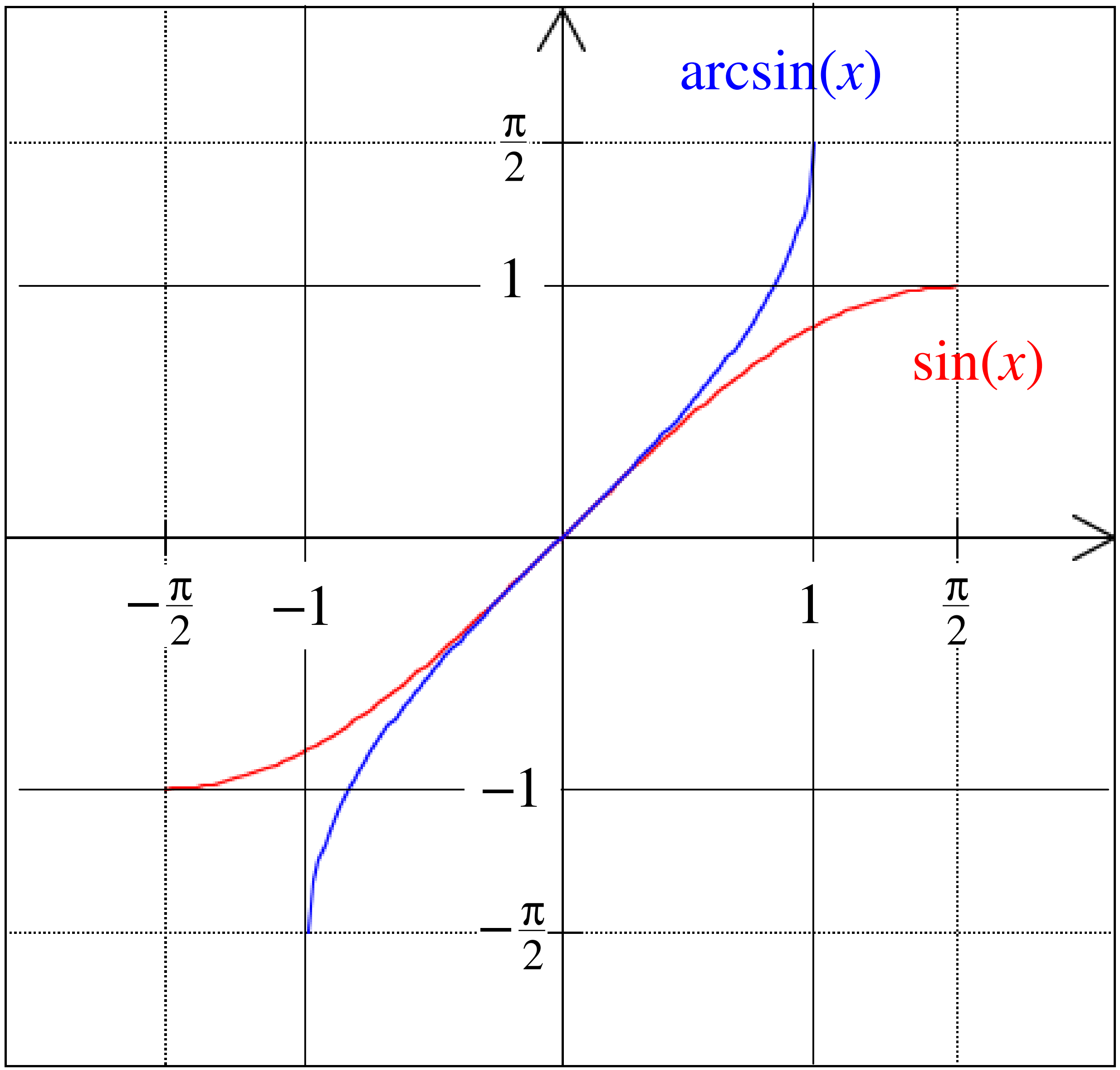
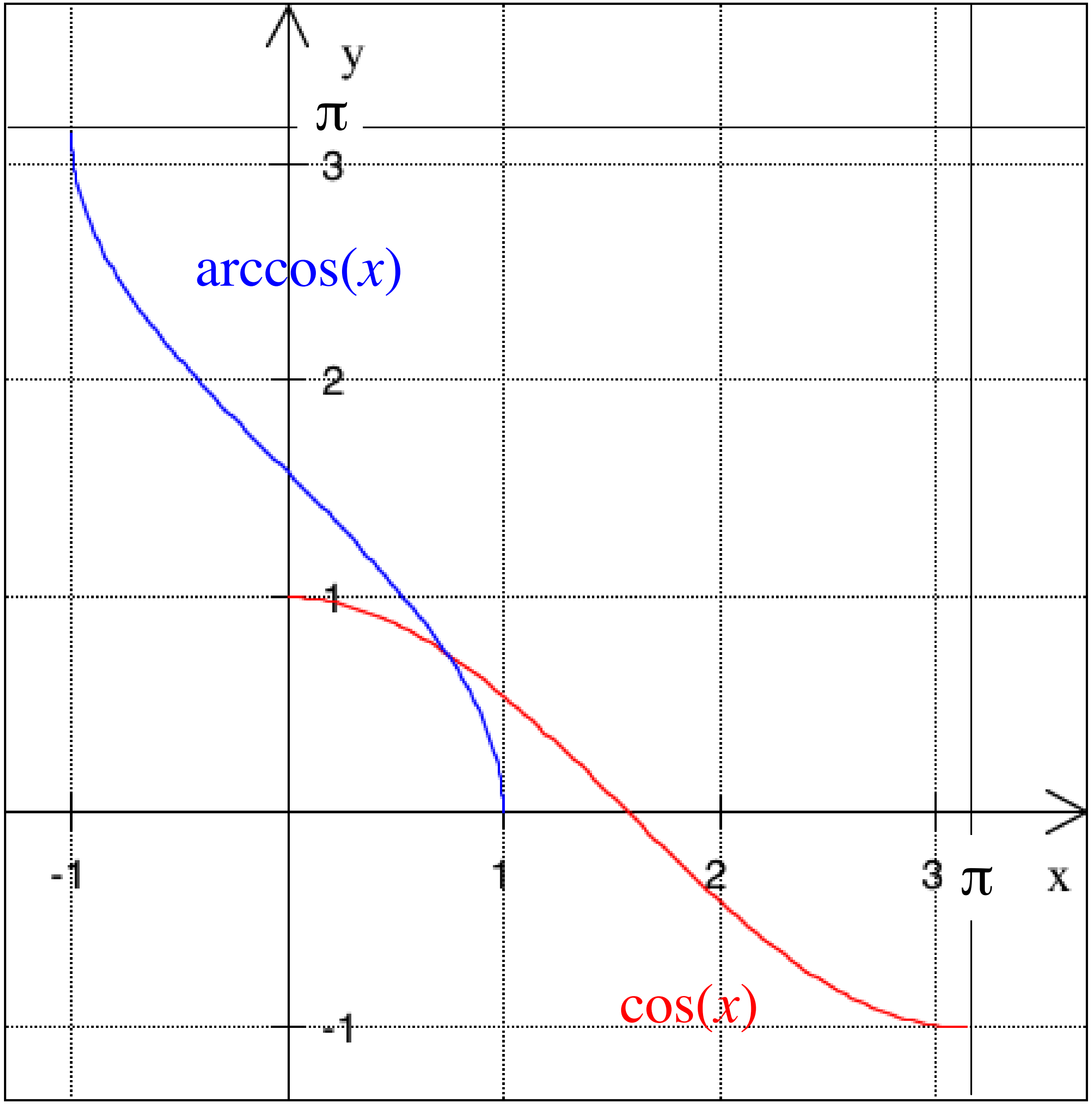
Eingeschränkte Sinus-Funktion



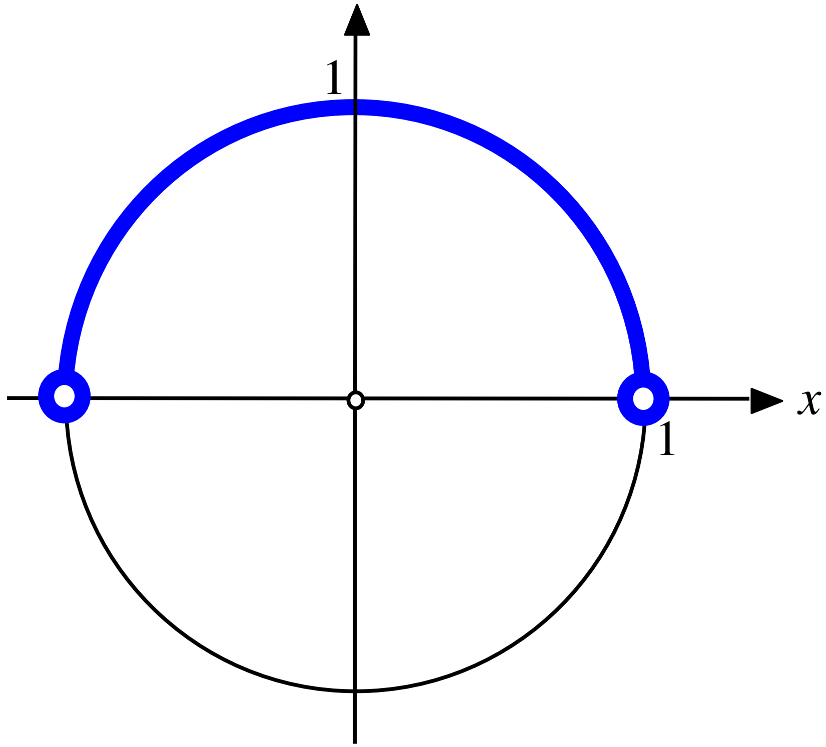
arcsin(x)
sin(x)
π/2
1
−1
−π/2
−π/2
−1
1
π/2



y
π
3
arccos(x)
2
1
-1
1
2
3
π
x
cos(x)
-1

Winkelfunktionen

Arcus, im Prinzip die Umkehrung

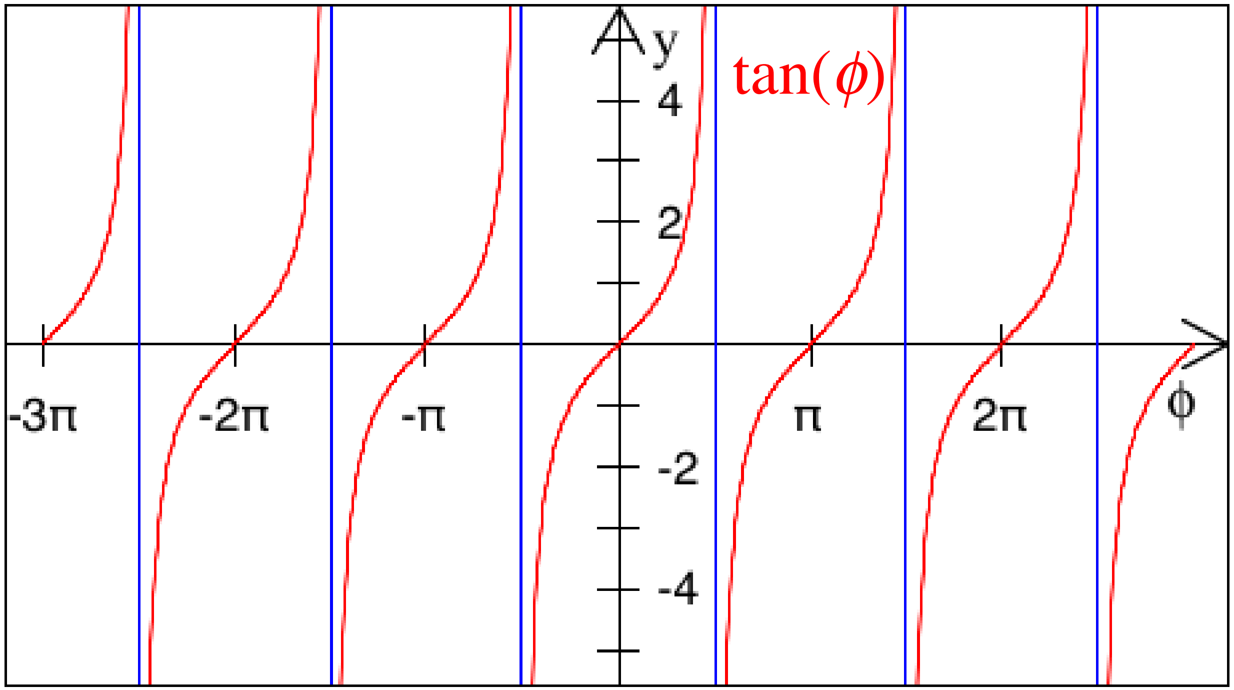


Arcuscosinus gibt die Lösung im Bereich $[0, \pi]$

Eingeschränkte Cosinus-Funktion

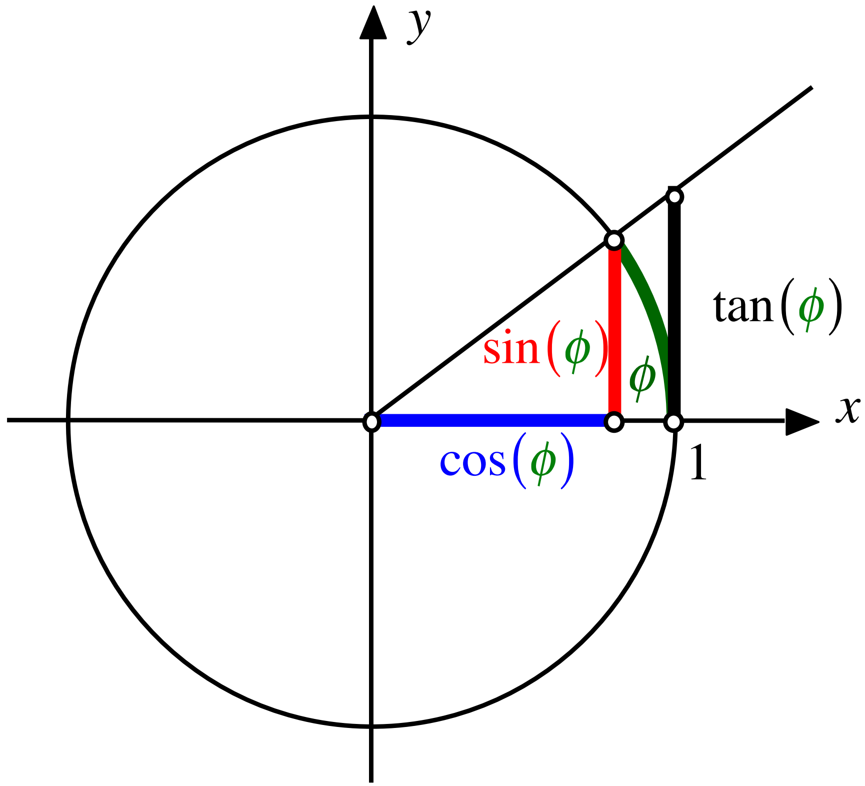
Tangens

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)}$$



Tangens

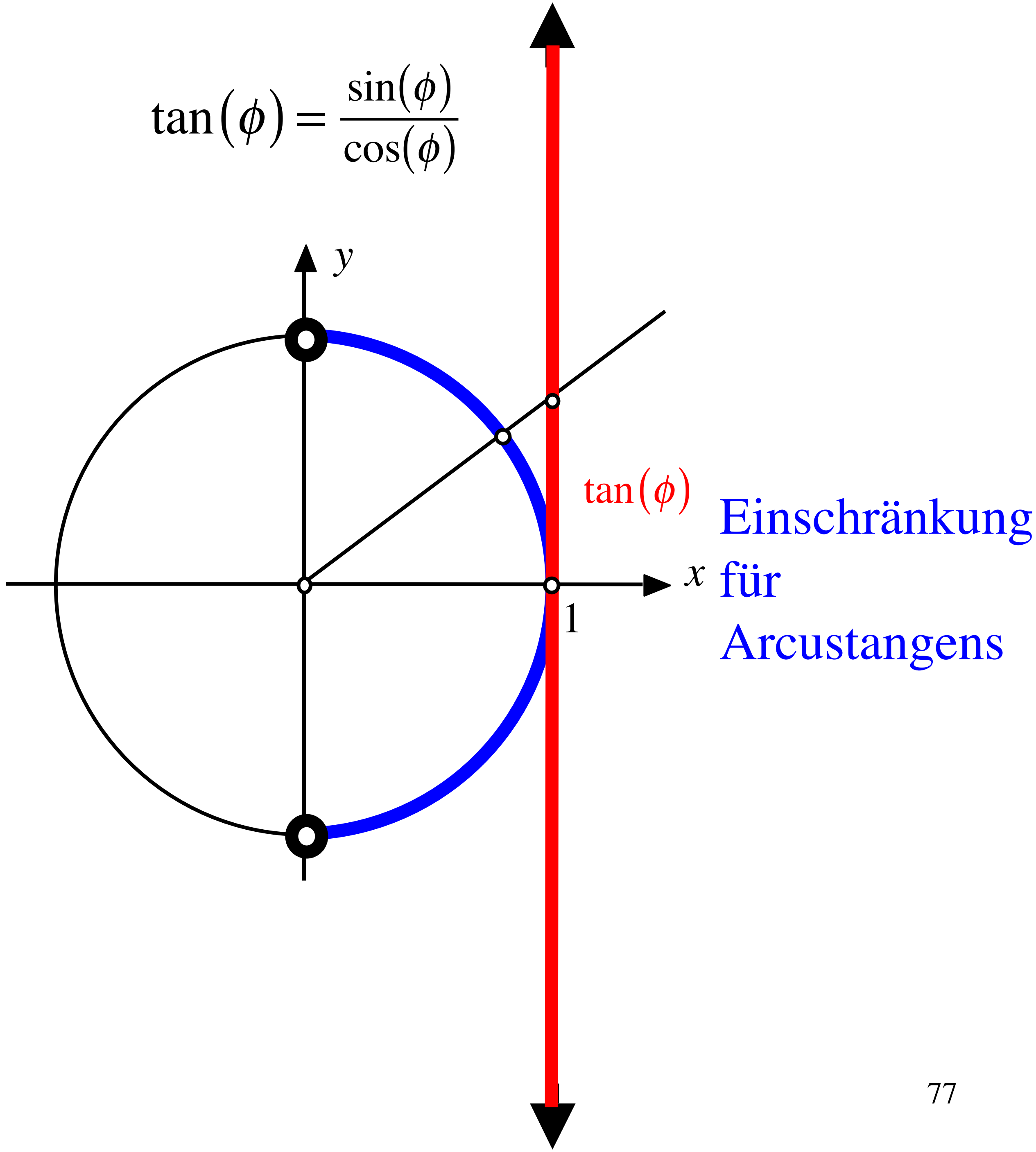
$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)}$$



Tangens auf der Tangente

Tangens

$$\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)}$$



Transformation!

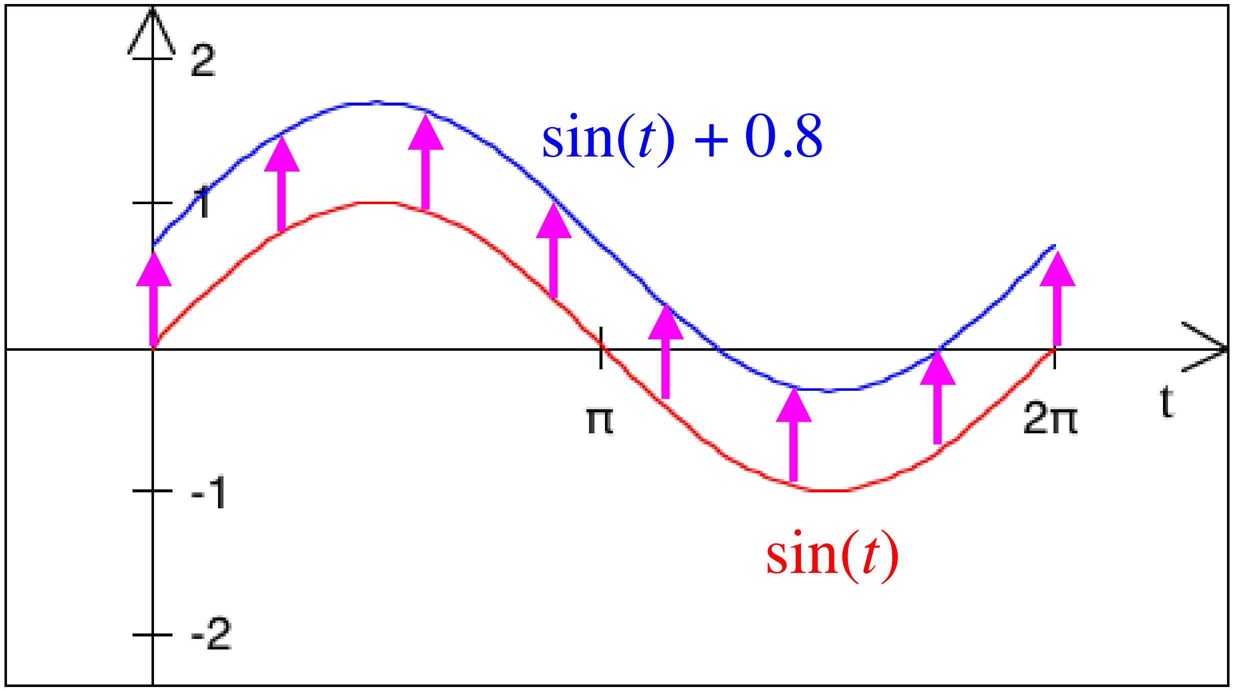
Dane R. Camp

If you add to a function, you'll give it a lift,
for the graph will be moved with a vertical shift.
But if you multiply, take a close look and see,
the graph's stretched by that factor vertically,
and negating the function will cause a reflection,
across the x-axis in an up-down direction.

But if you add before the function is used, hey!
The shift's horizontal — the opposite way!
And multiplication by a factor inside reveals,
the graph's being stretched by the reciprocal's.
And negating the values before f is applied,
reflects across the y-axis — it flips side to side!

Transformation!

If you add to a function, you'll give it a lift,
for the graph will be moved with a vertical shift.



Transformation!

But if you multiply, take a close look and see,
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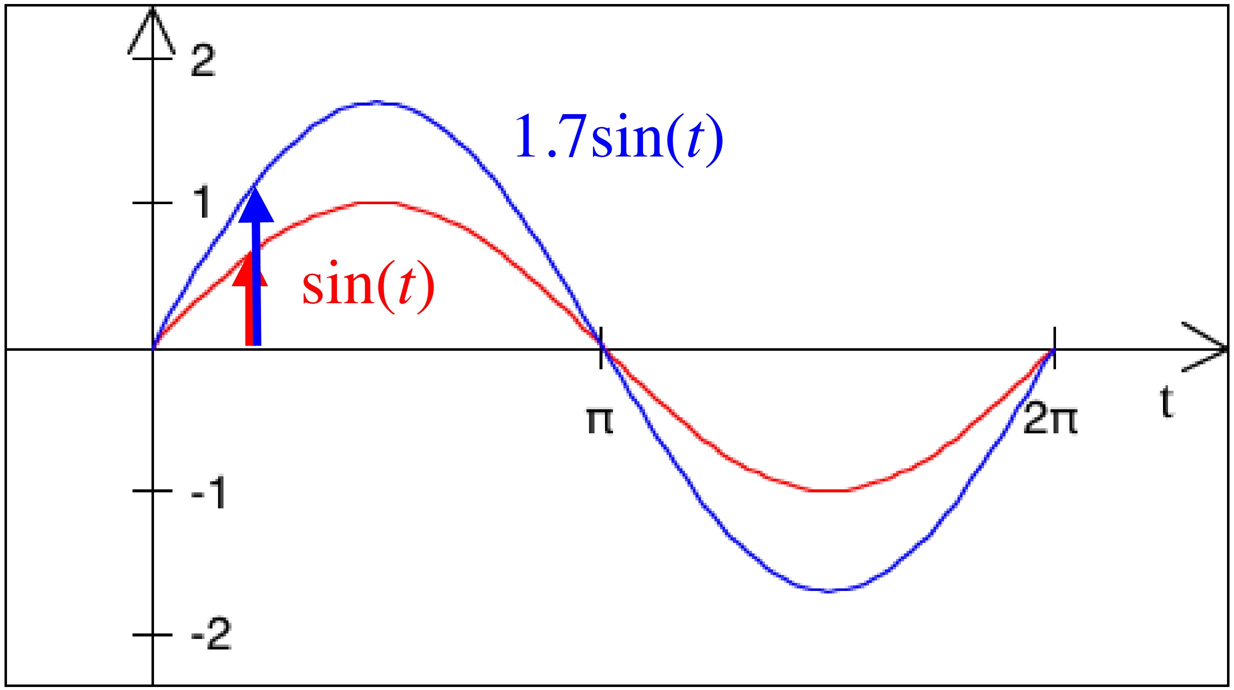
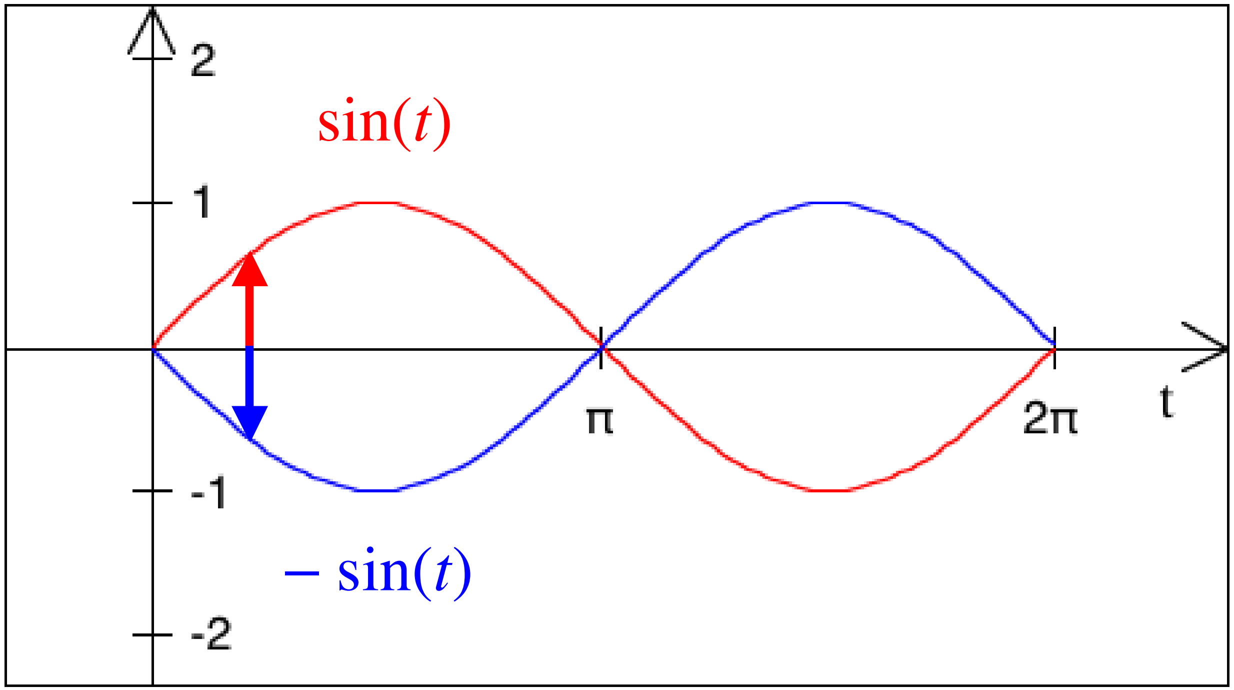


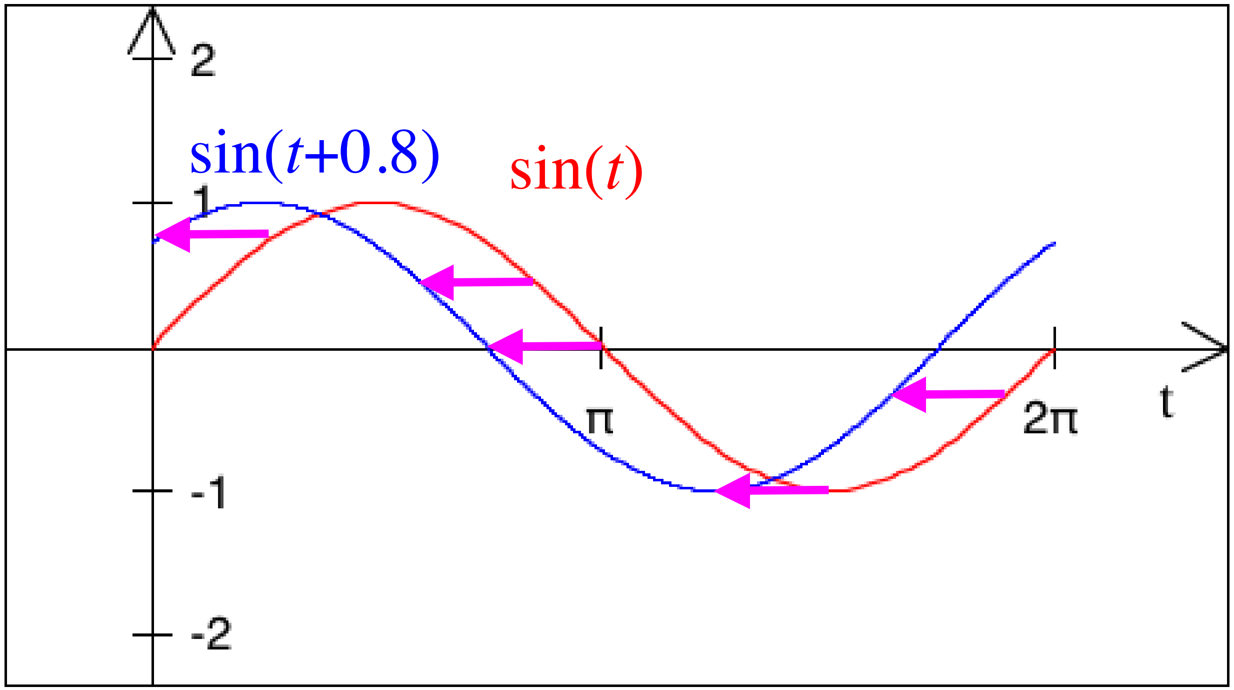
Transformation!

and negating the function will cause a reflection, across the x-axis in an up-down direction.



Transformation!

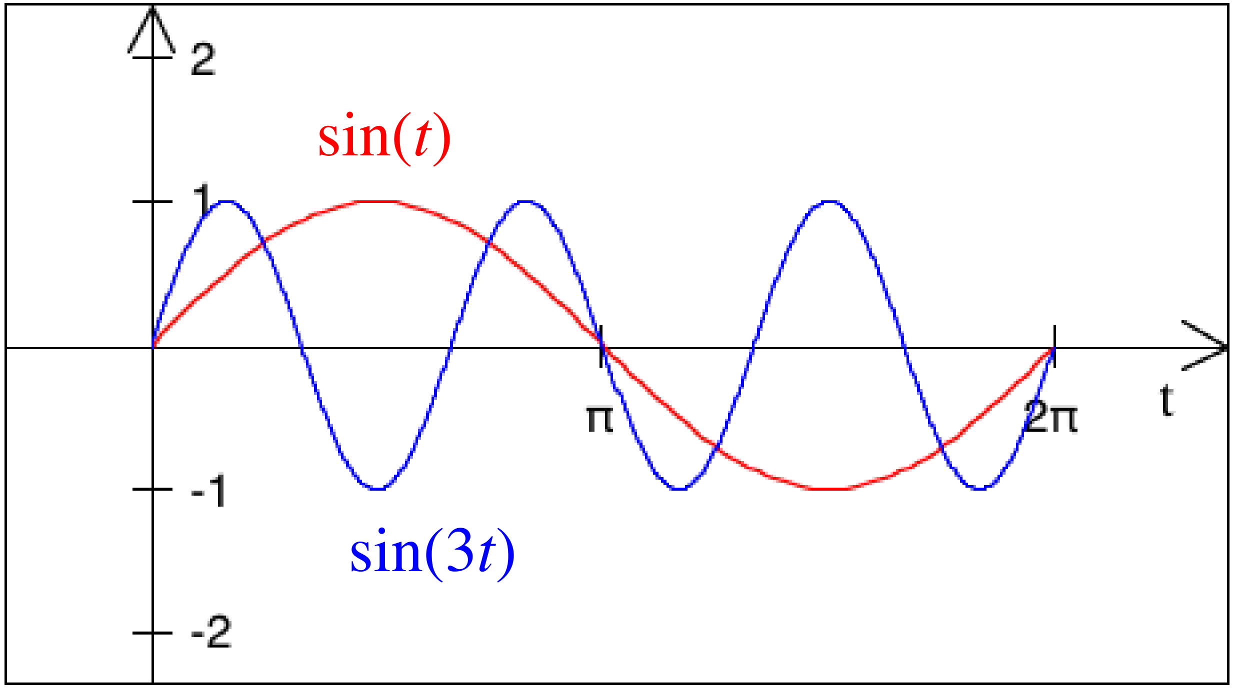
But if you add before the function is used, hey!
The shift's horizontal — the opposite way!



Wer langsam ist muss früher aufstehen.

Transformation!

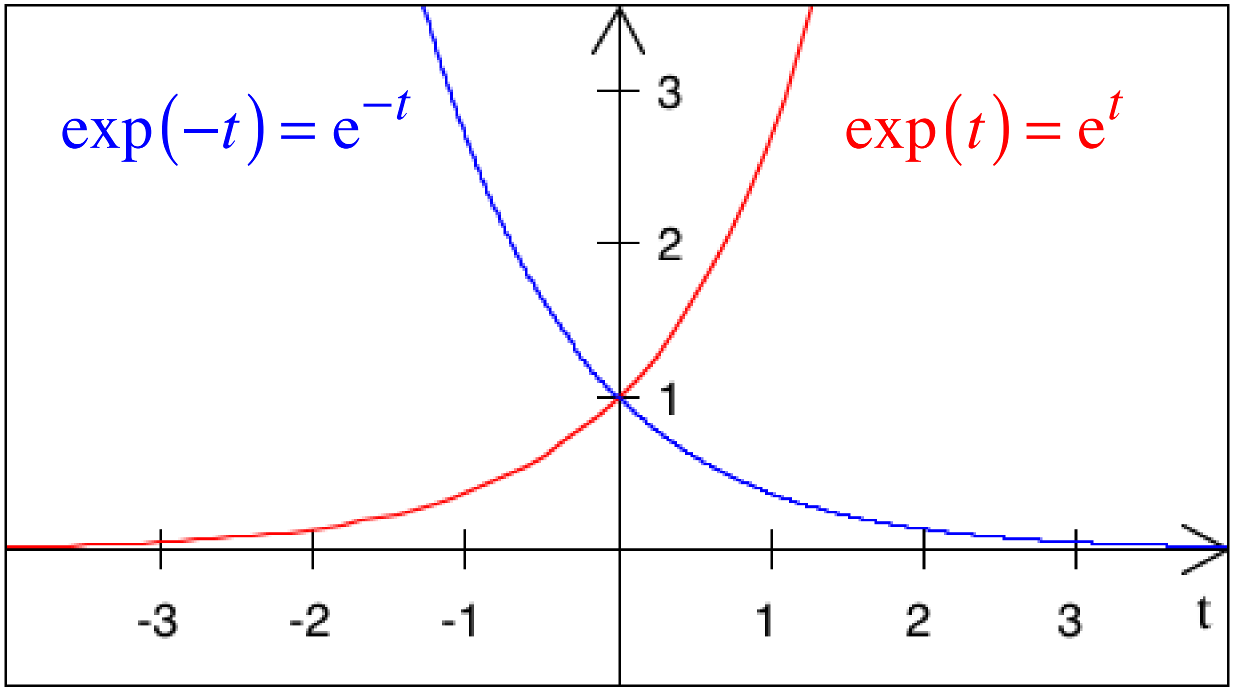
And multiplication by a factor inside reveals,
the graph's being stretched by the reciprocal's.



Schneller Puls
Große Frequenz, kleine Wellenlänge

Transformation!

And negating the values before f is applied, reflects across the y-axis — it flips side to side!



Vergangenheit wird Zukunft

Zeit

Amplitude

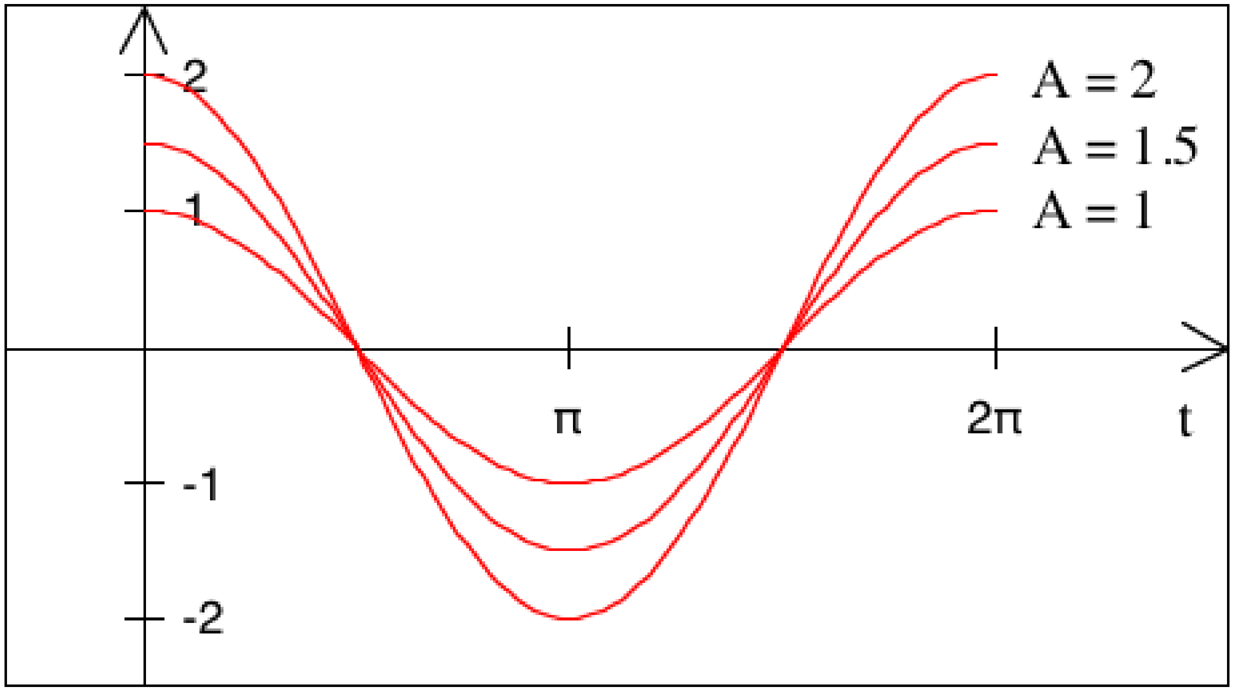
Zeit

$$f(t) = A\cos(\omega t + \varphi)$$

Kreis-
frequenz

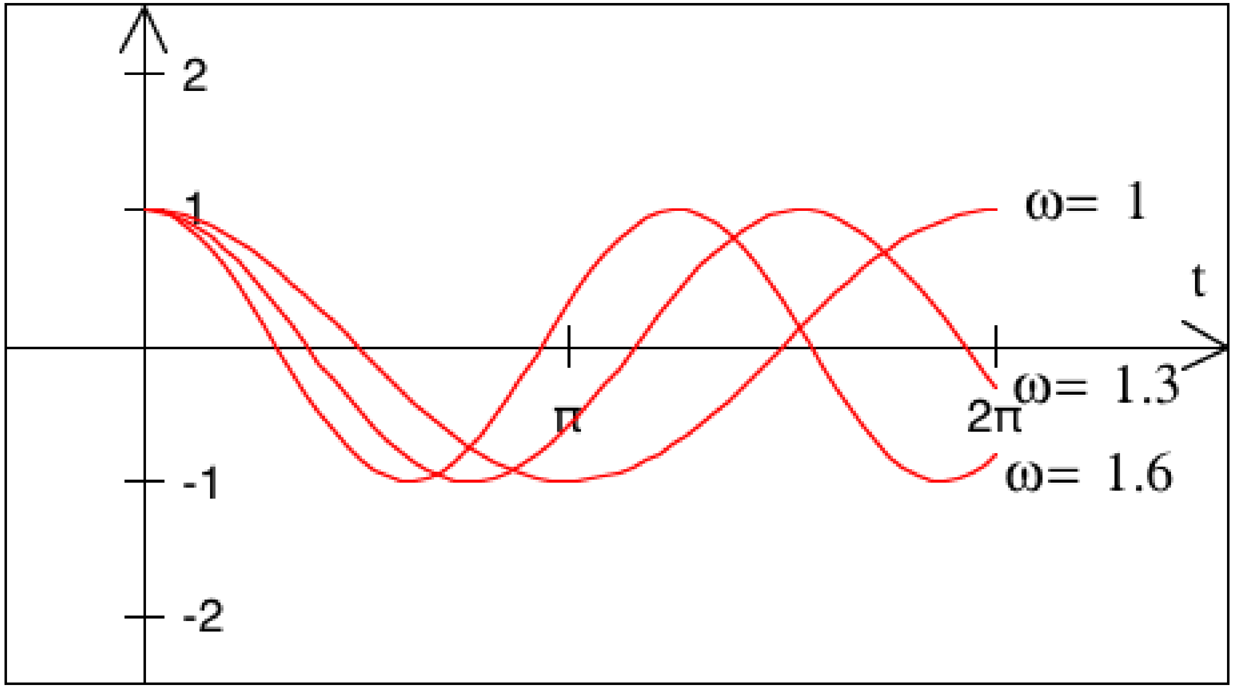
Phasen-
verschiebung

Amplitudenänderung



$$f(t) = A\cos(t) \quad \text{mit} \quad A = 1,\ 1.5,\ 2$$

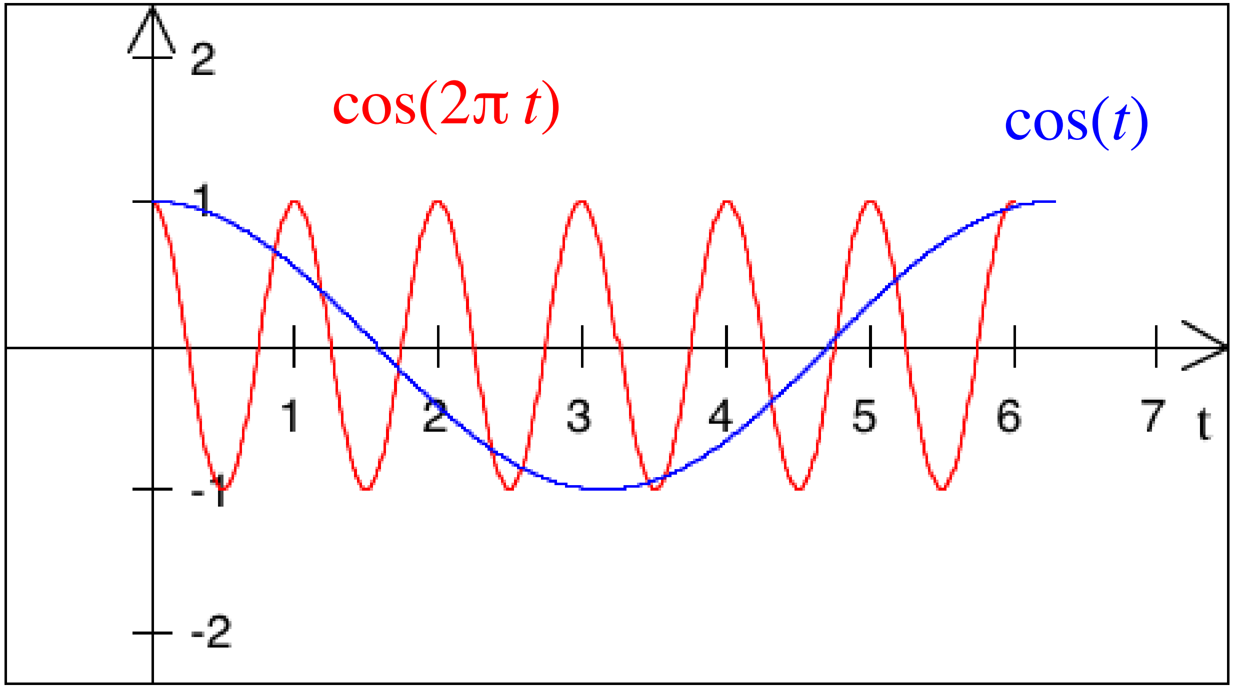
Kreisfrequenzänderung



$$f(t) = \cos(\omega t) \quad \text{mit} \quad \omega = 1,\ 1.3,\ 1.6$$

Kreisfrequenz

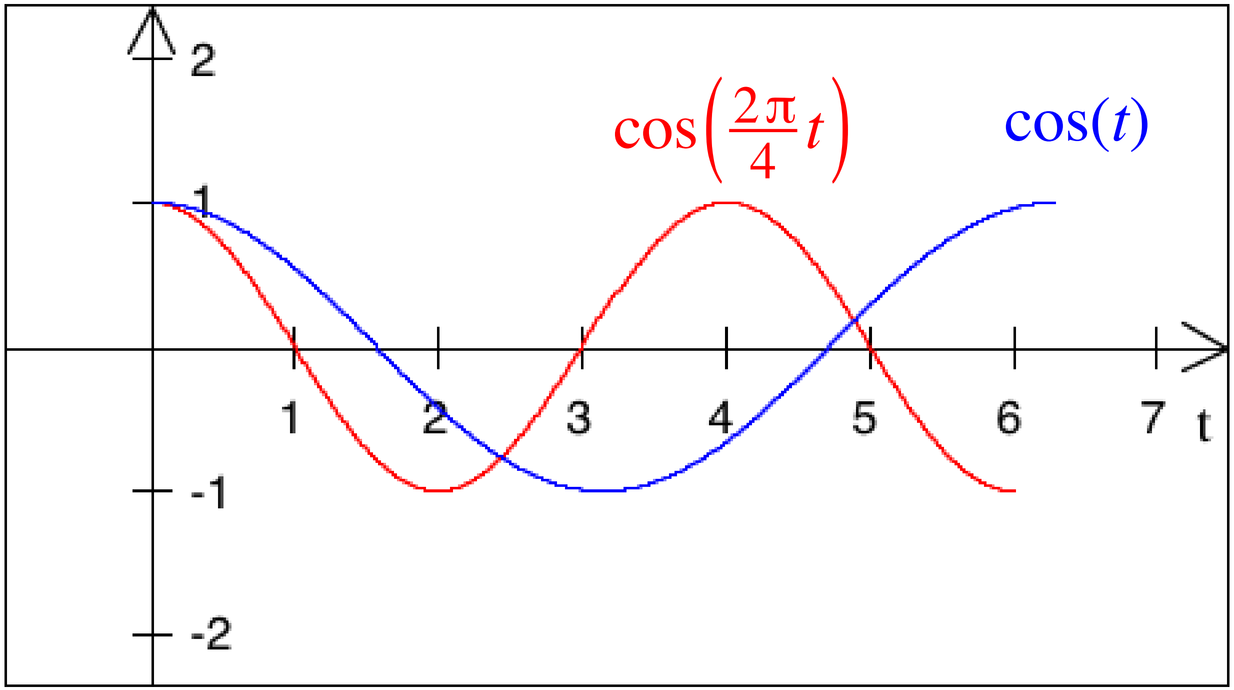
Frequenz und Kreisfrequenz



$\cos(t) \qquad \omega = 1 \qquad 1 \text{ Mal } / (2\pi)$

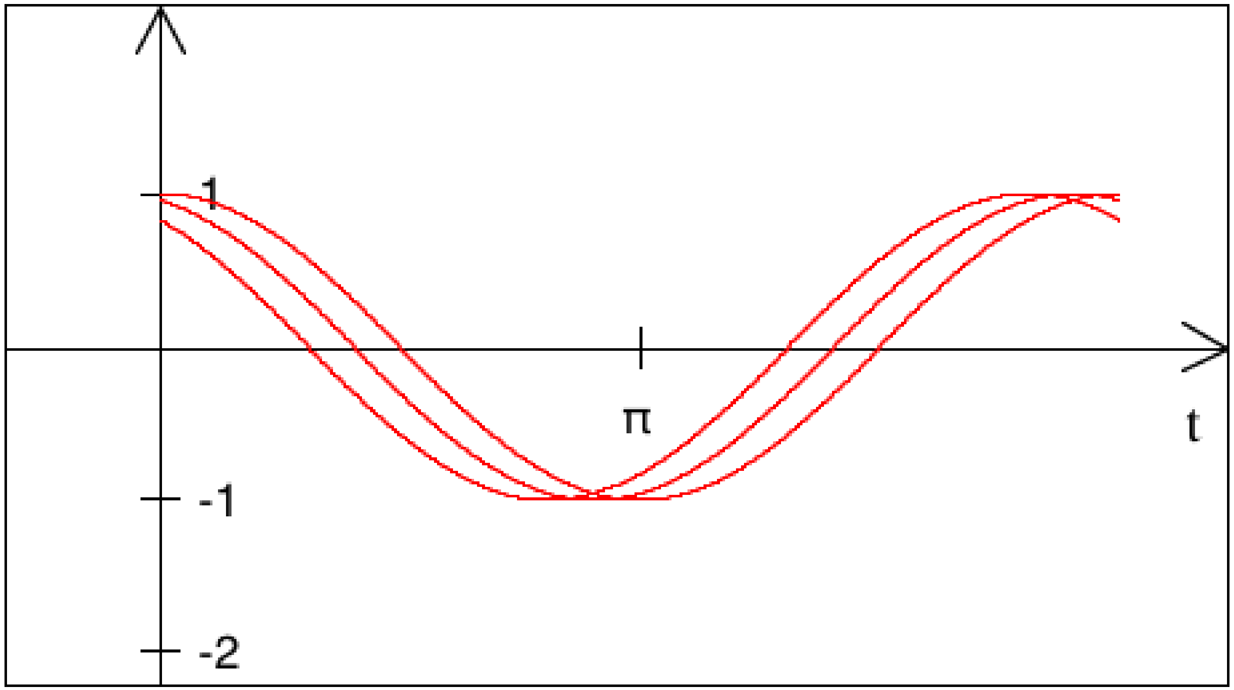
$\cos(2\pi t) \qquad \omega = 2\pi \qquad 1 \text{ Mal } / \text{ Zeiteinheit}$

Frequenz und Kreisfrequenz



$\cos\left(\frac{2\pi}{4}t\right)$ 1 Mal / 4 Zeiteinheiten

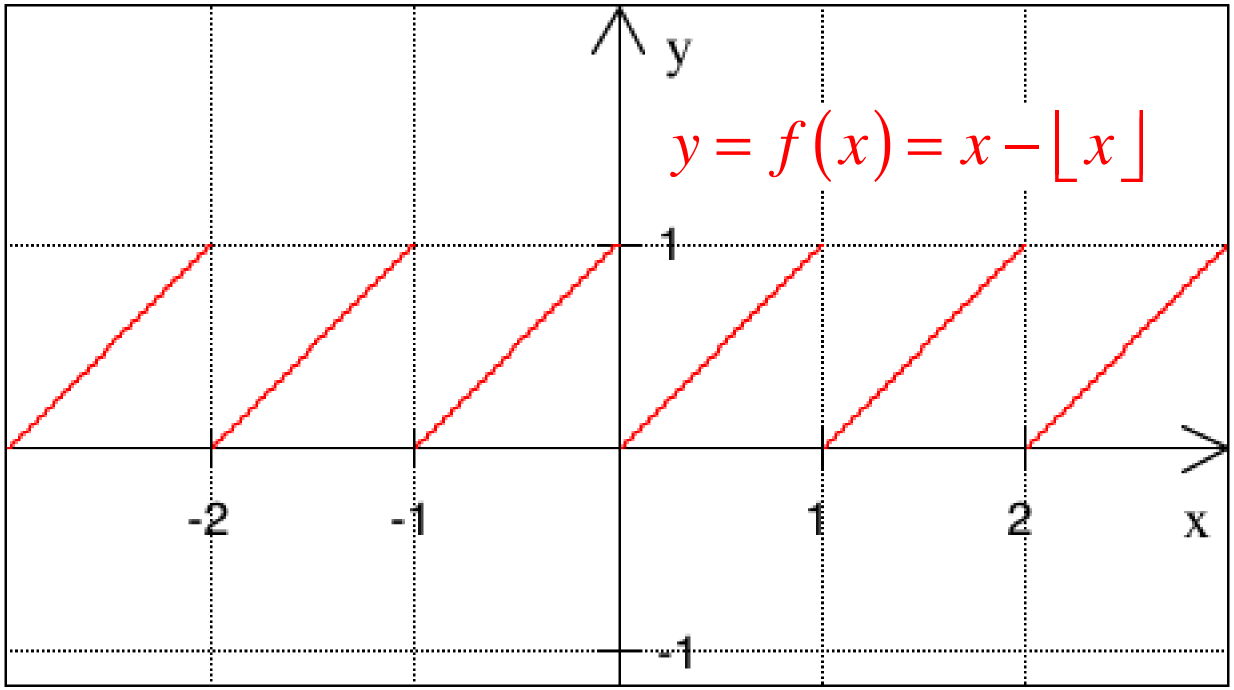
Phasenverschiebung



$$f(t) = \cos(t + \varphi) \quad \text{mit} \quad \varphi = 0,\ 0.3,\ 0.6$$

Phasenverschiebung

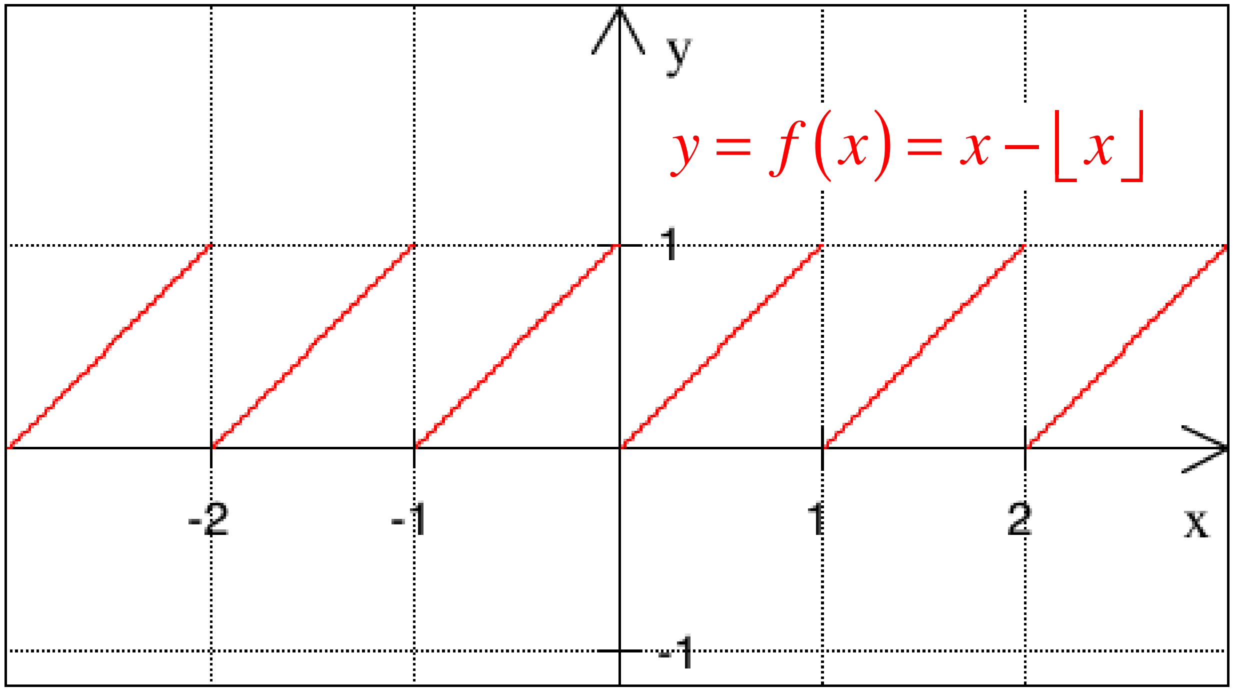
Fourier-Analyse



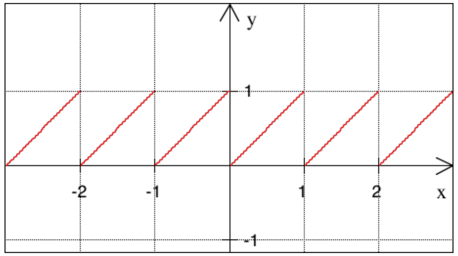
$$y = f(x) = x - \lfloor x \rfloor$$

Abrunden auf nächst kleinere ganze Zahl

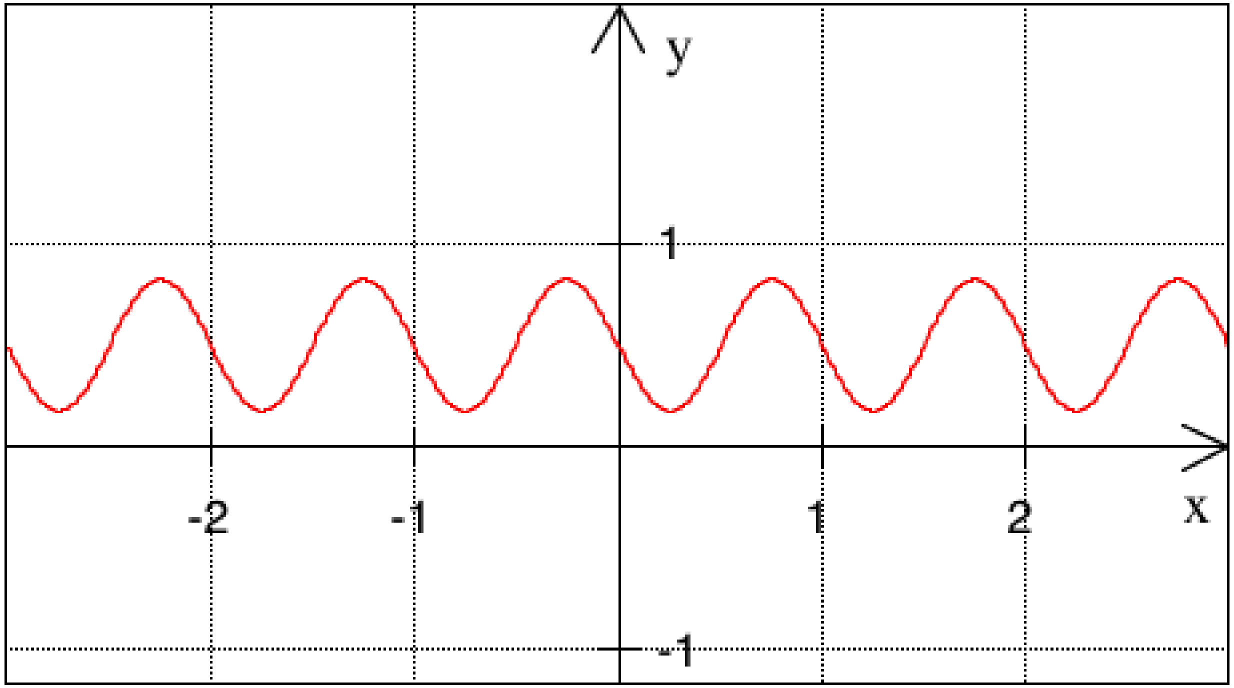
Fourier-Analyse



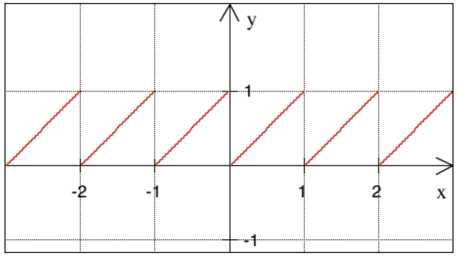
Darstellung durch „vertraute“ Funktionen



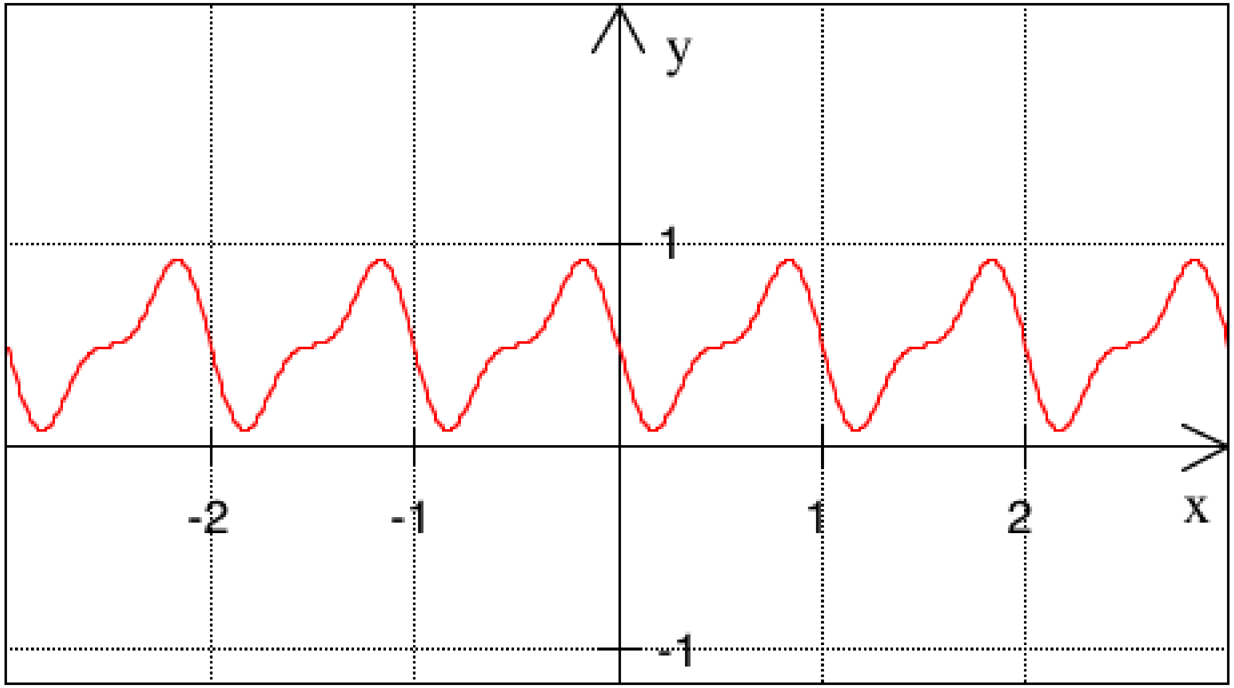
Fourier-Analyse



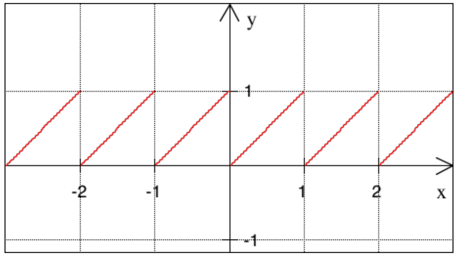
$$y = f(x) = \frac{1}{2} - \frac{1}{\pi}\sin(2\pi x)$$



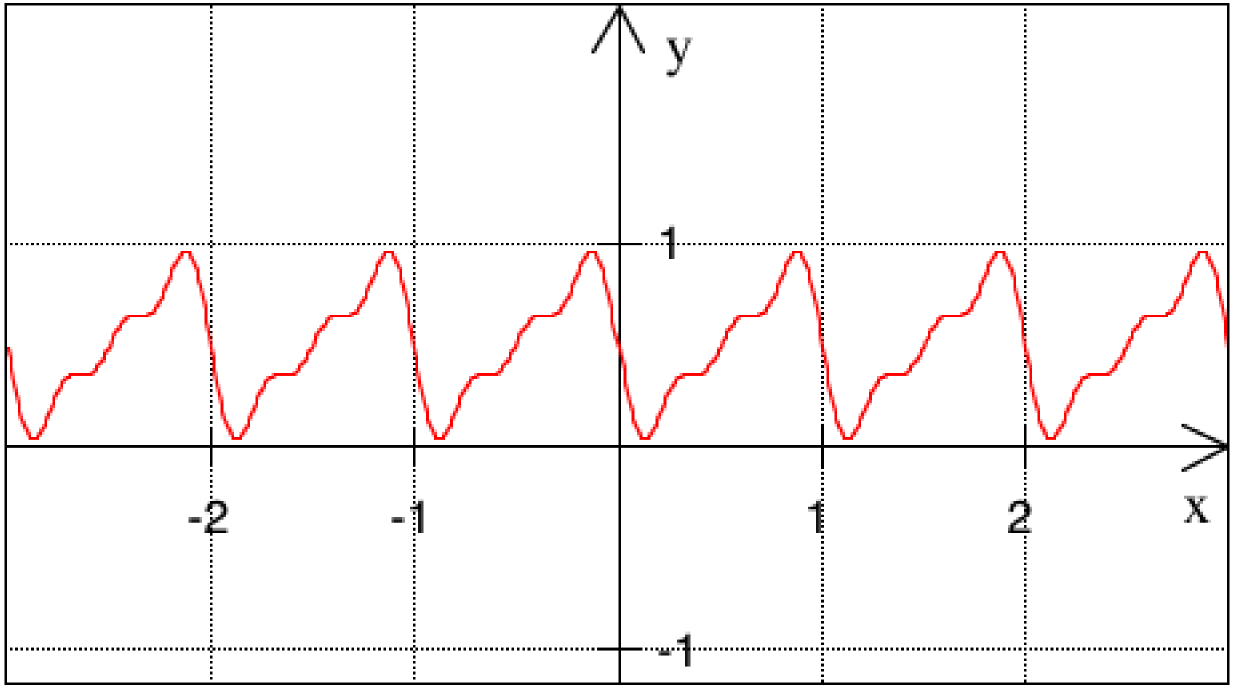
Fourier-Analyse



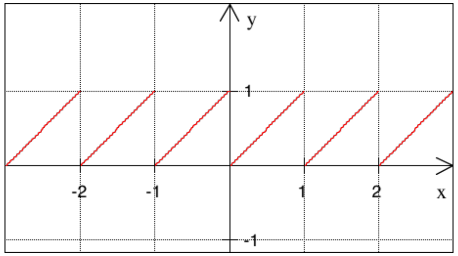
$$y = f(x) = \frac{1}{2} - \frac{1}{\pi}\left(\sin(2\pi x) + \frac{1}{2}\sin(2 * 2\pi x)\right)$$



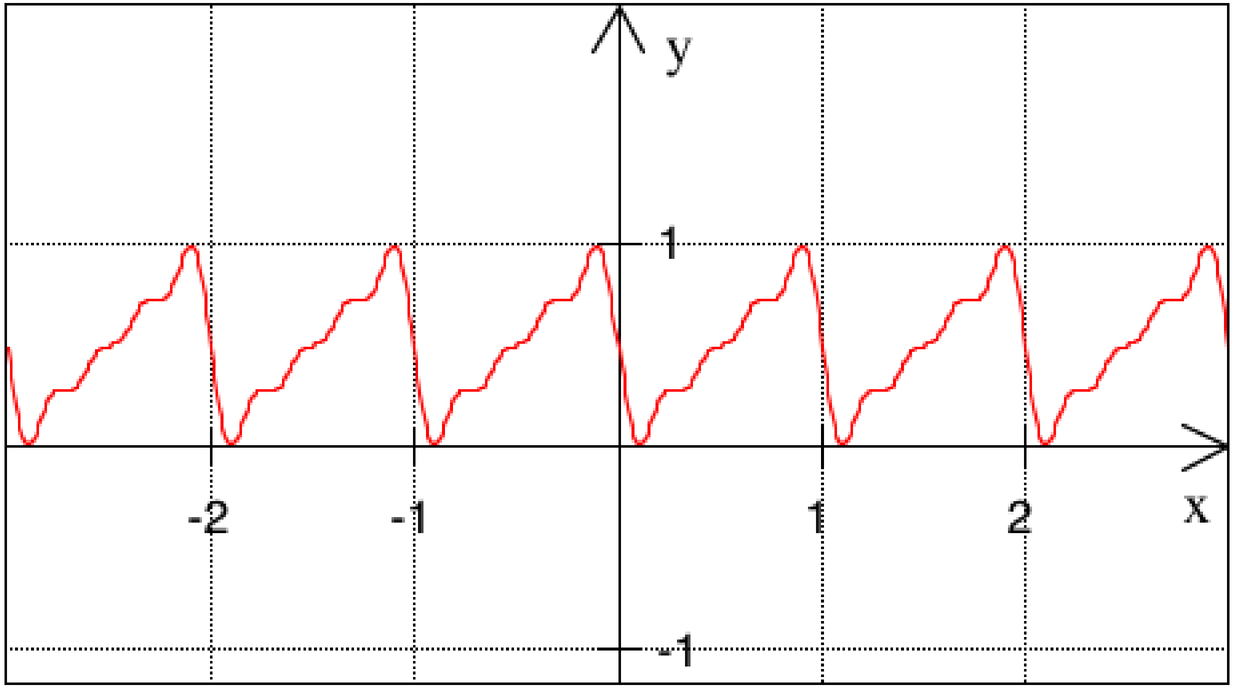
Fourier-Analyse



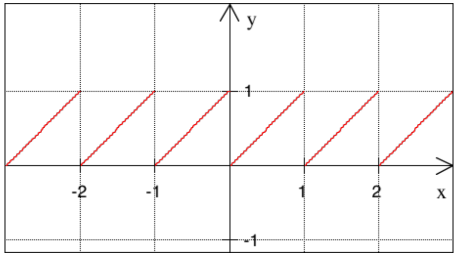
$$y = f(x) = \frac{1}{2} - \frac{1}{\pi}\left(\sin(2\pi x) + \frac{1}{2}\sin(2 * 2\pi x) + \frac{1}{3}\sin(3 * 2\pi x)\right)$$



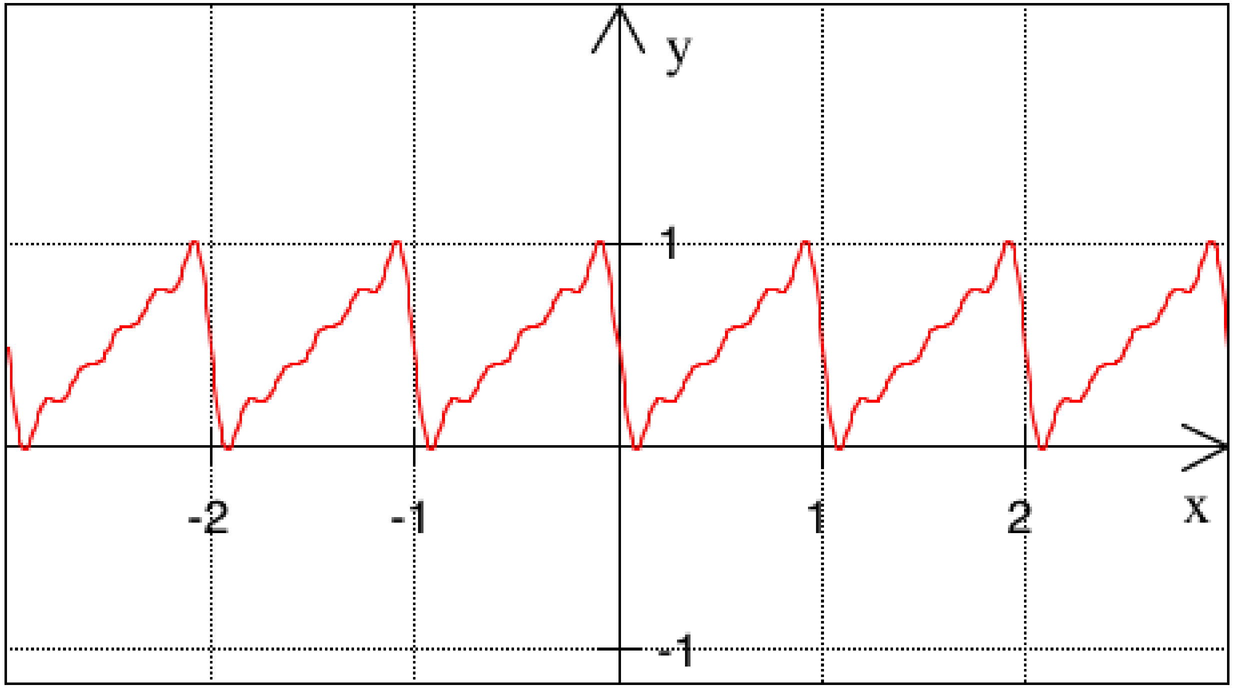
Fourier-Analyse



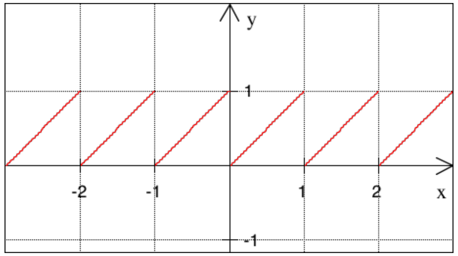
$$y = f(x) = \frac{1}{2} - \frac{1}{\pi}\left(\sin(2\pi x) + \frac{1}{2}\sin(2*2\pi x) + \frac{1}{3}\sin(3*2\pi x) + \frac{1}{4}\sin(4*2\pi x)\right)$$



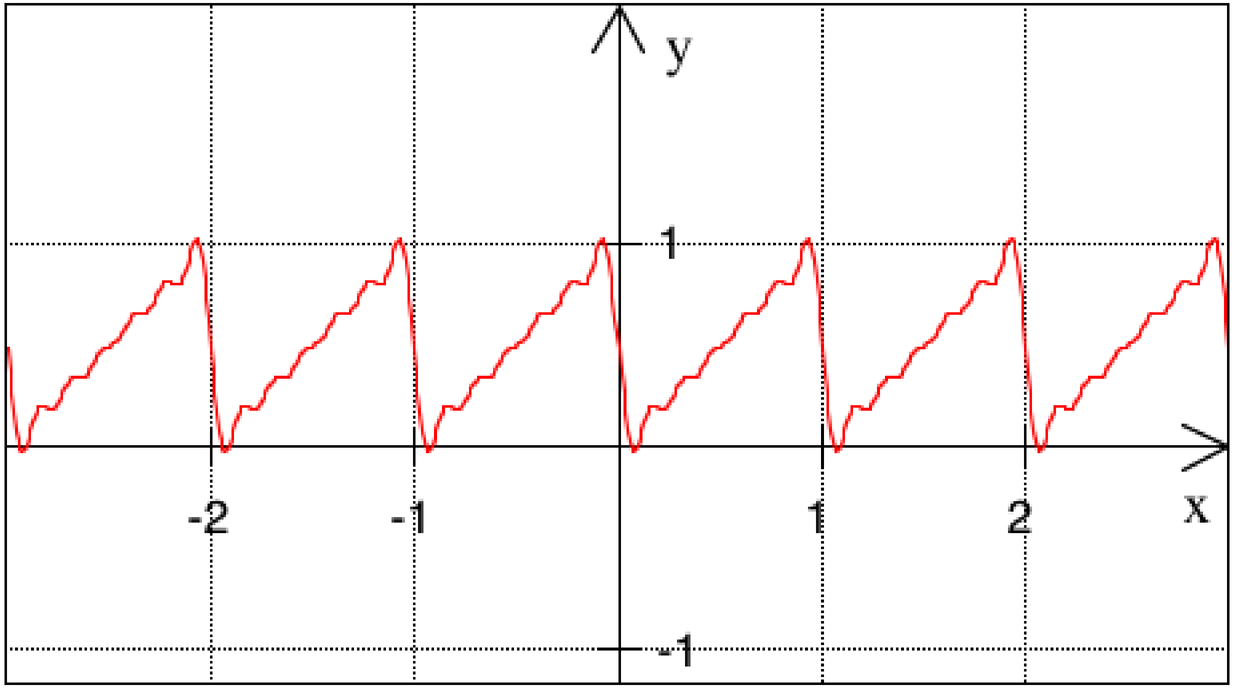
Fourier-Analyse



$$y = f(x) = \frac{1}{2} - \frac{1}{\pi}\left(\sin(2\pi x) + \cdots + \frac{1}{5}\sin(5 * 2\pi x)\right)$$



Fourier-Analyse



$$y = f(x) = \frac{1}{2} - \frac{1}{\pi}\left(\sin(2\pi x) + \cdots + \frac{1}{6}\sin(6 * 2\pi x)\right)$$

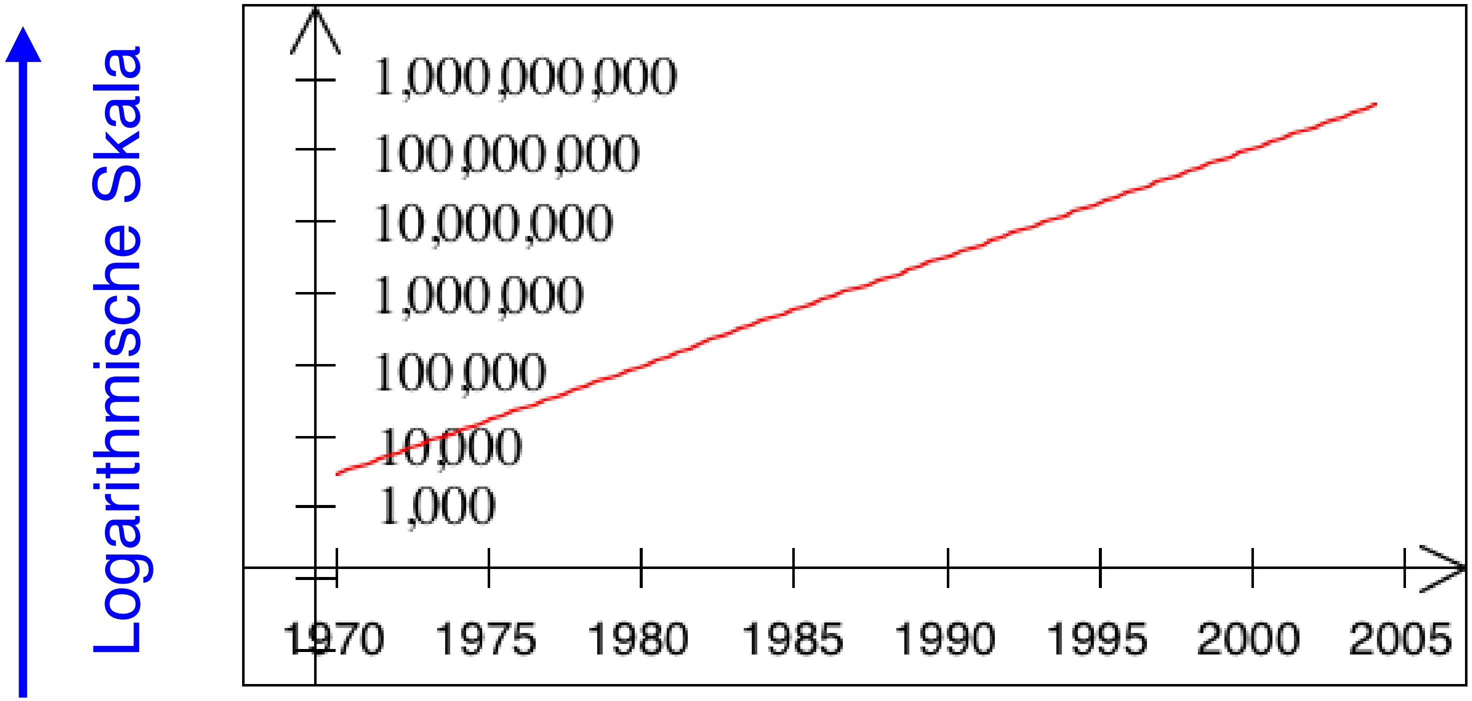
Exponentielles Wachstum

Gordon Moore war einer der Mitgründer von Intel und machte 1965 die berühmte Vorhersage, dass sich die Packungsdichte von Mikroprozessoren alle 18 bis 24 Monate verdoppeln würde.

Exponentielles Wachstum

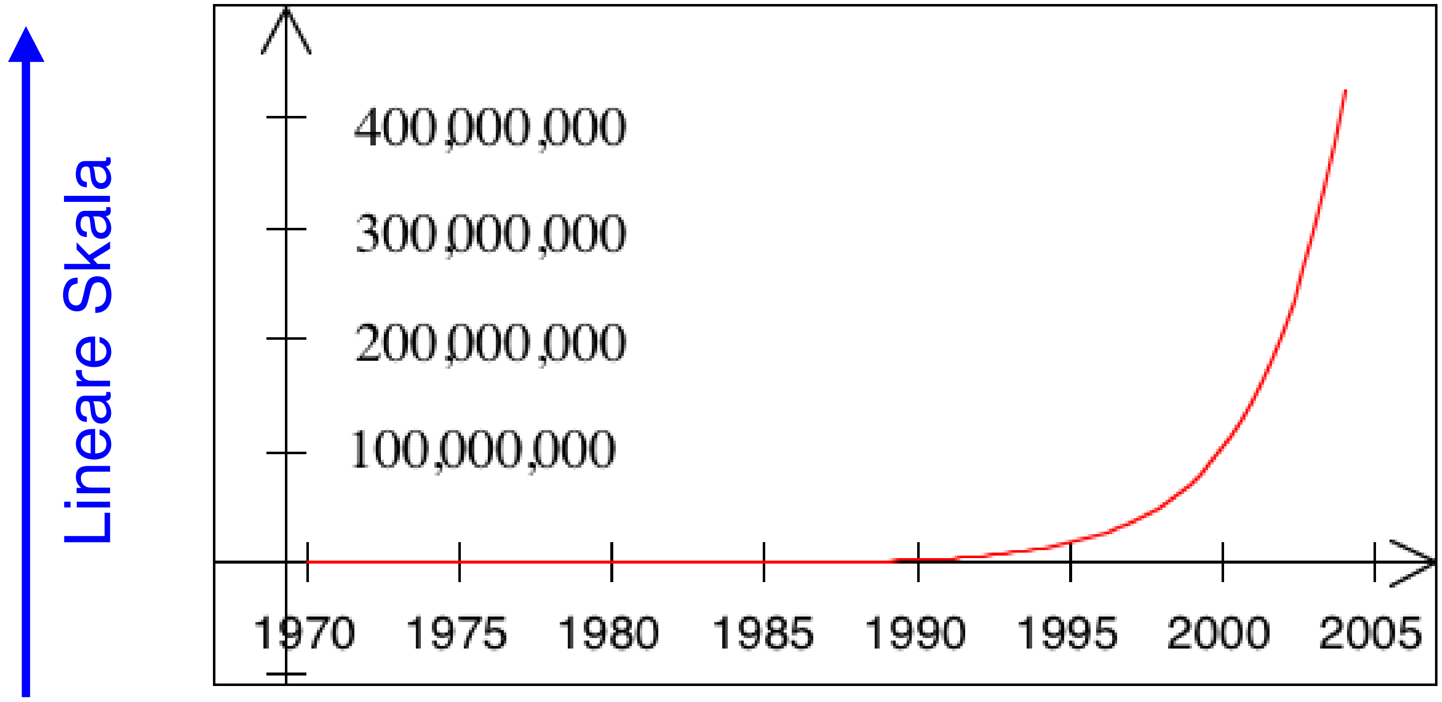
		Transistoren pro Chip
4004	1971	2,250
8008	1972	2,500
8080	1974	5,000
8086	1978	29,000
286	1982	120,000
Intel386™ processor	1985	275,000
Intel486™ processor	1989	1,180,000
Intel® Pentium® processor	1993	3,100,000
Intel® Pentium® II processor	1997	7,500,000
Intel® Pentium® III processor	1999	24,000,000
Intel® Pentium® 4 processor	2000	42,000,000
Intel® Itanium® processor	2002	220,000,000
Intel® Itanium® 2 processor	2003	410,000,000

Exponentielles Wachstum



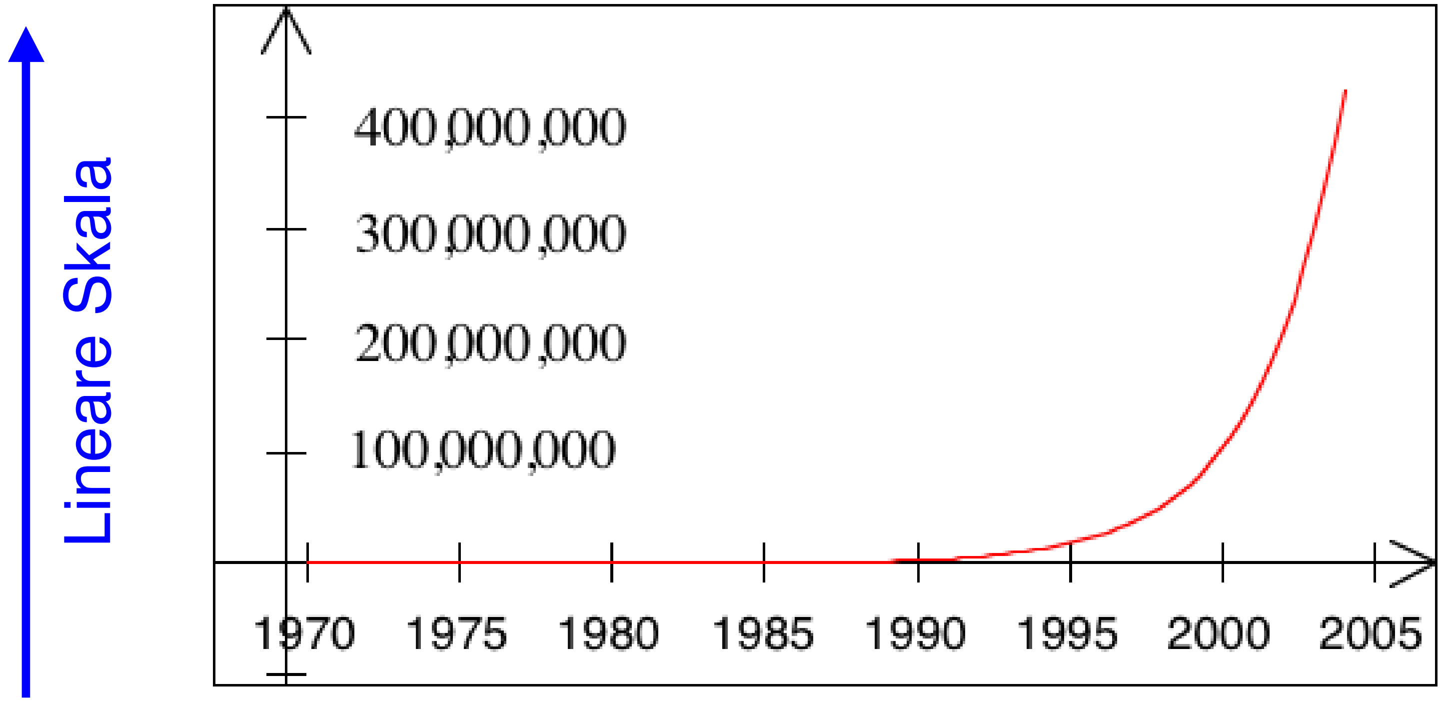
Regressionsgerade (Stochastik)

Exponentielles Wachstum



Exponentialkurve

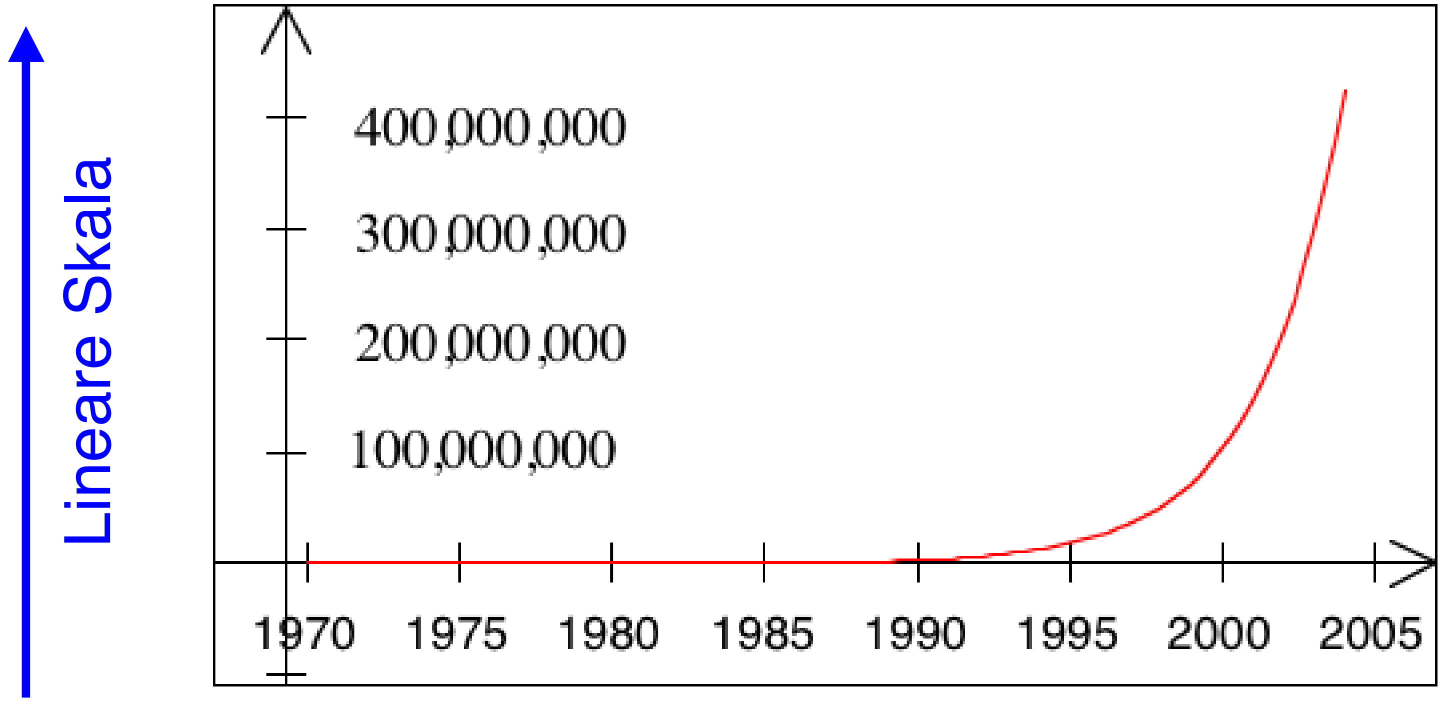
Exponentielles Wachstum $f(t) = 3000e^{0.349(t-1970)}$



Exponentialkurve

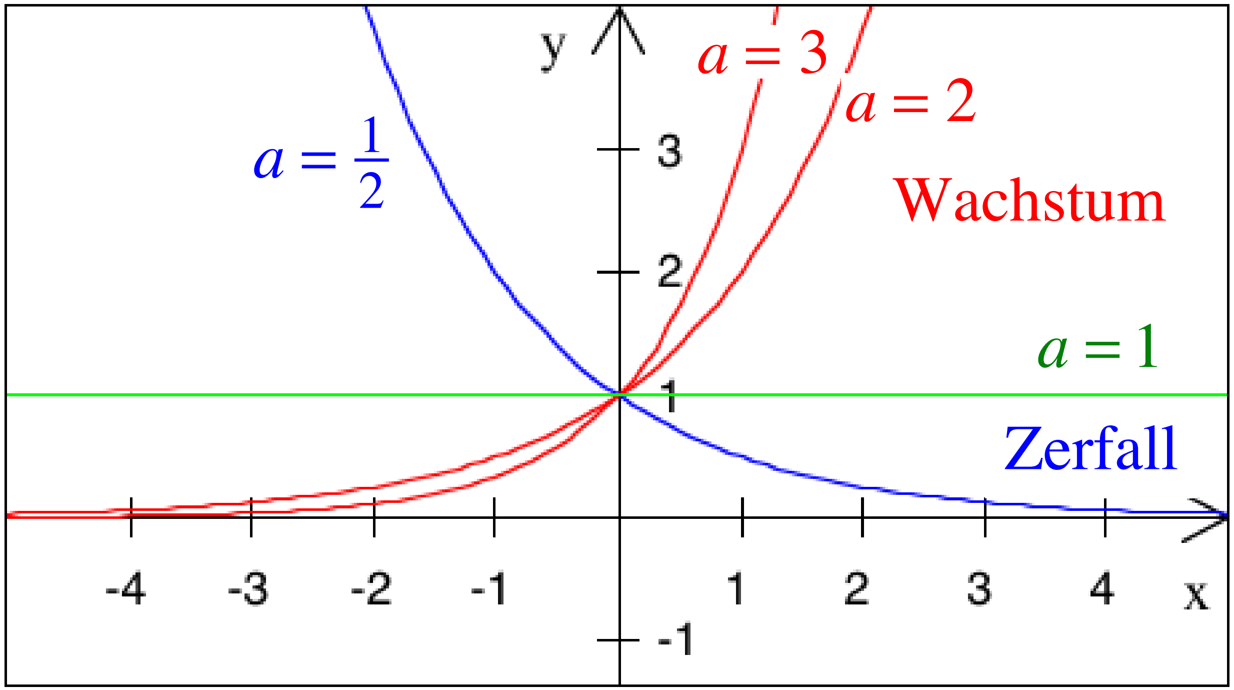
Exponentielles Wachstum $f(t) = 3000e^{0.349(t-1970)}$

Verdoppelungszeit = 1.986 Jahre



Exponentialkurve

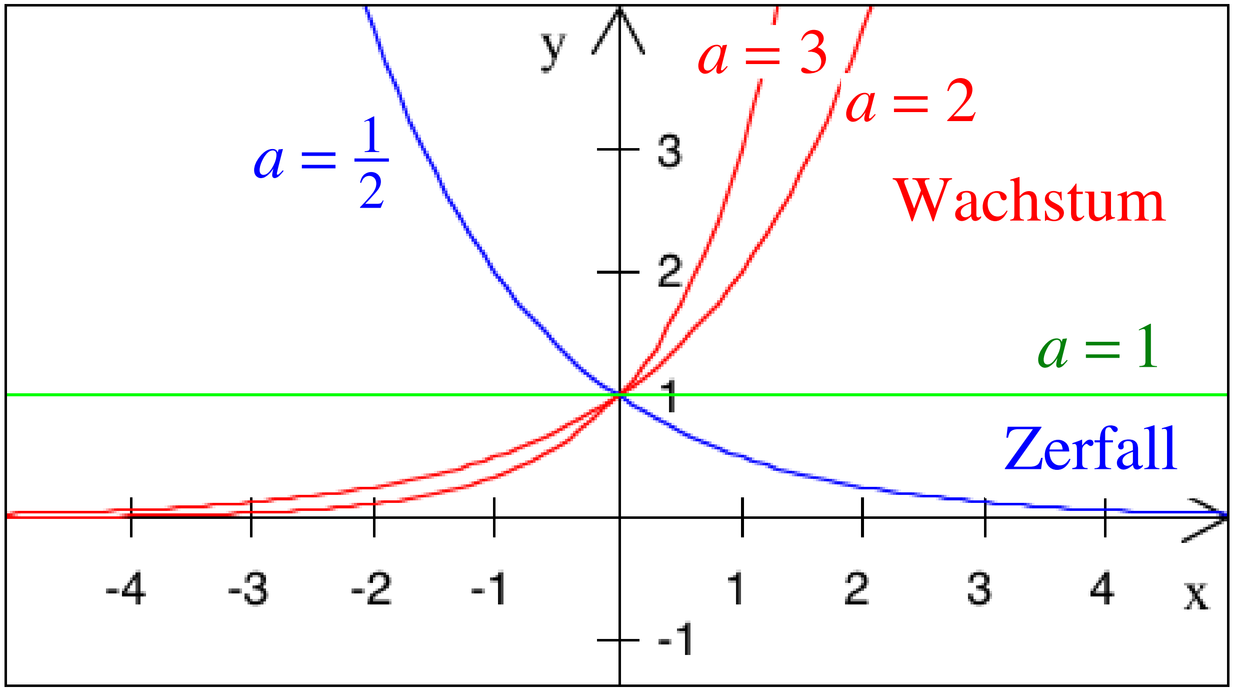
Exponentialfunktion $\qquad f(x) = a^x$



$a > 1 \qquad$ Wachstum

$0 < a < 1 \qquad$ Zerfall

Exponentialfunktion $\qquad f(x) = a^x$



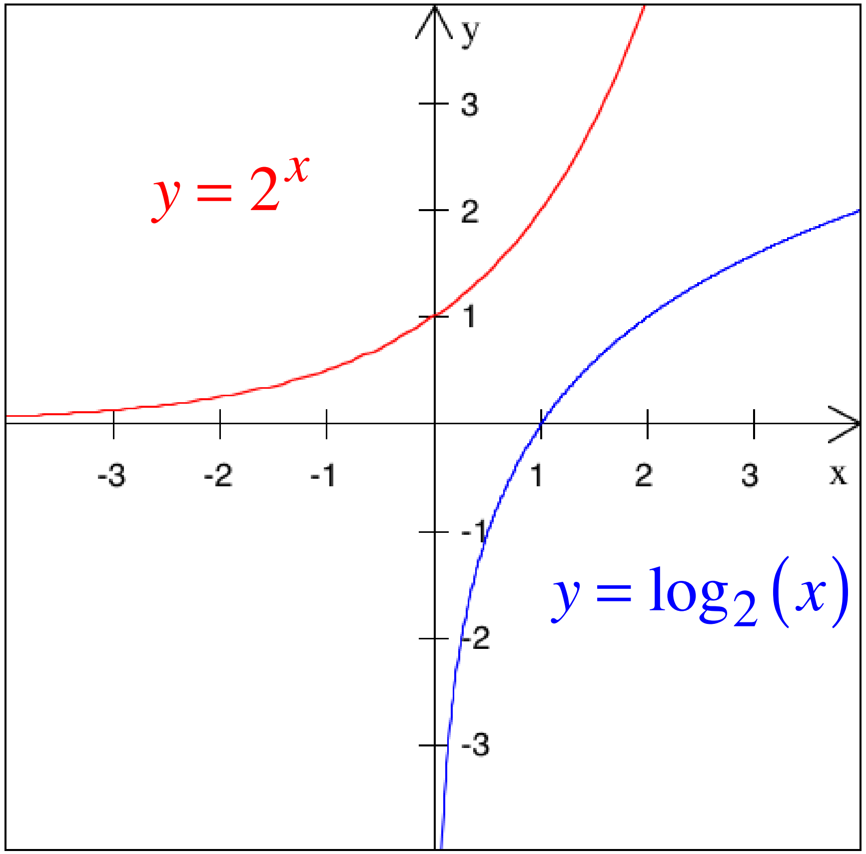
$a > 1$ Wachstum

$0 < a < 1$ Zerfall

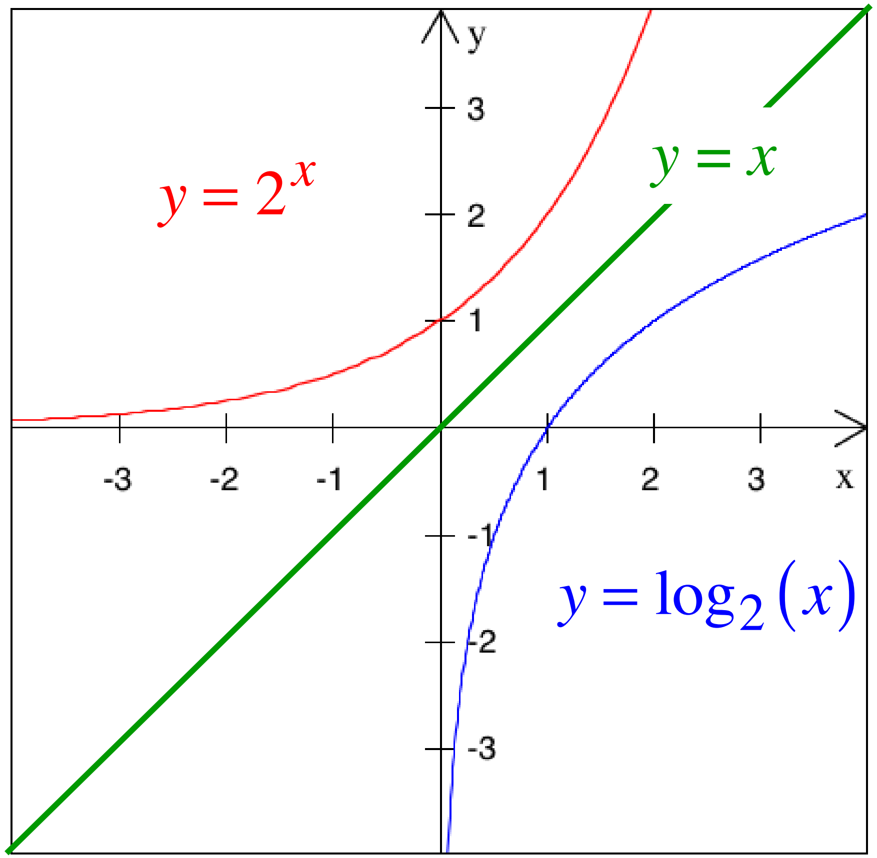
$a = 1$ konstante Funktion

a negativ ?

Die Logarithmusfunktion ist die Umkehrung der Exponentialfunktion

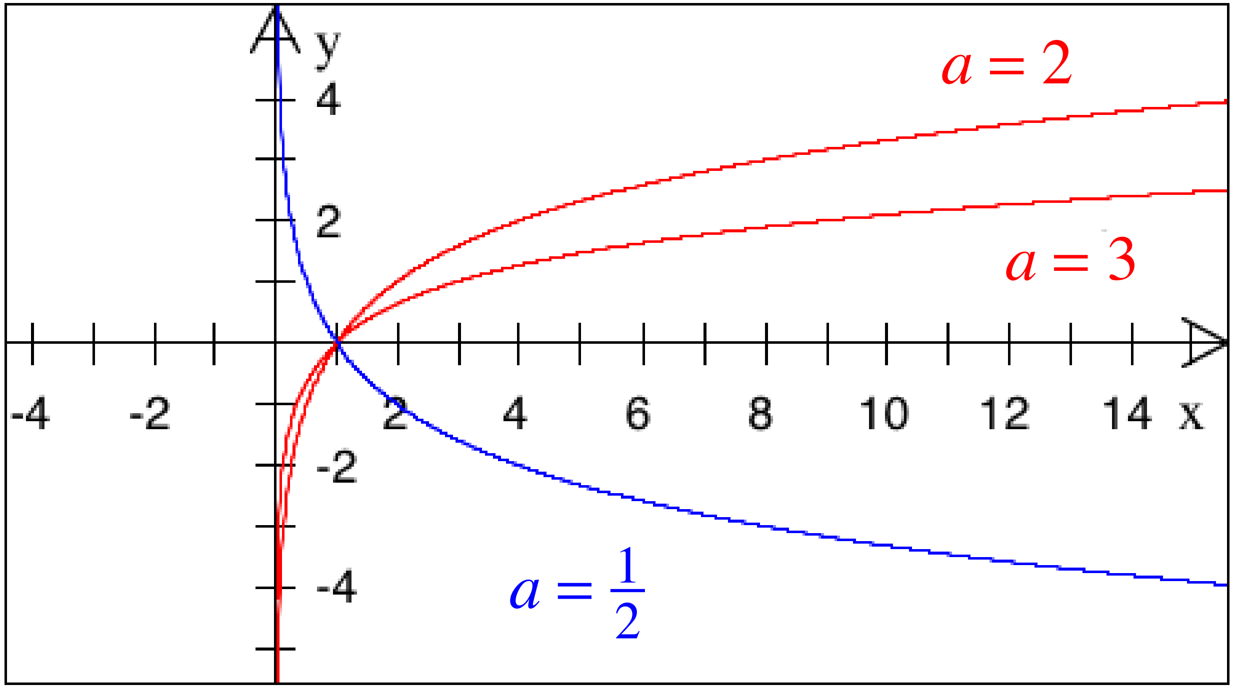


Die Logarithmusfunktion ist die Umkehrung der Exponentialfunktion



Spiegelachse $y = x$

Logarithmusfunktionen $\qquad f(x) = \log_a(x)$



Rechenregeln

$\log_a(xy) =$?	$\log_a\left(\frac{x}{y}\right) =$?
$\log_a\left(x^c\right) =$?	$\log_a\left(\sqrt[d]{x}\right) =$?

Basiswechsel: $\log_b(x) \underset{\substack{\uparrow \\ \text{Basis } a\ ?}}{=}$?

Rechenregeln

$$\log_a(xy) = \ ?$$

Rechenregeln

$$\log_a(xy) = \ ?$$

$$\log_a(x) = b$$

$$\log_a(y) = c$$

Rechenregeln

$$\log_a(xy) = \ ?$$

$$\log_a(x) = b \qquad \Leftrightarrow \qquad a^b = x$$

$$\log_a(y) = c \qquad \Leftrightarrow \qquad a^c = y$$

Rechenregeln

$$\log_a(xy) = \ ?$$

$$\left.\begin{array}{lcl} \log_a(x) = b & \Leftrightarrow & a^b = x \\ \log_a(y) = c & \Leftrightarrow & a^c = y \end{array}\right\}$$

$$a^{b+c} = xy$$

Rechenregeln

$$\log_a(xy) = \ ?$$

$$\left.\begin{aligned} \log_a(x) = b \quad &\Leftrightarrow \quad a^b = x \\ \log_a(y) = c \quad &\Leftrightarrow \quad a^c = y \end{aligned}\right\}$$

$$\log_a(xy) = b + c \quad \Leftrightarrow \quad a^{b+c} = xy$$

Rechenregeln

$$\log_a(xy) = \ ?$$

$$\left.\begin{array}{lcl} \log_a(x) = b & \Leftrightarrow & a^b = x \\[2em] \log_a(y) = c & \Leftrightarrow & a^c = y \end{array}\right\}$$

$$\log_a(xy) = \underset{\log_a(x)}{\underset{\uparrow}{b}} + \underset{\log_a(y)}{\underset{\uparrow}{c}} \quad \Leftrightarrow \quad a^{b+c} = xy$$

Rechenregeln

$$\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$$

$$\left.\begin{array}{lcl} \log_a(x) = b & \Leftrightarrow & a^b = x \\[2em] \log_a(y) = c & \Leftrightarrow & a^c = y \end{array}\right\}$$

$$\log_a(xy) = \underset{\substack{\uparrow \\ \log_a(x)}}{b} + \underset{\substack{\uparrow \\ \log_a(y)}}{c} \Leftrightarrow a^{b+c} = xy$$

Rechenregeln

$$\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$$

Großvaters Rechenschieber

Rechenregeln

$\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$	$\log_a\left(\frac{x}{y}\right) =$?
$\log_a\left(x^c\right) =$?	$\log_a\left(\sqrt[d]{x}\right) =$?

Basiswechsel: $\log_b(x) \underset{\substack{\uparrow \\ \text{Basis } a\,?}}{=}$?

Rechenregeln

	Inverse Operationen
$\log_a(xy) = \log_a(x) + \log_a(y)$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
$\log_a\left(x^c\right) =$?	$\log_a\left(\sqrt[d]{x}\right) =$?

Basiswechsel: $\log_b(x) \underset{\substack{\uparrow \\ \text{Basis } a\,?}}{=}$?

Rechenregeln

$\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
z.B. $c = 2$ $\log_a\left(x^c\right) = c\log_a(x)$	$\log_a\left(\sqrt[d]{x}\right) =$?

Basiswechsel: $\log_b(x) \underset{\uparrow}{=}$?

Basis a ?

Rechenregeln

$\boxed{\log_a(xy) = \log_a(x) + \log_a(y)}$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
$\log_a\left(x^c\right) = c\log_a(x)$	$\log_a\left(\sqrt[d]{x}\right) = \frac{1}{d}\log_a(x)$

Inverse Operationen

Basiswechsel: $\quad \log_b(x) \underset{\uparrow}{=} \; ?$

Basis a ?

Basiswechsel

$$\log_b(x) = ?$$

Was tun, wenn die Basis b auf dem Rechner nicht zur Verfügung steht?

Basiswechsel

$$\log_b(x) = ?$$

$$\log_b(x) = c$$

Basiswechsel

$$\log_b(x) = ?$$

$$\log_b(x) = c \quad \Leftrightarrow \quad b^c = x$$

Basiswechsel

$$\log_b(x) = ?$$

$$\log_b(x) = c \quad \Leftrightarrow \quad b^c = x$$

$$\log_a\left(b^c\right) = \log_a(x)$$

Basiswechsel

$$\log_b(x) = ?$$

$$\log_b(x) = c \quad \Leftrightarrow \quad b^c = x$$

$$\log_a\left(b^c\right) = \log_a(x)$$

$$c \log_a(b) = \log_a(x)$$

Basiswechsel

$$\log_b(x) = ?$$

$$\log_b(x) = c \quad \Leftrightarrow \quad b^c = x$$

$$\log_a\left(b^c\right) = \log_a(x)$$

$$c\log_a(b) = \log_a(x)$$

$$c = \frac{\log_a(x)}{\log_a(b)}$$

Basiswechsel

$$\log_b(x) = ?$$

$$\log_b(x) = c \quad \Leftrightarrow \quad b^c = x$$

$$\log_a\left(b^c\right) = \log_a(x)$$

$$c\log_a(b) = \log_a(x)$$

$$c = \frac{\log_a(x)}{\log_a(b)}$$

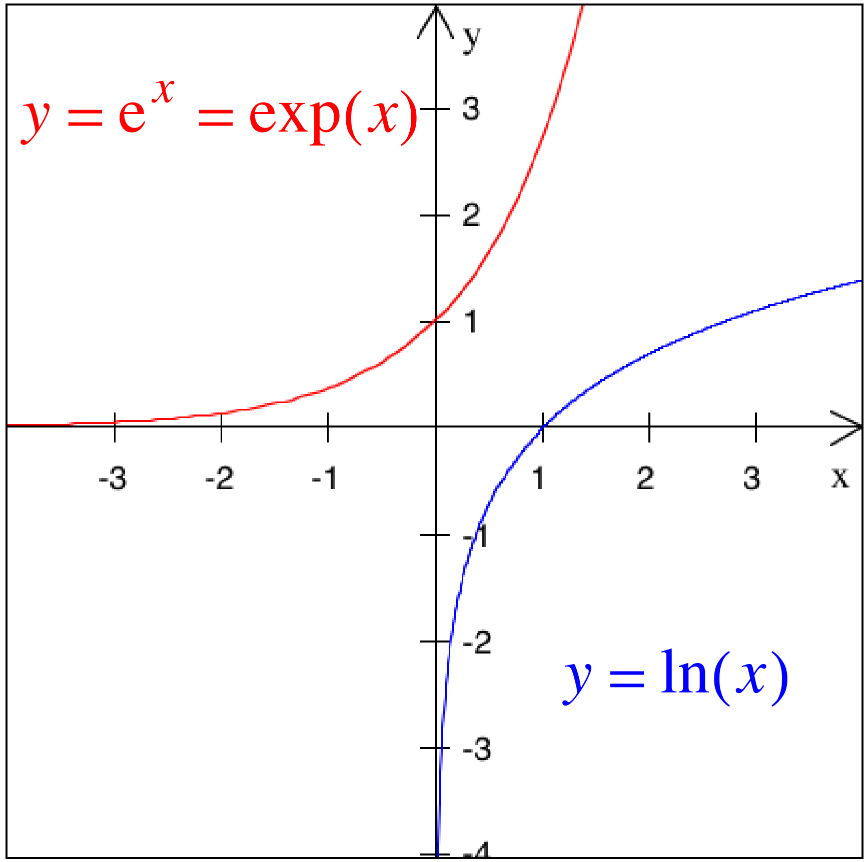
$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Rechenregeln

$\log_a(xy) = \log_a(x) + \log_a(y)$	$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
$\log_a\left(x^c\right) = c\log_a(x)$	$\log_a\left(\sqrt[d]{x}\right) = \frac{1}{d}\log_a(x)$

Basiswechsel: $\log_b(x) \underset{\substack{\uparrow \\ \text{Basis } a}}{=} \frac{\log_a(x)}{\log_a(b)}$

(Natürliche) Exponentialfunktion und natürlicher Logarithmus



Basis $\quad e \approx 2.718 \quad e \notin \mathbb{Q}$

Eulersche Zahl

(Natürliche) Exponentialfunktion und natürlicher Logarithmus

Basis $\quad e \approx 2.718 \quad e \notin \mathbb{Q}$

Eulersche Zahl

Folge

Folge $\qquad a_n = \frac{1}{n}$

Folge $\qquad a_n = \frac{1}{n}$

Funktion $\qquad$ Definitionsbereich $= \mathbb{N}$

$x \quad \to \quad \frac{1}{x} \qquad$ oder $\qquad f(x) = \frac{1}{x}$

$n \quad \to \quad \frac{1}{n} \qquad$ oder $\qquad f(n) = \frac{1}{n}$

---

alles dasselbe

Name

Folge $a_n = \frac{1}{n}$ Wert

Nummer

Name

Folge $a_n = \dfrac{1}{n}$ Wert

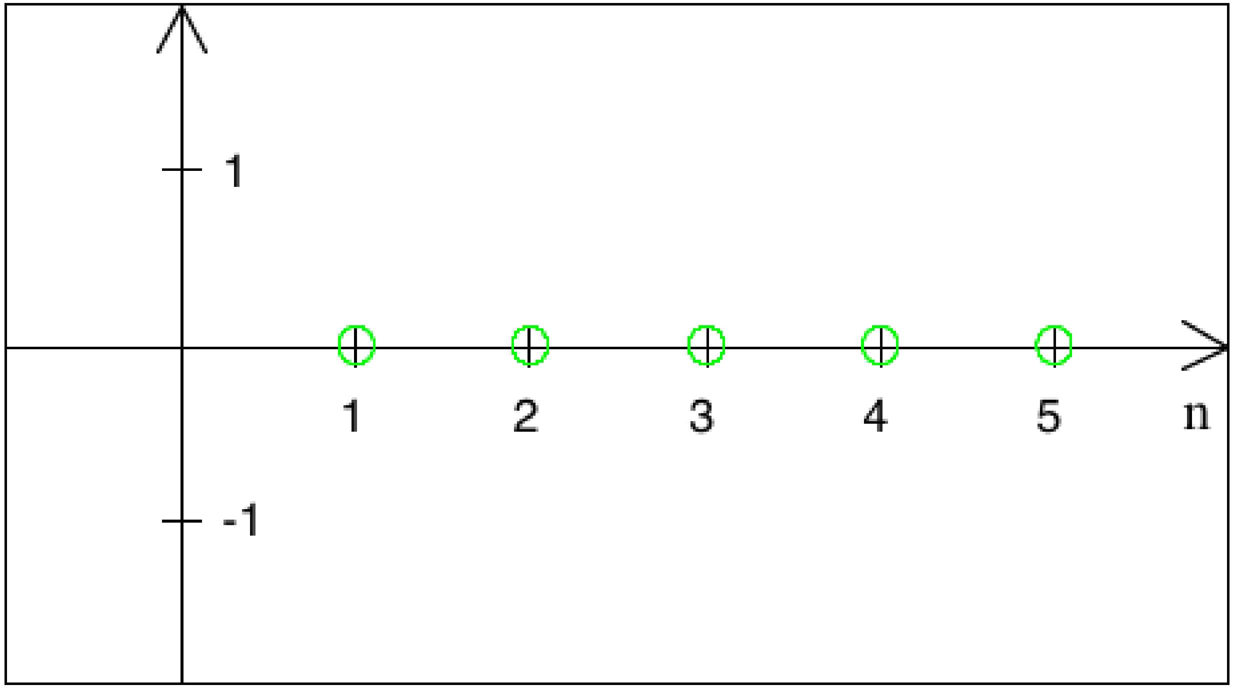
Nummer

Wertetabelle

n	1	2	3	4	5	$\cdots$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$

$$a_n = \frac{1}{n}$$

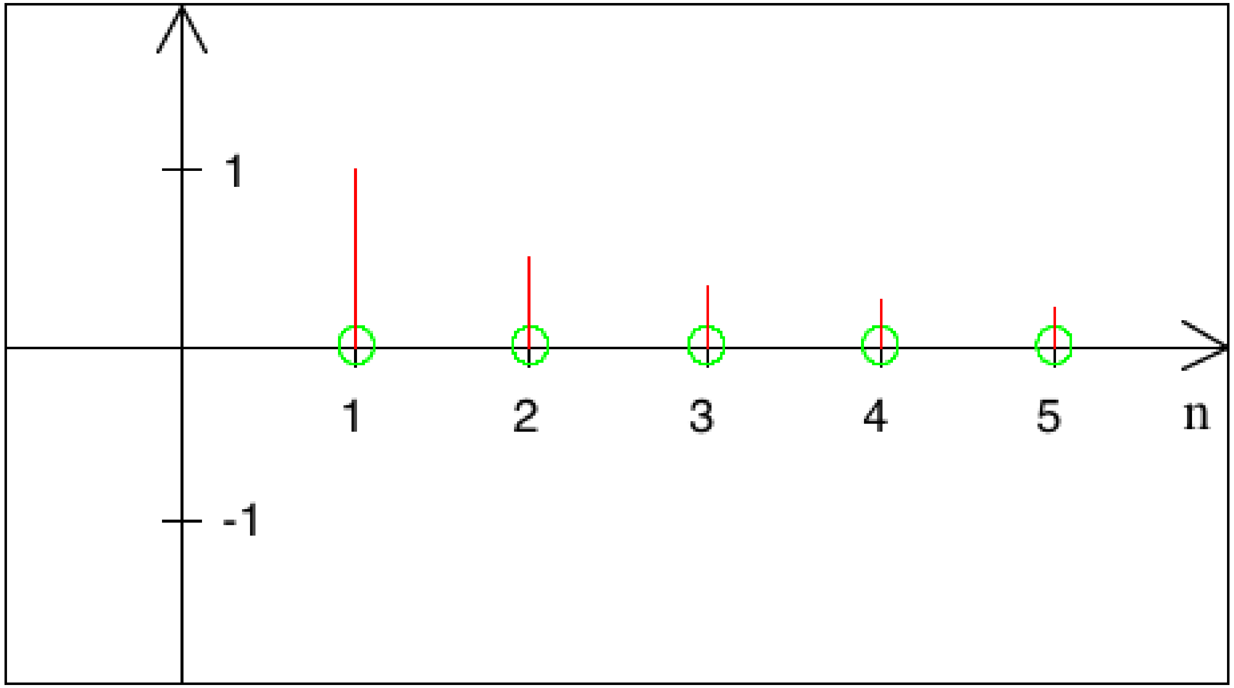
Definitionsbereich



n	1	2	3	4	5	$\cdots$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$

$$a_n = \frac{1}{n}$$

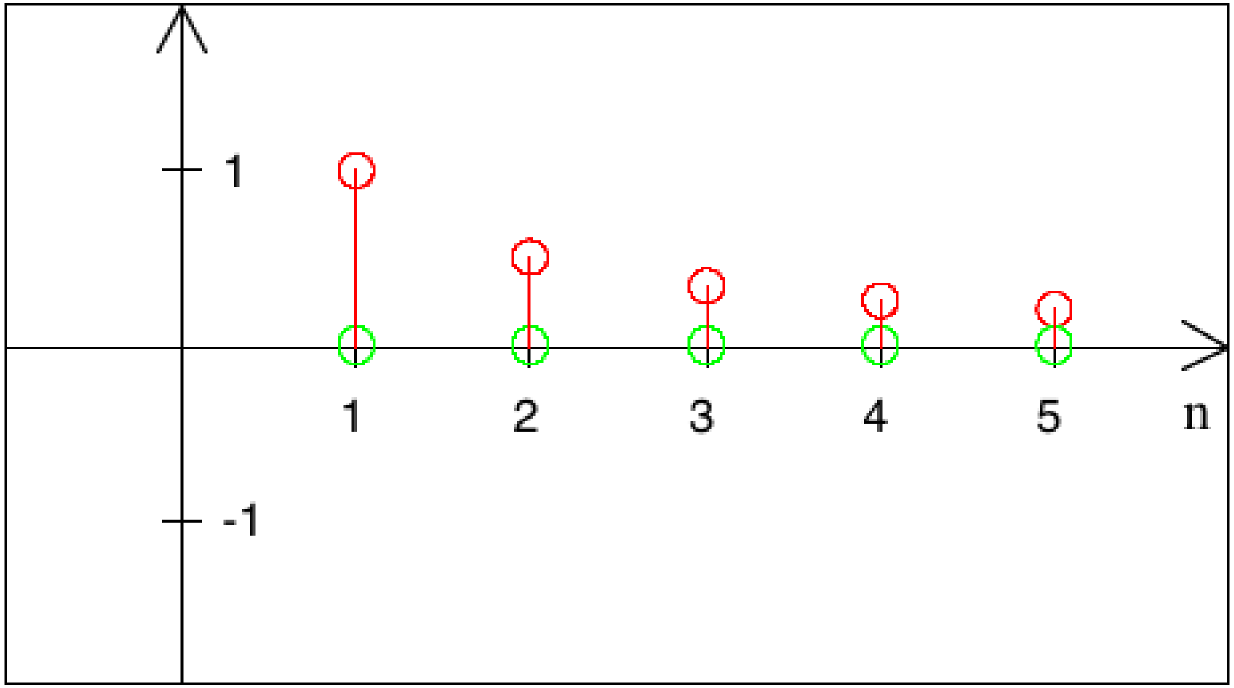
Werte



n	1	2	3	4	5	$\cdots$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$

$a_n = \frac{1}{n}$

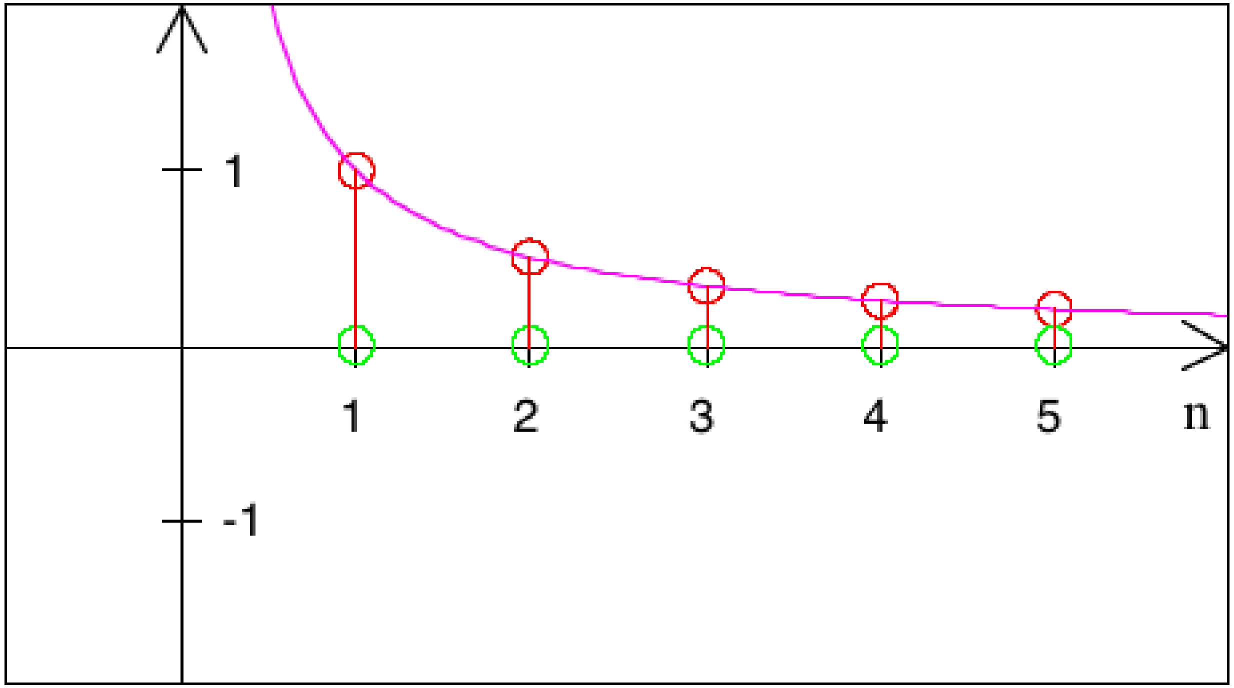
Graph



n	1	2	3	4	5	$\cdots$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$

$a_n = \frac{1}{n}$

„Kurve“



n	1	2	3	4	5	$\cdots$
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\cdots$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

Limes, Grenzwert

$a_n = \dfrac{1}{n}$ ist eine Nullfolge

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right)=0$$

Limes, Grenzwert

$a_n = \frac{1}{n}$ ist eine Nullfolge

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

Unendlich

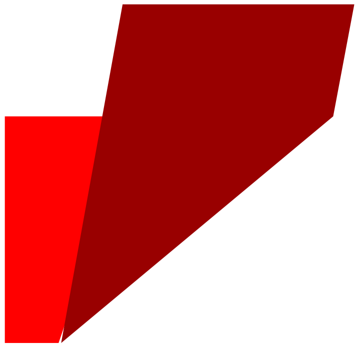
Vergiss
Deine Grenzen
Wandre aus
Das Niemandsland
Unendlich
Nimmt dich auf

Rose Ausländer (1901-1988)

Falten

1. Wir beginnen mit einem langen Streifen.

Falten



2. Wir falten in irgend einer Richtung nach OBEN.

Falten

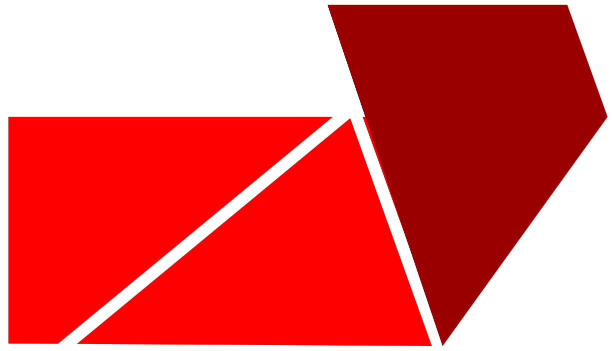
3. Auffalten

4. Wir falten nach UNTEN – nun *genau* wie dargestellt.

Falten

5. Auffalten

Falten

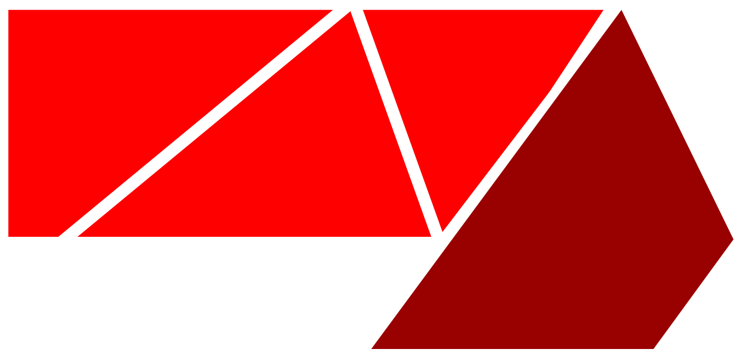


6. Wir falten nach OBEN – *genau* wie dargestellt.

Falten

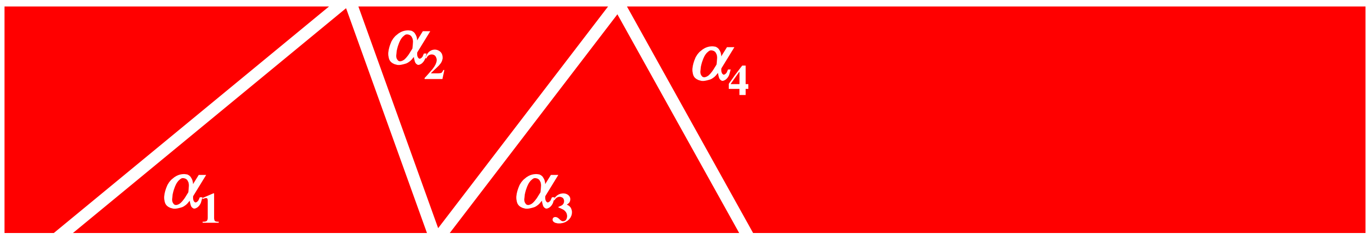
7. Auffalten

8. Wir falten nach UNTEN – nun *genau* wie dargestellt.



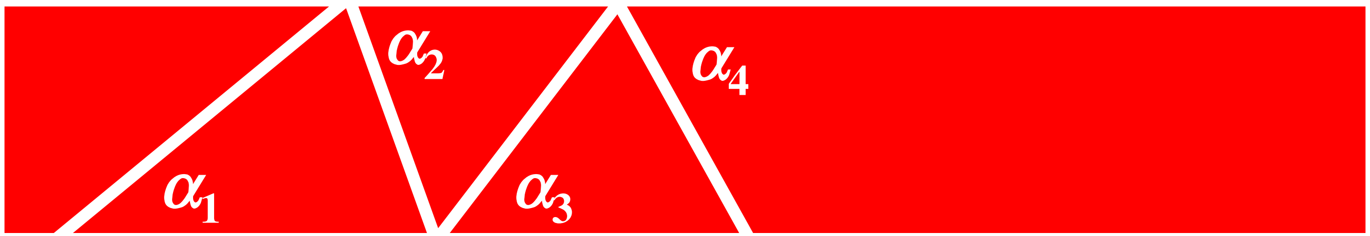
Falten

9. Auffalten



Falten

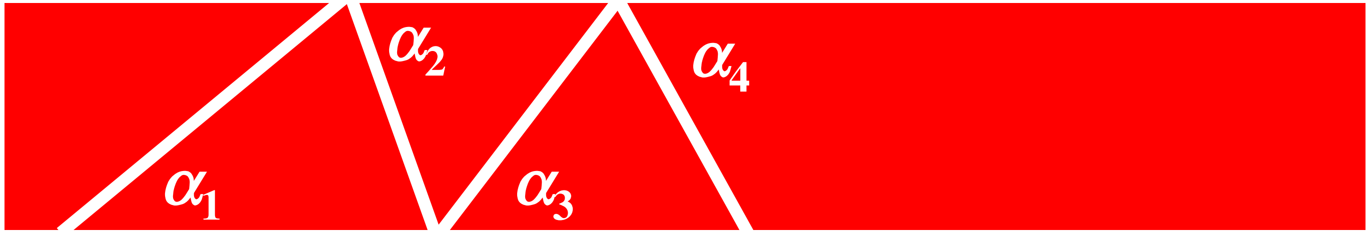
9. Auffalten



Vermutung: $$\lim_{n\to\infty}\left(\alpha_n\right) = 60°$$

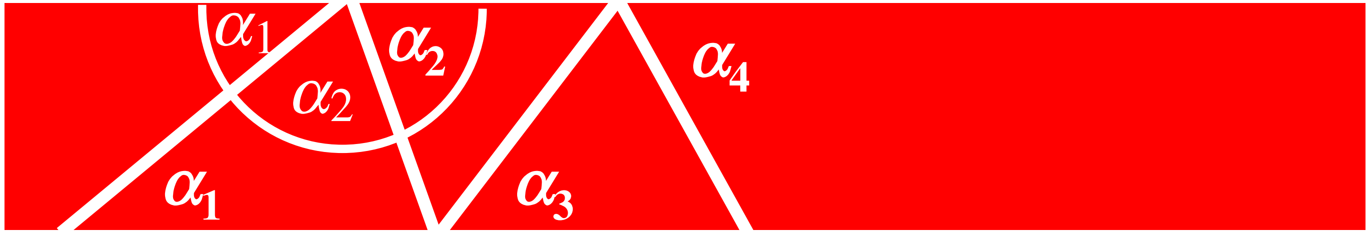
„unendlich"

Beispiel



$$\alpha_1 \quad = \quad 36° \qquad \text{Startwert}$$

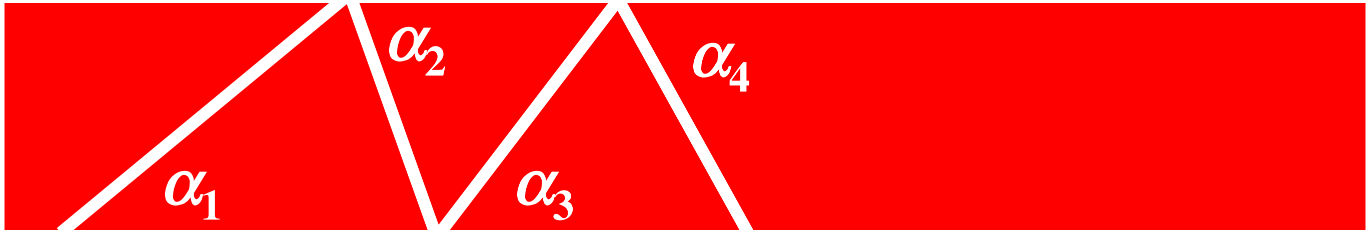
Beispiel



$$\alpha_1 = 36^\circ \quad \text{Startwert}$$

$$\alpha_2 = \frac{180^\circ - \alpha_1}{2} = 72^\circ$$

Beispiel

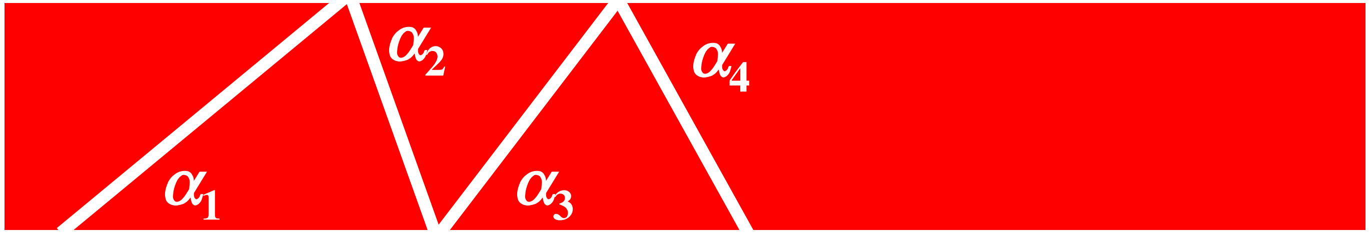


$$\alpha_1 = 36° \quad \text{Startwert}$$

$$\alpha_2 = \frac{180° - \alpha_1}{2} = 72°$$

$$\alpha_3 = \frac{180° - \alpha_2}{2} = 54°$$

Beispiel



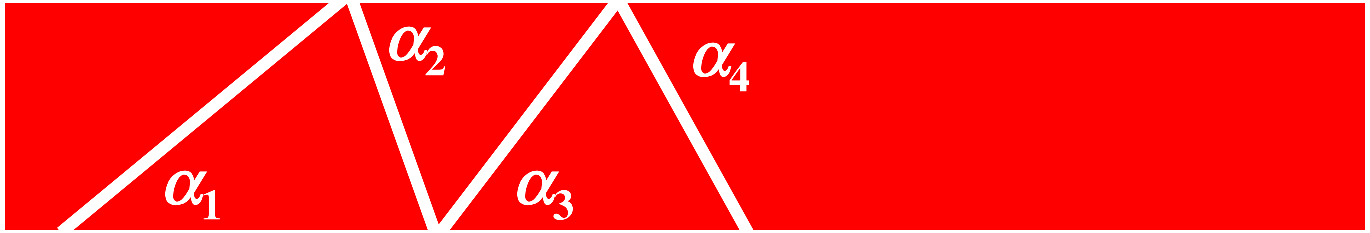
$$\alpha_1 = 36° \quad \text{Startwert}$$

$$\alpha_2 = \frac{180° - \alpha_1}{2} = 72°$$

$$\alpha_3 = \frac{180° - \alpha_2}{2} = 54°$$

$$\alpha_4 = \frac{180° - \alpha_3}{2} = 63°$$

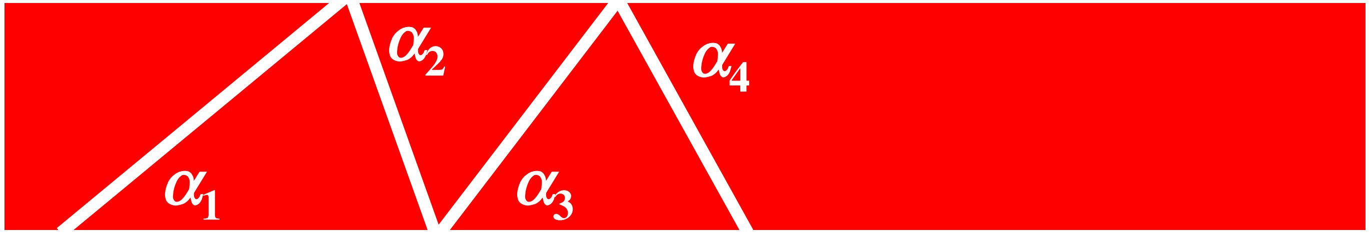
Beispiel



$$
\begin{aligned}
\alpha_1 &= 36^\circ && \text{Startwert} && = 60^\circ - 24^\circ \\
\alpha_2 &= \frac{180^\circ - \alpha_1}{2} &&= 72^\circ && = 60^\circ + 12^\circ \\
\alpha_3 &= \frac{180^\circ - \alpha_2}{2} &&= 54^\circ && = 60^\circ - 6^\circ \\
\alpha_4 &= \frac{180^\circ - \alpha_3}{2} &&= 63^\circ && = 60^\circ + 3^\circ
\end{aligned}
$$

Der Fehler wird jedes Mal halbiert: $\lim\limits_{n\to\infty} (\alpha_n) = 60^\circ$

Beispiel

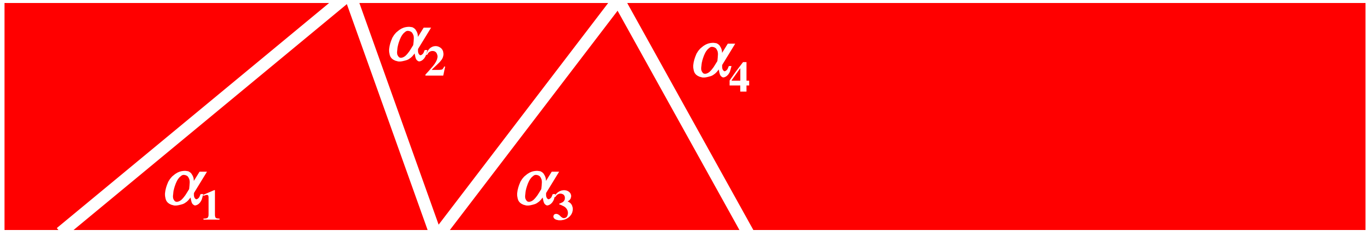


$\alpha_1 = 36°$ Startwert

$$\boxed{\alpha_n = \frac{180° - \alpha_{n-1}}{2}}$$

Rekursionsformel

Beispiel



α_1 Startwert beliebig

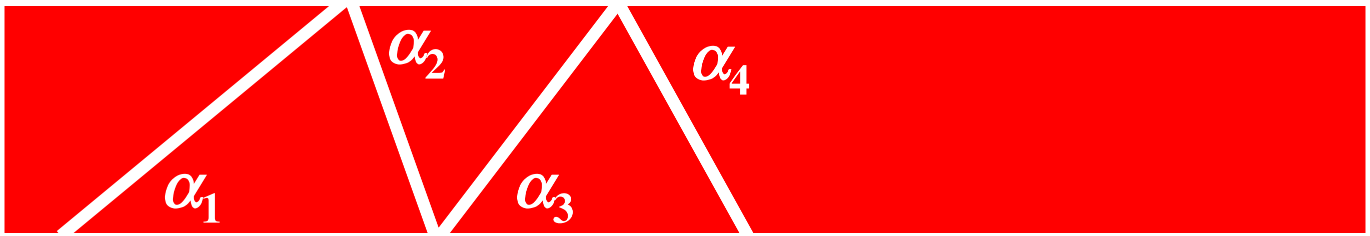
$$\boxed{\alpha_n = \frac{180° - \alpha_{n-1}}{2}}$$

Rekursionsformel

Der Startwert spielt keine Rolle

n	α_n	α_n	α_n	α_n
1	10.0000000°	10000.0000000°	70.0000000°	-200.0000000°
2	85.0000000°	-4910.0000000°	55.0000000°	190.0000000°
3	47.5000000°	2545.0000000°	62.5000000°	-5.0000000°
4	66.2500000°	-1182.5000000°	58.7500000°	92.5000000°
5	56.8750000°	681.2500000°	60.6250000°	43.7500000°
6	61.5625000°	-250.6250000°	59.6875000°	68.1250000°
7	59.2187500°	215.3125000°	60.1562500°	55.9375000°
8	60.3906250°	-17.6562500°	59.9218750°	62.0312500°
9	59.8046875°	98.8281250°	60.0390625°	58.9843750°
10	60.0976563°	40.5859375°	59.9804688°	60.5078125°
11	59.9511719°	69.7070313°	60.0097656°	59.7460938°
12	60.0244141°	55.1464844°	59.9951172°	60.1269531°
13	59.9877930°	62.4267578°	60.0024414°	59.9365234°
14	60.0061035°	58.7866211°	59.9987793°	60.0317383°
15	59.9969482°	60.6066895°	60.0006104°	59.9841309°

Beispiel



α_1 Startwert beliebig

$$\boxed{\alpha_n = \frac{180^\circ - \alpha_{n-1}}{2}}$$

Rekursionsformel

Wie finden wir den Limes?

(1) Annahme: Es gibt einen Limes

$$\alpha = \lim_{n \to \infty} (\alpha_n)$$

(1) Annahme: Es gibt einen Limes

$$\alpha = \lim_{n \to \infty} \left(\alpha_n \right)$$

(2) In Rekursionsformel einsetzen:

$$\alpha = \frac{180° - \alpha}{2}$$

(1) Annahme: Es gibt einen Limes

$$\alpha = \lim_{n\to\infty} \left(\alpha_n\right)$$

(2) In Rekursionsformel einsetzen:

$$\alpha = \frac{180° - \alpha}{2}$$

(3) Nach α auflösen:

$$2\alpha = 180° - \alpha$$

$$3\alpha = 180°$$

$$\alpha = 60°$$

Konvergenz

$$\left.\begin{array}{l} \lim\limits_{n\to\infty}\left(a_n\right) = a \\[2em] a_n \to a \quad \text{für} \quad n \to \infty \\[2em] a_n \xrightarrow{\;n\to\infty\;} a \end{array}\right\}$$

Die Folge a_n konvergiert gegen den Grenzwert a

Konvergenz

$$\left.\begin{array}{l} \lim\limits_{n\to\infty}\left(a_n\right) = a \\[3em] a_n \to a \quad \text{für} \quad n \to \infty \\[3em] a_n \xrightarrow{n\to\infty} a \end{array}\right\}$$

$\boxed{\text{Die Folge } a_n \text{ konvergiert gegen den Grenzwert } a}$

Gegenteil: Divergenz

Rechenregeln

$$\lim_{n\to\infty}\left(a_n+b_n\right)=\lim_{n\to\infty}\left(a_n\right)+\lim_{n\to\infty}\left(b_n\right)$$

$$\lim_{n\to\infty}\left(\lambda a_n\right)=\lambda\lim_{n\to\infty}\left(a_n\right)$$

$$\lim_{n\to\infty}\left(a_n\cdot b_n\right)=\lim_{n\to\infty}\left(a_n\right)\cdot\lim_{n\to\infty}\left(b_n\right)$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\frac{\lim\limits_{n\to\infty}\left(a_n\right)}{\lim\limits_{n\to\infty}\left(b_n\right)}\qquad\text{falls}\qquad\lim_{n\to\infty}\left(b_n\right)\neq 0$$

$$a_n = \frac{n+2}{5n^2-2n+1}$$

$$\lim_{n\to\infty}(a_n) = \ ?$$

$$a_n = \frac{n+2}{5n^2-2n+1}$$

Explizite Angabe der Folge
(keine Rekursionsformel)

$$\lim_{n\to\infty}(a_n) = \ ?$$

$$a_n = \frac{n+2}{5n^2-2n+1}$$

Explizite Angabe der Folge
(keine Rekursionsformel)

$$\lim_{n\to\infty}(a_n) =$$ Vermutung?

$$a_n = \frac{n+2}{5n^2-2n+1} \underset{\uparrow}{=} \frac{\frac{1}{n}+\frac{2}{n^2}}{5-\frac{2}{n}+\frac{1}{n^2}}$$

Trick: $\boxed{\text{Erweitern mit } \frac{1}{n^2}}$

$$\lim_{n\to\infty}\left(a_n\right) = \text{Vermutung?}$$

$$a_n = \frac{n+2}{5n^2-2n+1} \underset{\uparrow}{=} \frac{\frac{1}{n}+\frac{2}{n^2}}{5-\frac{2}{n}+\frac{1}{n^2}}$$

Trick: $\boxed{\text{Erweitern mit } \frac{1}{n^2}}$

$$\lim_{n\to\infty}\left(a_n\right) = \frac{0+0}{5-0+0} = 0$$

$$a_n = \frac{n^2 - 2n}{5n^2 + 1}$$

$$\lim_{n \to \infty} \left(a_n \right) = ?$$

$$a_n = \frac{n^2-2n}{5n^2+1} \underset{\uparrow}{=} \frac{1-\frac{2}{n}}{5+\frac{1}{n^2}}$$

Methode: $\boxed{\text{Erweitern mit } \frac{1}{n^2}}$

$$\lim_{n\to\infty}(a_n) = ?$$

Wird ein Trick zwei Mal angewendet,
ist es eine Methode.

$$a_n = \frac{n^2 - 2n}{5n^2 + 1} \underset{\uparrow}{=} \frac{1 - \frac{2}{n}}{5 + \frac{1}{n^2}}$$

Methode: $\boxed{\text{Erweitern mit } \frac{1}{n^2}}$

$$\lim_{n \to \infty} (a_n) = \frac{1-0}{5+0} = \frac{1}{5}$$

Wird ein Trick zwei Mal angewendet,
ist es eine Methode.

$$a_n = \frac{n^4 - 2n^3 + 1}{5n^2 + 1}$$

Womit erweitern?

$$a_n = \frac{n^4 - 2n^3 + 1}{5n^2 + 1} \underset{\uparrow}{=} \frac{1 - \frac{2}{n} + \frac{1}{n^4}}{\frac{5}{n^2} + \frac{1}{n^4}}$$

$$\boxed{\begin{array}{l}\text{Erweitern} \\ \text{mit } \quad \dfrac{1}{n^4}\end{array}}$$

Womit erweitern?

$$a_n = \frac{n^4 - 2n^3 + 1}{5n^2 + 1} \underset{\uparrow}{=} \frac{1 - \frac{2}{n} + \frac{1}{n^4}}{\frac{5}{n^2} + \frac{1}{n^4}}$$

$$\boxed{\text{Erweitern mit } \frac{1}{n^4}}$$

Womit erweitern?

$$\lim_{n \to \infty} \left(a_n\right) = \frac{1 - 0 + 0}{0 + 0}$$

Division durch Null

$$a_n = \frac{n^4 - 2n^3 + 1}{5n^2 + 1} \underset{\uparrow}{=} \frac{n^2\left(1 - \frac{2}{n} + \frac{1}{n^4}\right)}{5 + \frac{1}{n^2}}$$

Neuer Versuch

Erweitern

mit $\frac{1}{n^2}$

$$a_n = \frac{n^4 - 2n^3 + 1}{5n^2 + 1} \underset{\uparrow}{=} \frac{n^2\left(1 - \frac{2}{n} + \frac{1}{n^4}\right)}{5 + \frac{1}{n^2}}$$

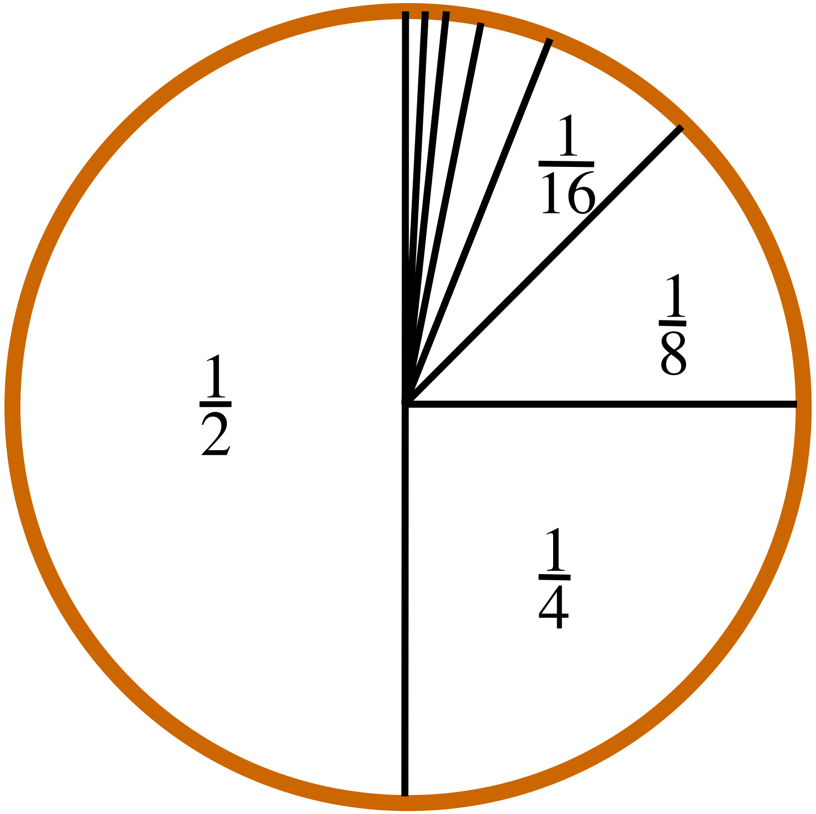
Erweitern mit $\frac{1}{n^2}$

Neuer Versuch

$$\lim_{n\to\infty}(a_n) = \frac{\infty(1-0+0)}{5+0} = \infty$$

Die Folge divergiert.

So ein Käse!



Portion pro Mahlzeit: $a_n = \left(\frac{1}{2}\right)^n$

Was gegessen ist:

$$s_1 = \frac{1}{2}$$

$$s_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

$$s_4 = \text{ wie geht es weiter?}$$

Summenfolge

Folge :

$a_1, a_2, a_3, \ldots$

Reihe :

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

$s_1, s_2, s_3, \ldots$ Folge von Partialsummen

Käsebeispiel (Swiss cheese)

Folge $\quad a_n = \left(\frac{1}{2}\right)^n \qquad a = \lim_{n\to\infty}\left(a_n\right) = 0$

Reihe $\quad s_n = \sum_{k=1}^{n}\left(\frac{1}{2}\right)^k \qquad s = \lim_{n\to\infty}\left(s_n\right) = \sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^k = \underset{\uparrow}{1}$

ganzer Käse

Harmonische Reihe

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = ?$$

Harmonische Reihe

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = ?$$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

Harmonische Reihe

$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = ?$$

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{>\frac{1}{2}} + \cdots$$

Harmonische Reihe

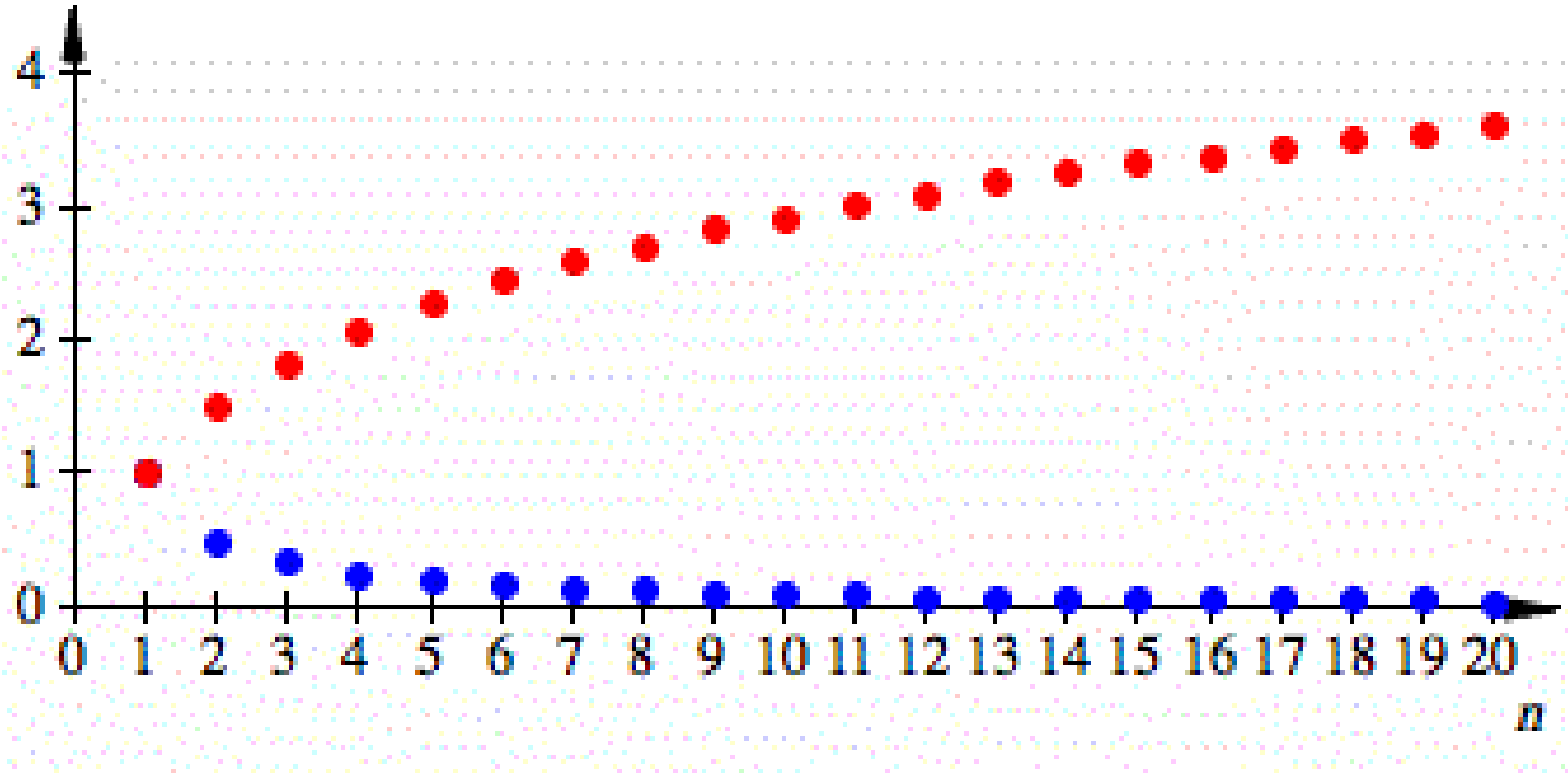
$$s_n = \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = ?$$

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{>\frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{>\frac{1}{2}} + \cdots$$

Die harmonische Reihe $\sum_{k=1}^{\infty} \frac{1}{k}$ ist divergent

Harmonische Reihe



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Neues Beispiel

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}$$

$$\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots = ?$$

Summe der Kehrwerte der Quadratzahlen

$$s_n = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \cdots + \frac{1}{n \cdot n}$$

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Vergleich mit:

$$t_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n}$$

$$s_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{n\cdot n}$$

Vergleich mit:

$$t_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n}$$

t_n ist größer als s_n (Warum?)

$$s_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{n\cdot n}$$

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Weiter : $\frac{1}{2\cdot 3} = \frac{1}{2} - \frac{1}{3}$

$$s_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{n\cdot n}$$

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$$t_n = 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

Erstes Glied unbehandelt stehen lassen

$$s_n = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{n\cdot n}$$

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$$t_n = 1 + \Big(\frac{1}{1} \underbrace{- \frac{1}{2}\Big) + \Big(\frac{1}{2}}_{0} \underbrace{- \frac{1}{3}\Big) + \Big(\frac{1}{3}}_{0} \underbrace{- \frac{1}{4}\Big) + \quad}_{0} \cdots \underbrace{\quad + \Big(\frac{1}{n-1}}_{0} - \frac{1}{n}\Big)$$

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$$t_n = 1 + 1 + \text{lange nichts mehr} - \frac{1}{n} = 2 - \frac{1}{n}$$

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$$\lim_{n\to\infty} t_n = 2$$

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$$t_n = 1 + 1 + \text{lange nichts mehr } - \frac{1}{n} = 2 - \frac{1}{n}$$

$$\lim_{n\to\infty} t_n = 2 \qquad \Rightarrow \qquad \lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} \frac{1}{k^2} < 2$$

$$\boxed{\text{Die Reihe} \quad \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{ist konvergent}}$$

Die Reihe $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ ist konvergent

Wir: $\displaystyle s = \sum_{k=1}^{\infty} \frac{1}{k^2} < 2$ Abschätzung

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Leonhard Euler (1736): $s = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.6449$

Exakte Angabe, und wie!

Leonhard Euler

1707 - 1783

Leonhard Euler (1736): $$s = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.6449$$

Exakte Angabe, und wie!

Geometrische Folge :

$$\boxed{a_{n+1} = q \cdot a_n} \quad \text{Rekursionsformel}$$

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$$a_1 = q \cdot a_0$$ Nummerierung ab Null sinnvoll

$$a_2 = q \cdot a_1 = q^2 \cdot a_0$$

$$a_3 = q \cdot a_2 = q^3 \cdot a_0$$

$$\boxed{a_n = q^n \cdot a_0}$$ explizite Formel

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$q > 1 \quad \Rightarrow$ exponentielles Wachstum

$0 < q < 1 \quad \Rightarrow$ exponentielle Abnahme

q negativ?

Geometrische Reihe

$$\underbrace{\sum_{k=0}^{n} a_0 q^k}_{\text{mit } a_0} = a_0 \sum_{k=0}^{n} q^k \quad \text{oder} \quad \underbrace{\sum_{k=1}^{n} a_0 q^k}_{\text{ohne } a_0} = a_0 \sum_{k=1}^{n} q^k$$

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Sonderfall: $a_0 = 1$

$$s_n = \sum_{k=0}^{n} q^k \quad \text{oder} \quad s_n = \sum_{k=1}^{n} q^k$$

Käsebeispiel $q = \frac{1}{2}$

$$s_n = \sum_{k=0}^{n} q^k$$ Geometrische Reihe

$$s_n = 1 + q + q^2 + \cdots + q^n$$

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$$\begin{array}{lcccccccccccc} s_n & = & 1 & + & q & + & q^2 & + & \cdots & + & q^n & & \\ q s_n & = & & & q & + & q^2 & + & \cdots & + & q^n & + & q^{n+1} \\ \hline \end{array}$$

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$$\begin{array}{rcl} s_n & = & 1 + q + q^2 + \cdots + q^n \\ q s_n & = & q + q^2 + \cdots + q^n + q^{n+1} \\ \hline s_n(1-q) & = & 1 - q^{n+1} \end{array}$$

$$s_n = \sum_{k=0}^{n} q^k$$ Geometrische Reihe

$$\begin{array}{rcccccccccccc}
s_n & = & 1 & + & q & + & q^2 & + & \cdots & + & q^n & & \\
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$$s_n = \sum_{k=0}^{n} q^k \qquad \text{Geometrische Reihe}$$

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falls $|q| < 1 \quad \Rightarrow \quad \lim\limits_{n\to\infty}\left(q^n\right) = 0$

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s_n & = & 1 + q + q^2 + \cdots + q^n \\
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\hline
s_n(1-q) & = & 1 \phantom{{}+ q + q^2 + \cdots + q^n} - q^{n+1}
\end{array}$$

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Erstes Glied

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Geometrische Reihe $$s_n = \sum_{k=1}^{n} q^k$$

$$s_n = q + q^2 + \cdots + q^n$$

$\boxed{\text{fängt erst bei } q \text{ an}}$

Geometrische Reihe

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$$\begin{array}{rclllllllll} s_n & = & q & + & q^2 & + & \cdots & + & q^n & & \\ q s_n & = & & & q^2 & + & \cdots & + & q^n & + & q^{n+1} \\ \hline \end{array}$$

Geometrische Reihe $\qquad s_n = \sum_{k=1}^{n} q^k$

$$
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Geometrische Reihe

$$a_0 + a_1 + a_2 + \cdots = a_0 \sum_{k=0}^{\infty} q^k = \frac{a_0}{1-q}$$

$$a_1 + a_2 + a_3 + \cdots = a_0 \sum_{k=1}^{\infty} q^k = \frac{a_0 q}{1-q} = \frac{a_1}{1-q}$$

falls $|q| < 1$

$$a_n \xrightarrow[\bullet q]{} a_{n+1} \quad \Rightarrow \quad \boxed{\sum_{k=\text{Start}}^{\infty} a_k = \frac{\text{Startglied}}{1-q}}$$

Heißer Fall: $q = -1$

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Beispiel: $a_n = 7 \cdot (-1)^n$

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$$a_1 = -7, \quad a_2 = +7, \quad a_3 = -7, \quad a_4 = +7, \quad \ldots$$

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$a_1 = -7, \quad a_2 = +7, \quad a_3 = -7, \quad a_4 = +7, \quad \ldots$

$-7 + 7 - 7 + 7 - 7 + 7 \quad \cdots \quad = ?$

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Silvio: $\underbrace{-7 + 7}_{0}\underbrace{-7 + 7}_{0}\underbrace{-7 + 7}_{0} \quad \cdots \quad = 0$

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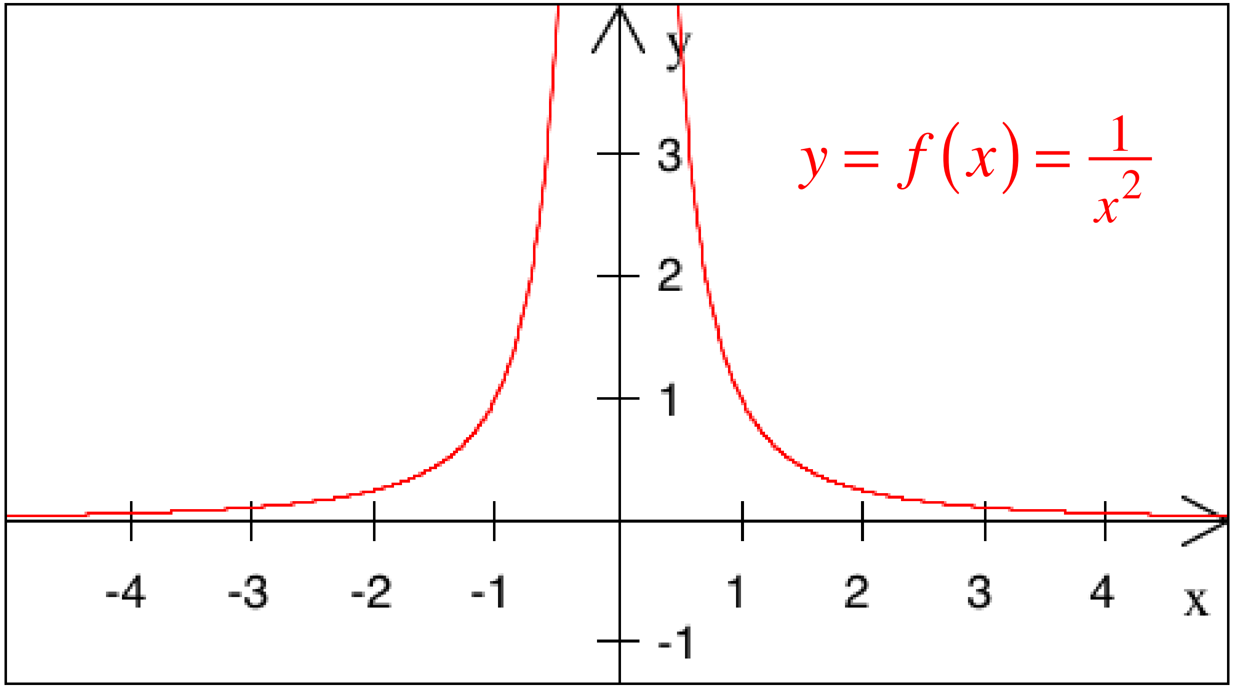
Silvia: $-7 + \underbrace{7 - 7}_{0} + \underbrace{7 - 7}_{0} + \underbrace{7 \quad}_{0} \cdots \quad = -7$

Wer hat recht?

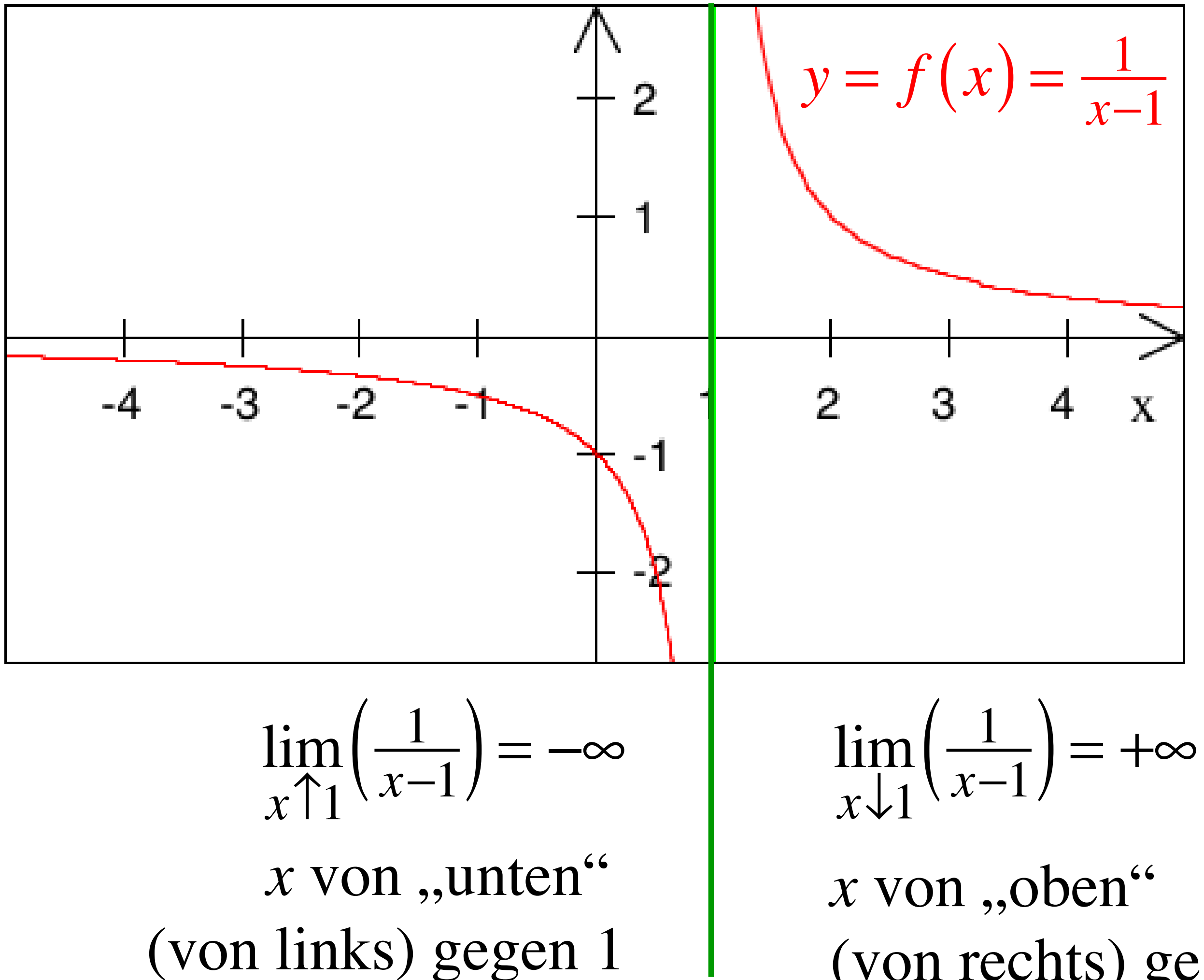
Grenzwerte bei Funktionen



$$\lim_{x \to -\infty} \left(\frac{1}{x^2} \right) = 0 \qquad \lim_{x \to 0} \left(\frac{1}{x^2} \right) = \infty \qquad \lim_{x \to \infty} \left(\frac{1}{x^2} \right) = 0$$

divergent

Grenzwerte bei Funktionen



$$\lim_{x \uparrow 1} \left(\frac{1}{x-1} \right) = -\infty$$

x von „unten“
(von links) gegen 1

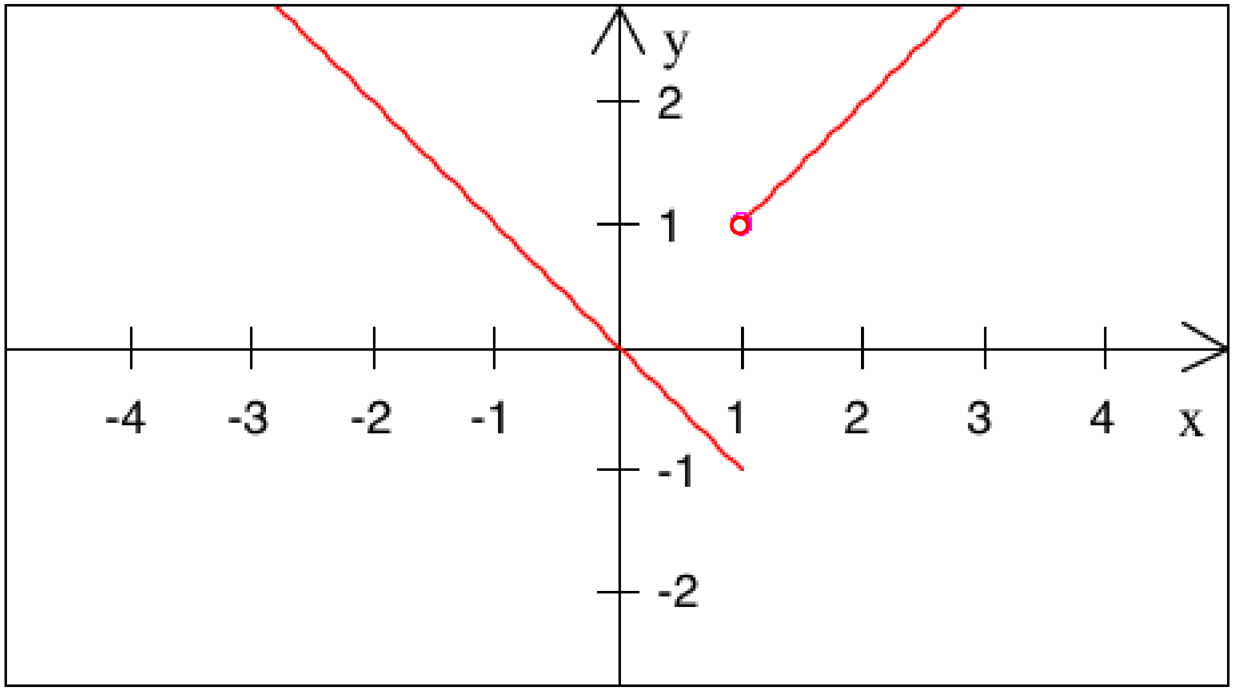
$$\lim_{x \downarrow 1} \left(\frac{1}{x-1} \right) = +\infty$$

x von „oben“
(von rechts) gegen 1

$$y = f(x) = \begin{cases} x & \text{für } x \geq 1 \\ -x & \text{für } x < 1 \end{cases}$$

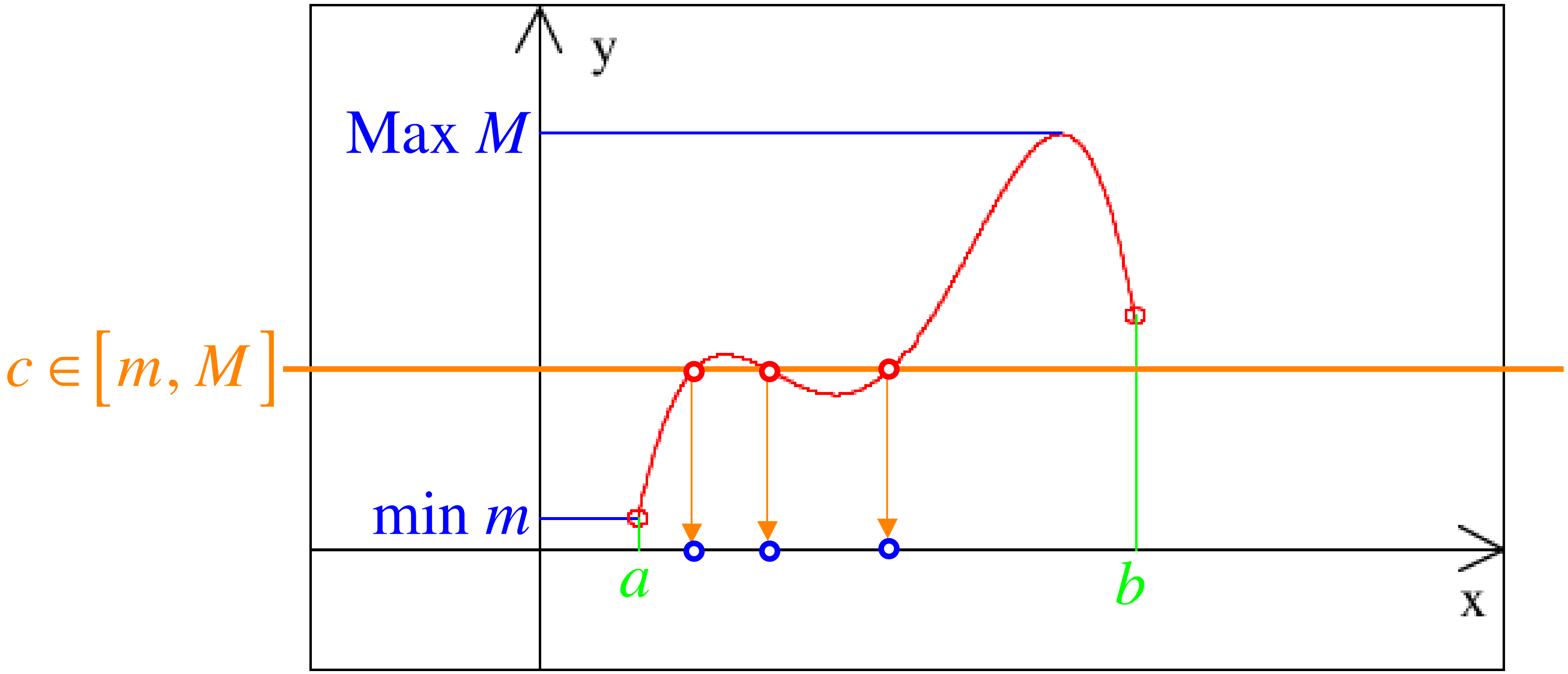
Sprungstellen bei Funktionen

Sprungstelle (Unstetigkeitsstelle) bei $x_0 = 1$



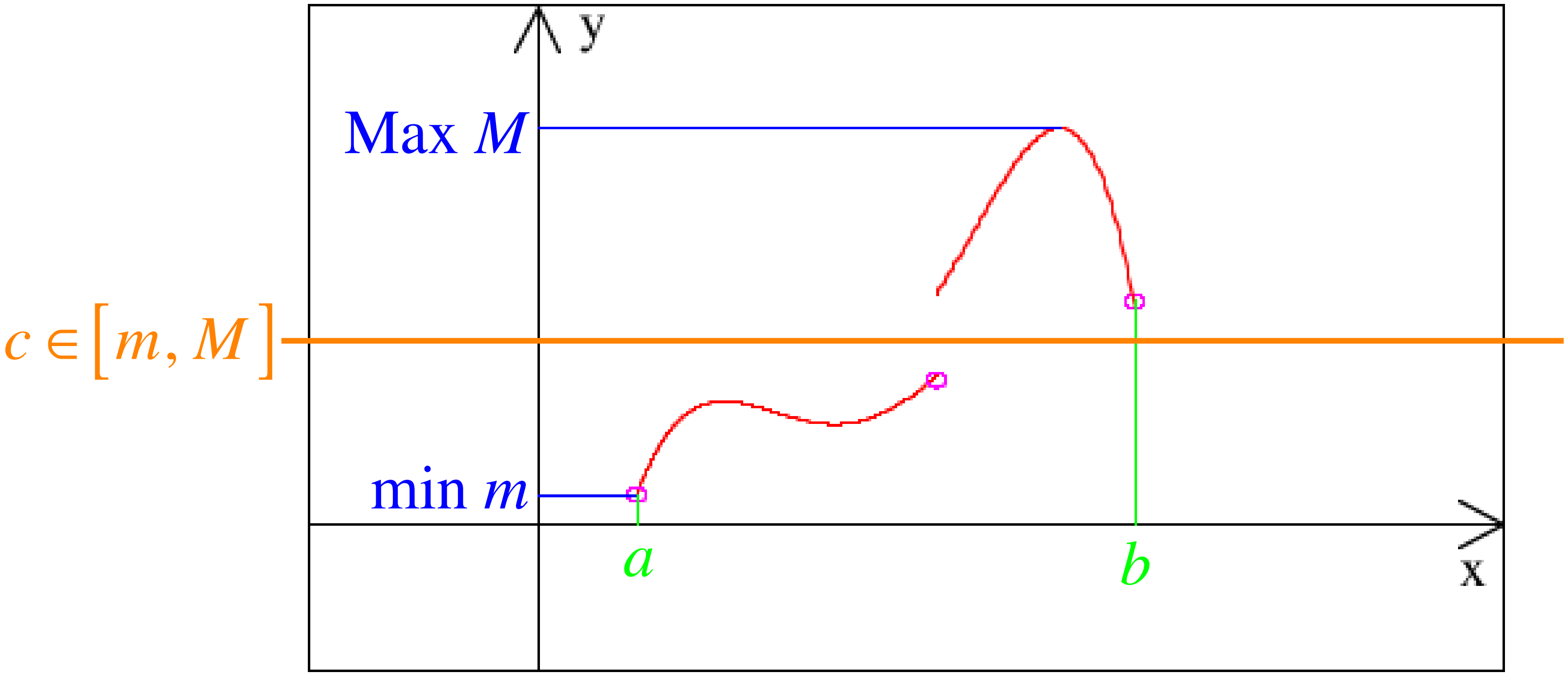
$$\lim_{x \uparrow 1} \left(f(x) \right) = -1 \qquad \lim_{x \downarrow 1} \left(f(x) \right) = +1$$

Hauptsatz über stetige Funktionen



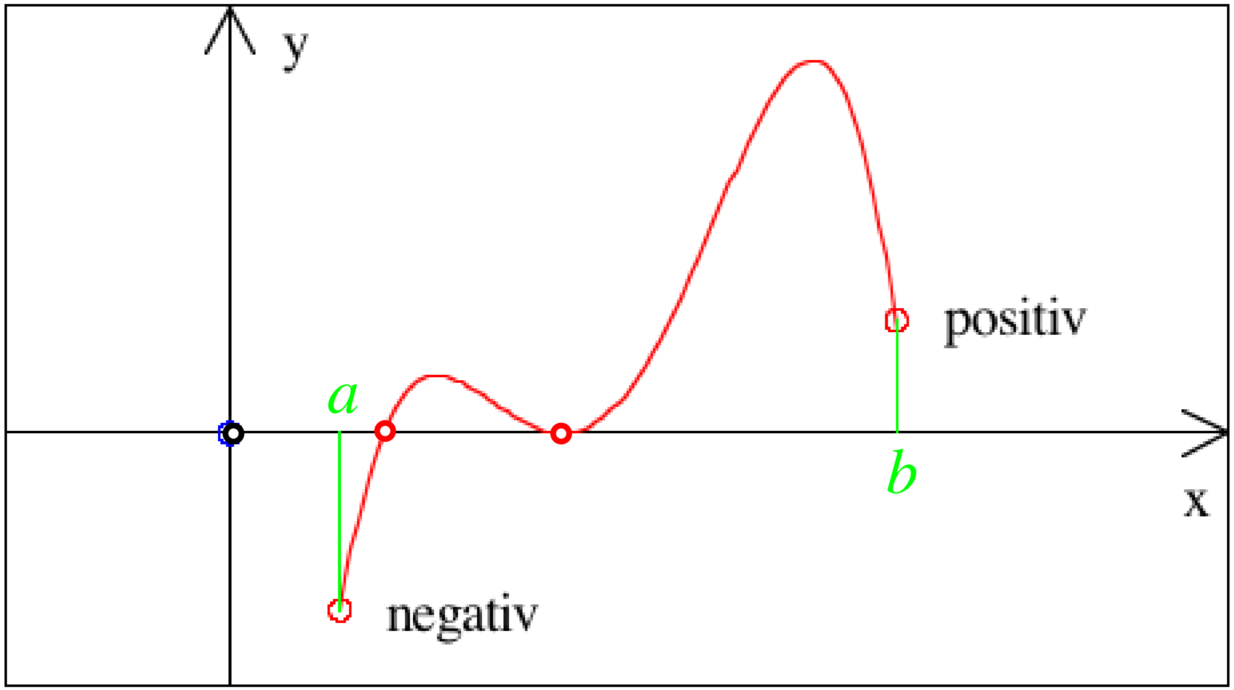
Mindestens ein passendes x mit $f(x) = c$

Gegenbeispiel: Unstetige Funktion



Kein passendes x mit $f(x) = c$

Nullstelle



Mindestens eine Nullstelle