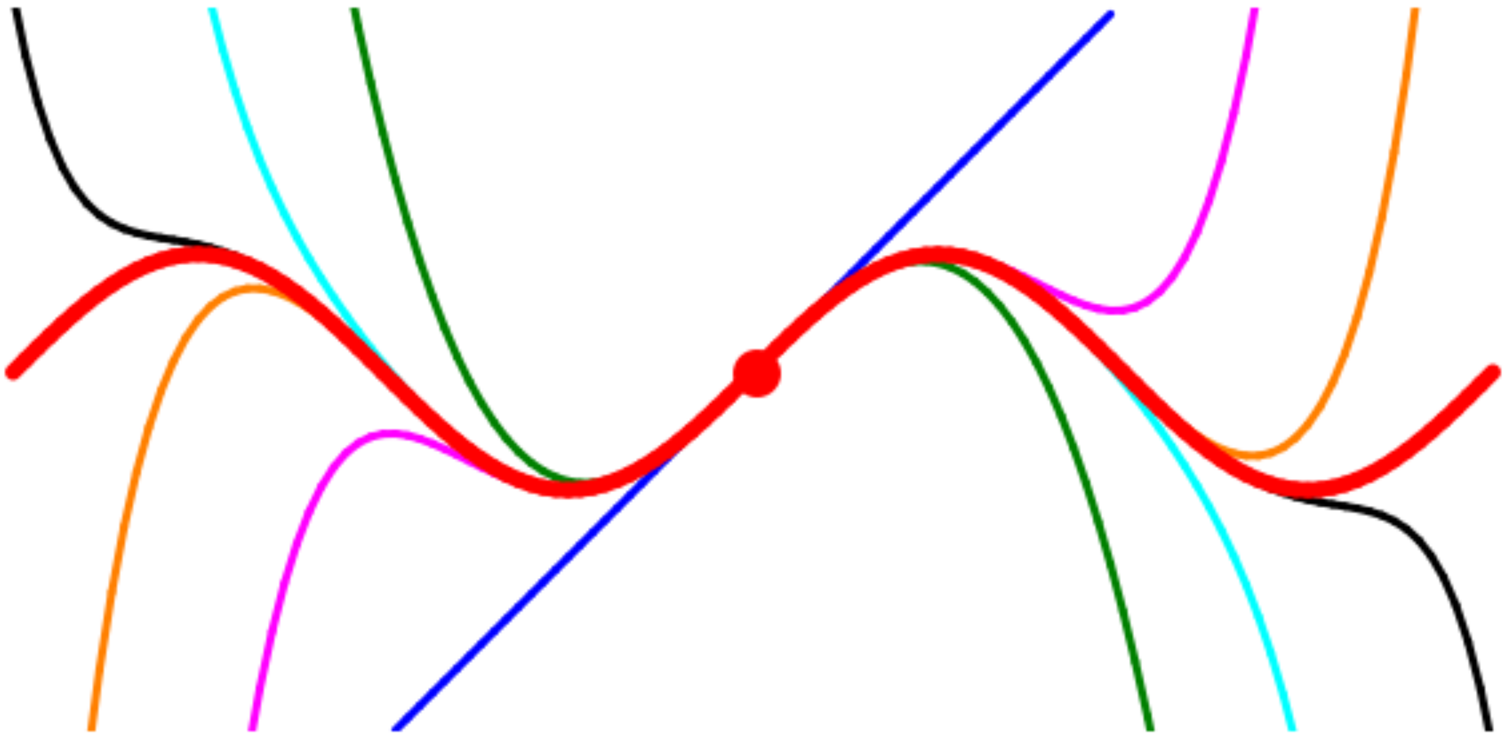
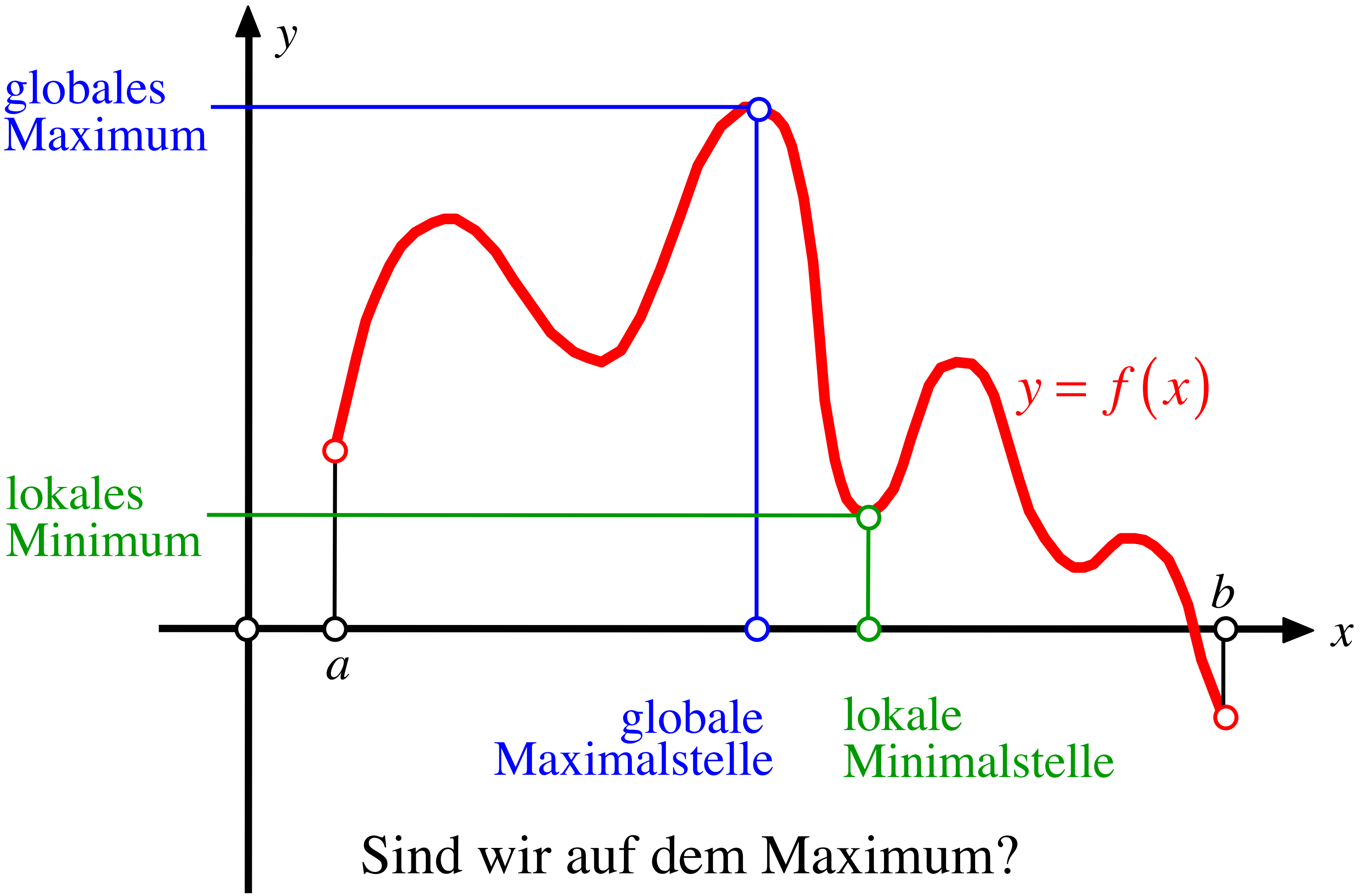
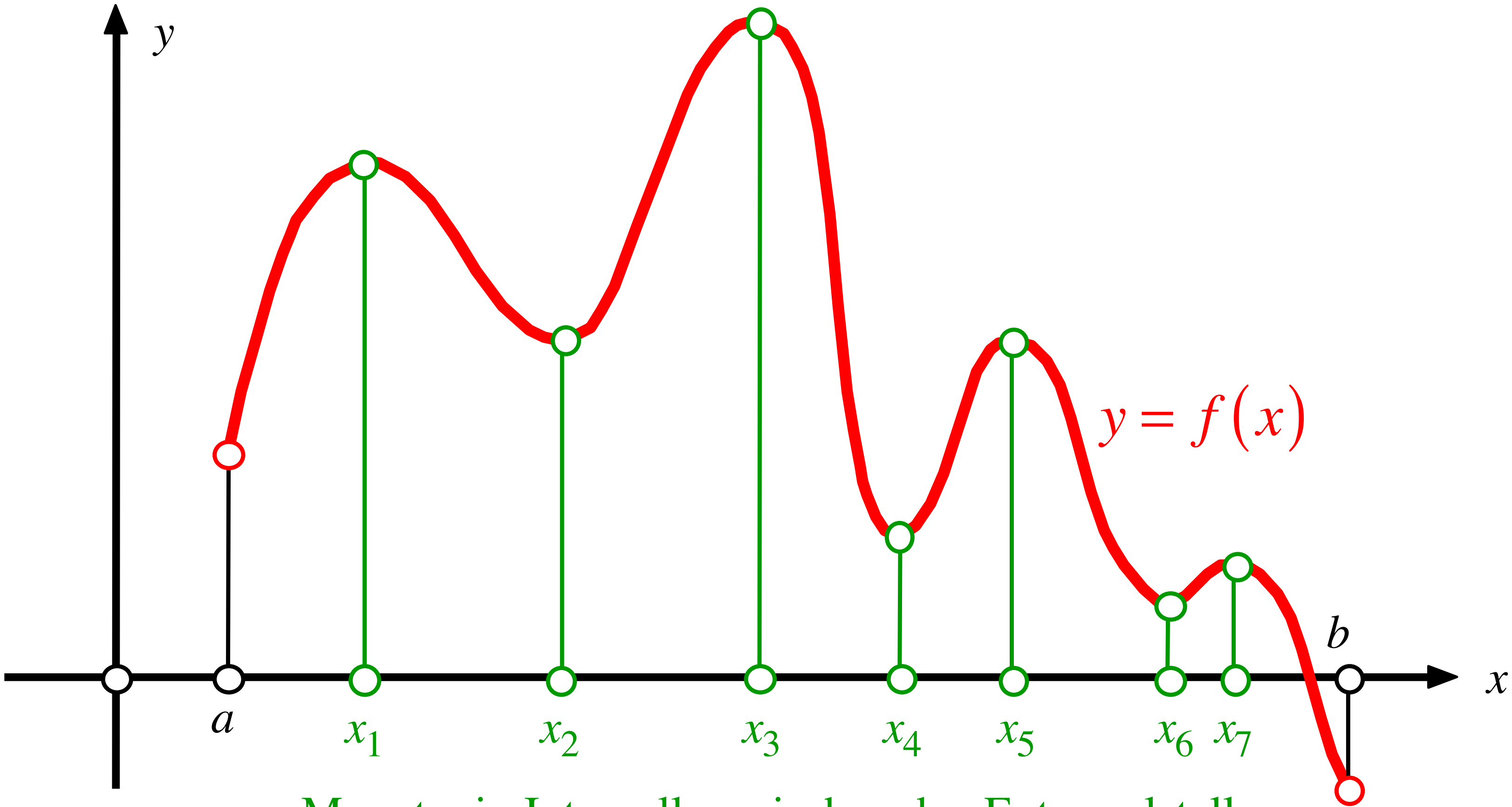


Modul 105 Taylor

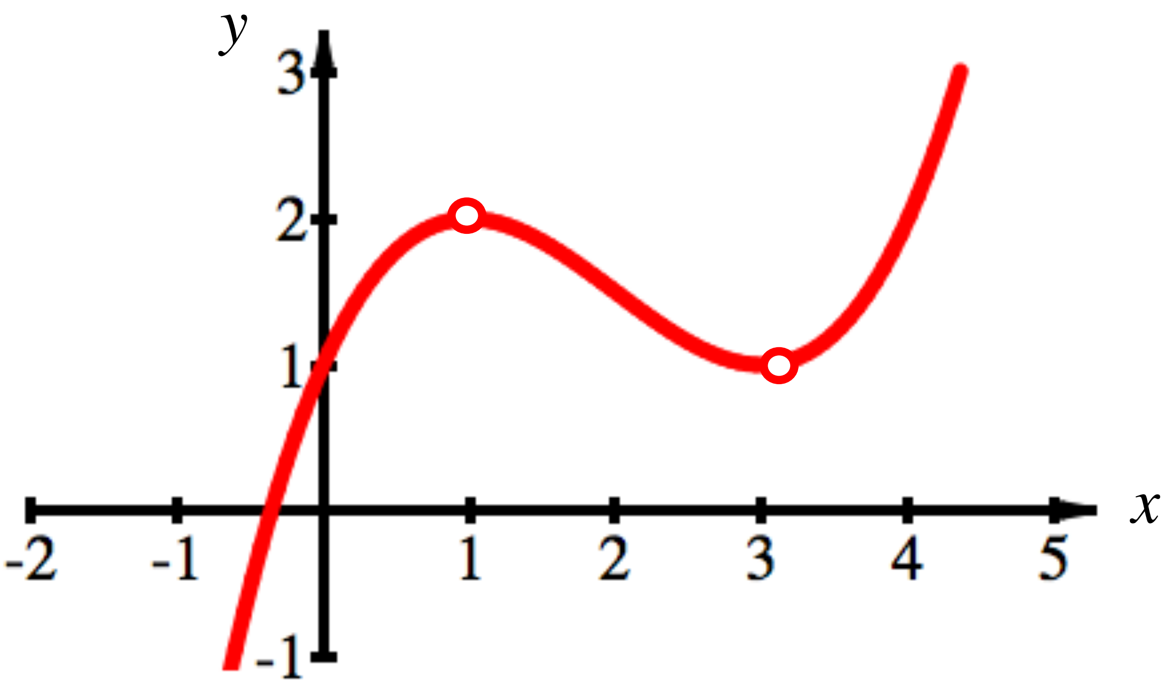


Sind wir auf dem Maximum?



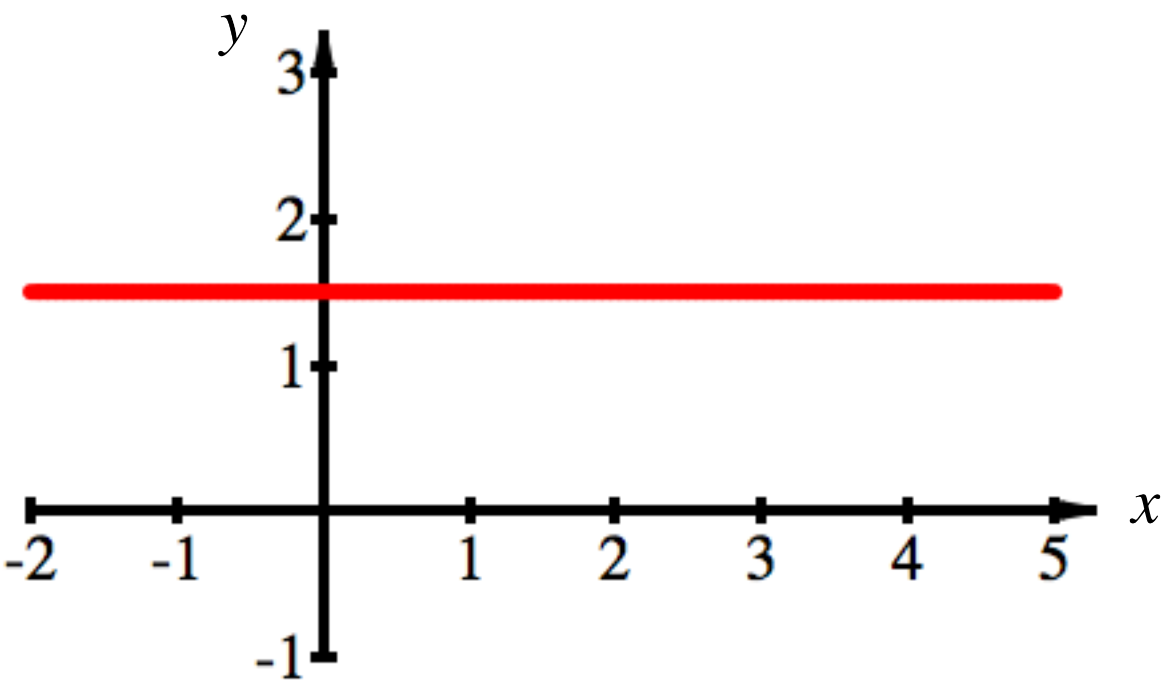
Monotonie-Intervalle zwischen den Extremalstellen

Stationäre Stellen: $f'(x_0) = 0$



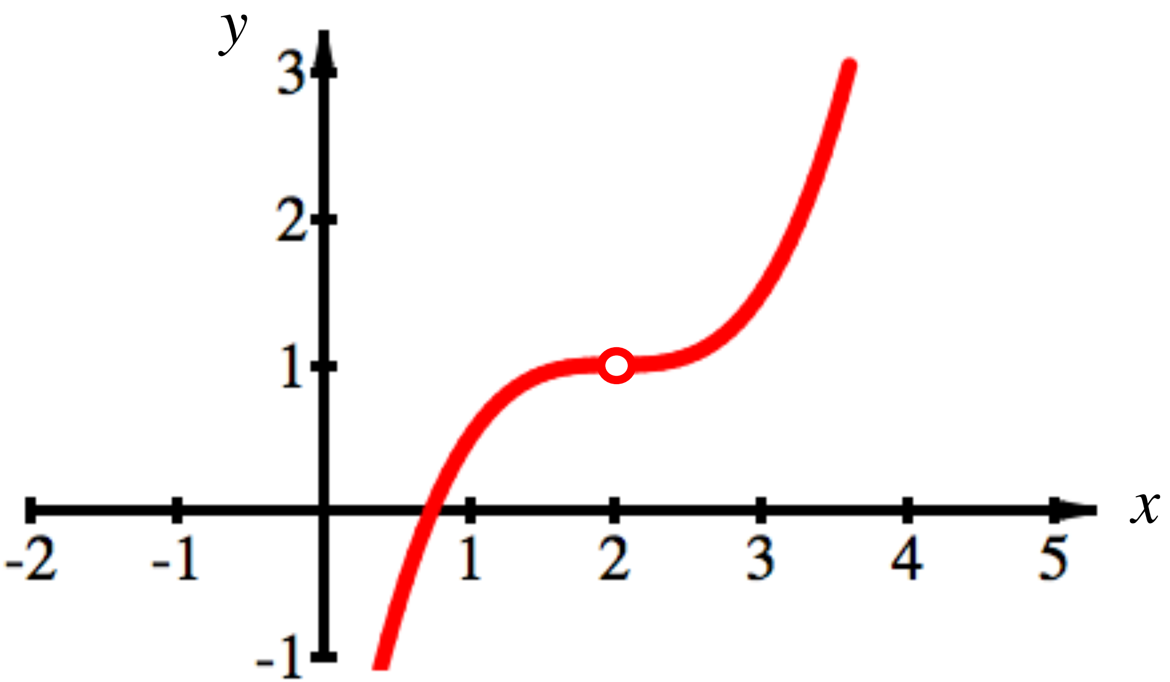
Extrema

Stationäre Stellen: $f'(x_0) = 0$



Konstante Funktion

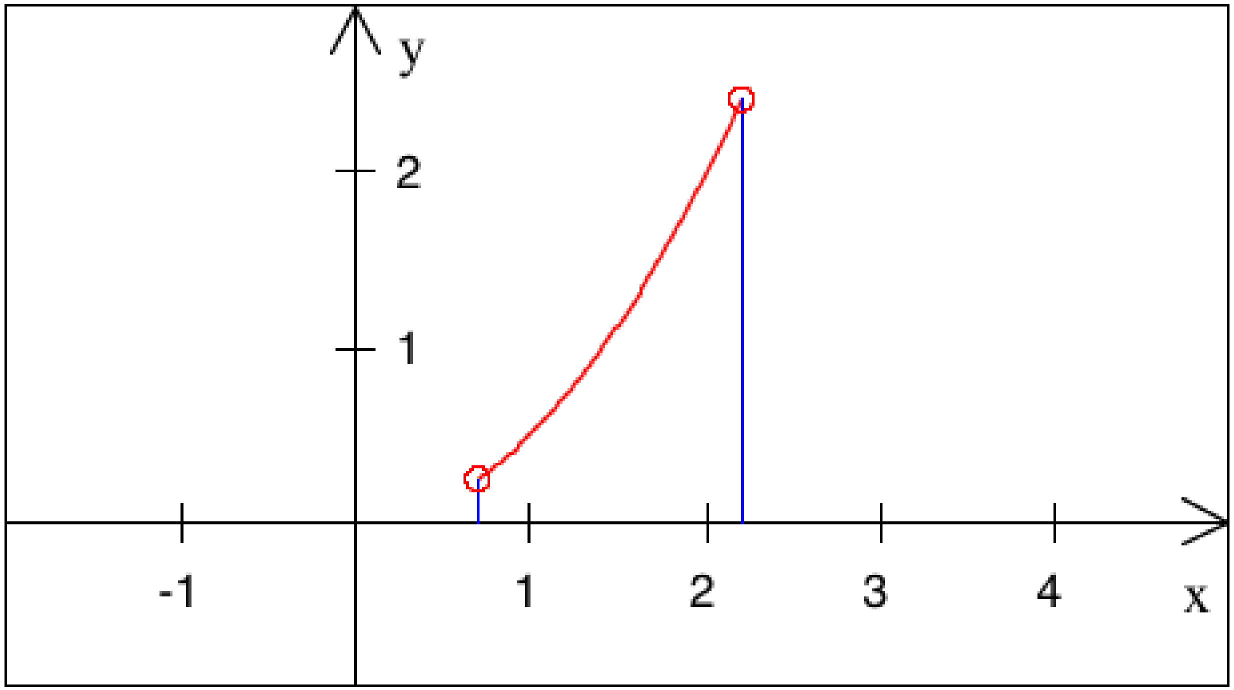
Stationäre Stellen: $f'(x_0) = 0$



Terrassenpunkt

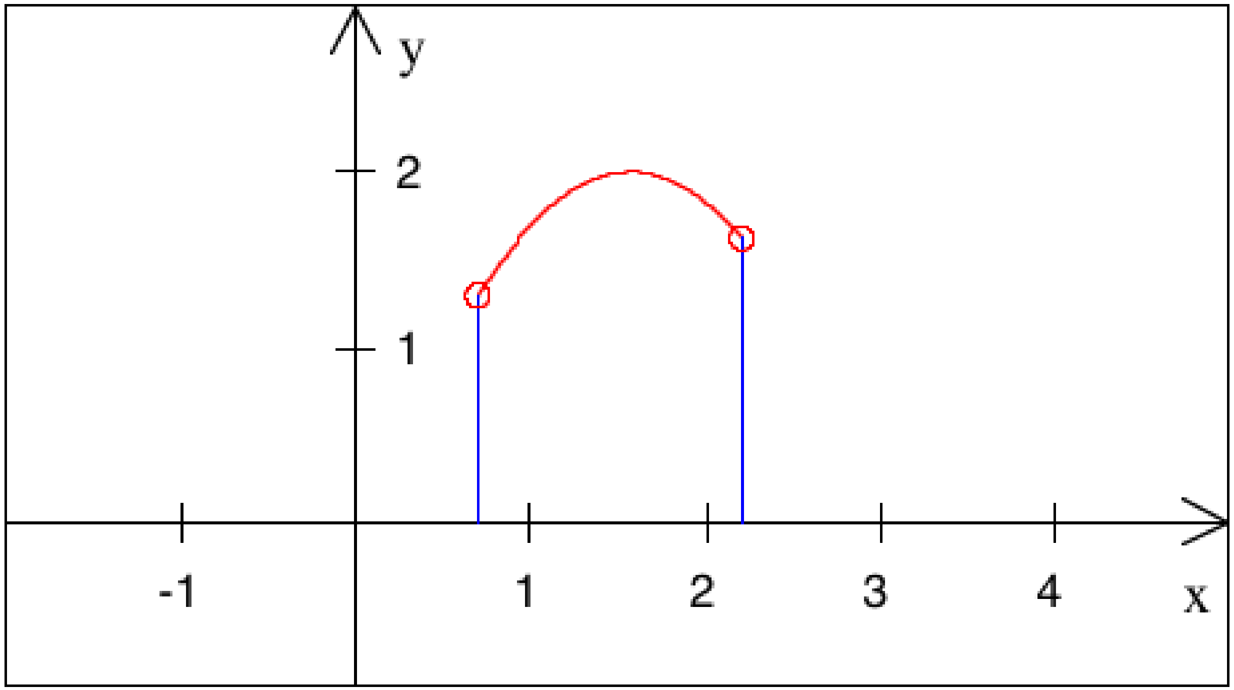
Auf der Suche nach Extremstellen:

Auf der Suche nach Extremstellen:



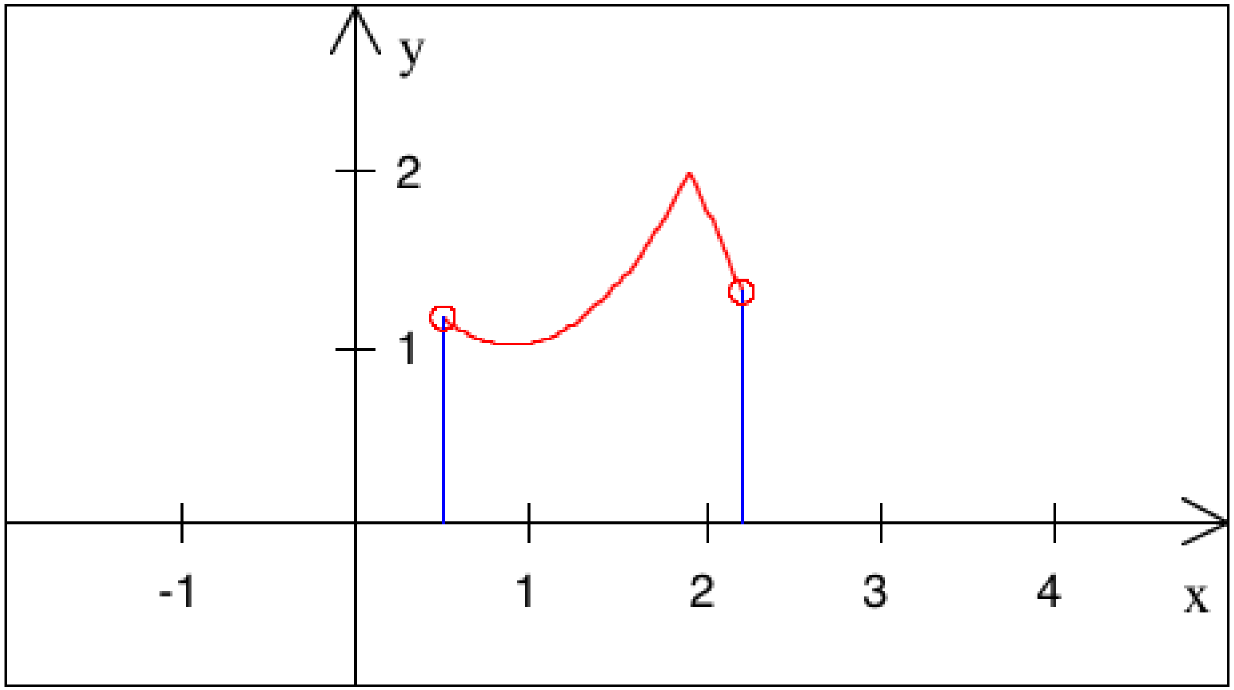
Randextrema

Auf der Suche nach Extremstellen:



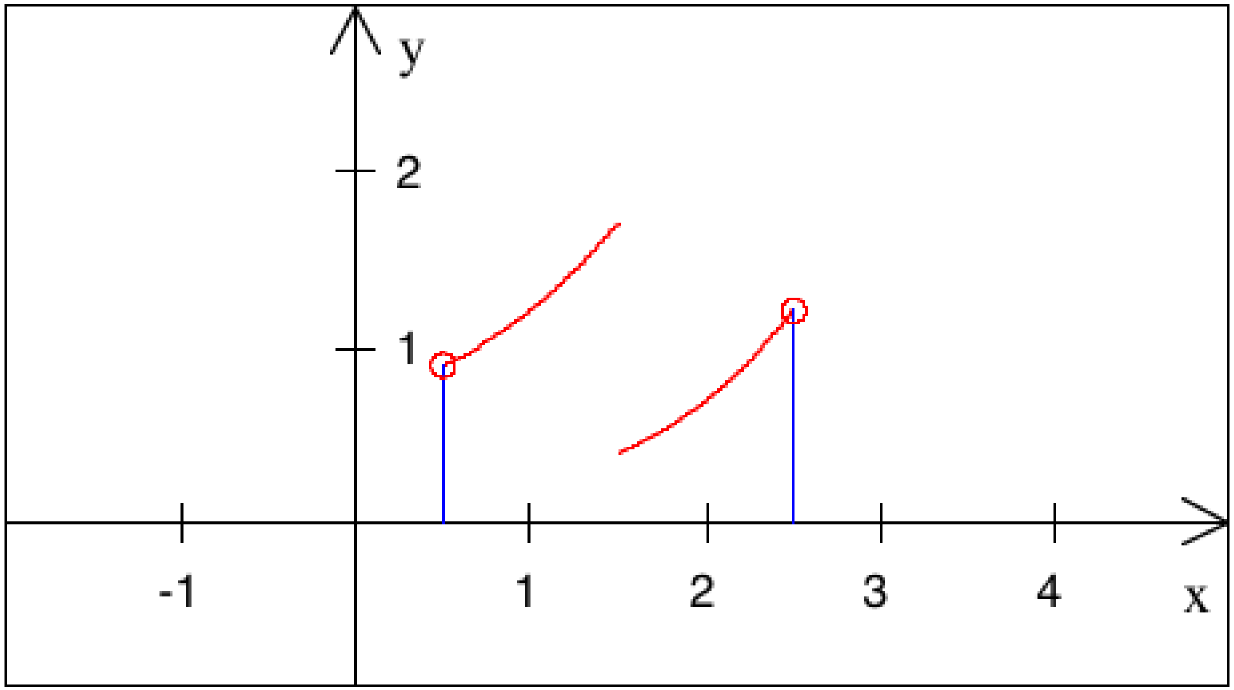
(Relative) Randextrema

Auf der Suche nach Extremstellen:



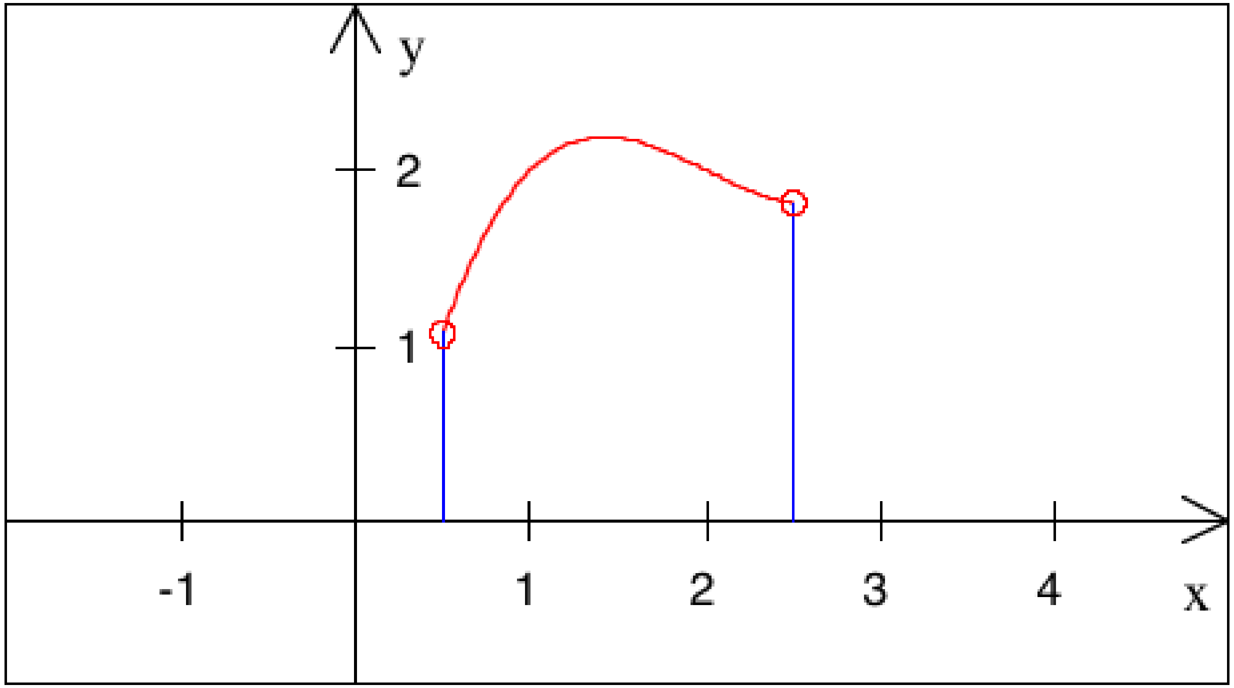
Spitze, nicht differenzierbar

Auf der Suche nach Extremstellen:



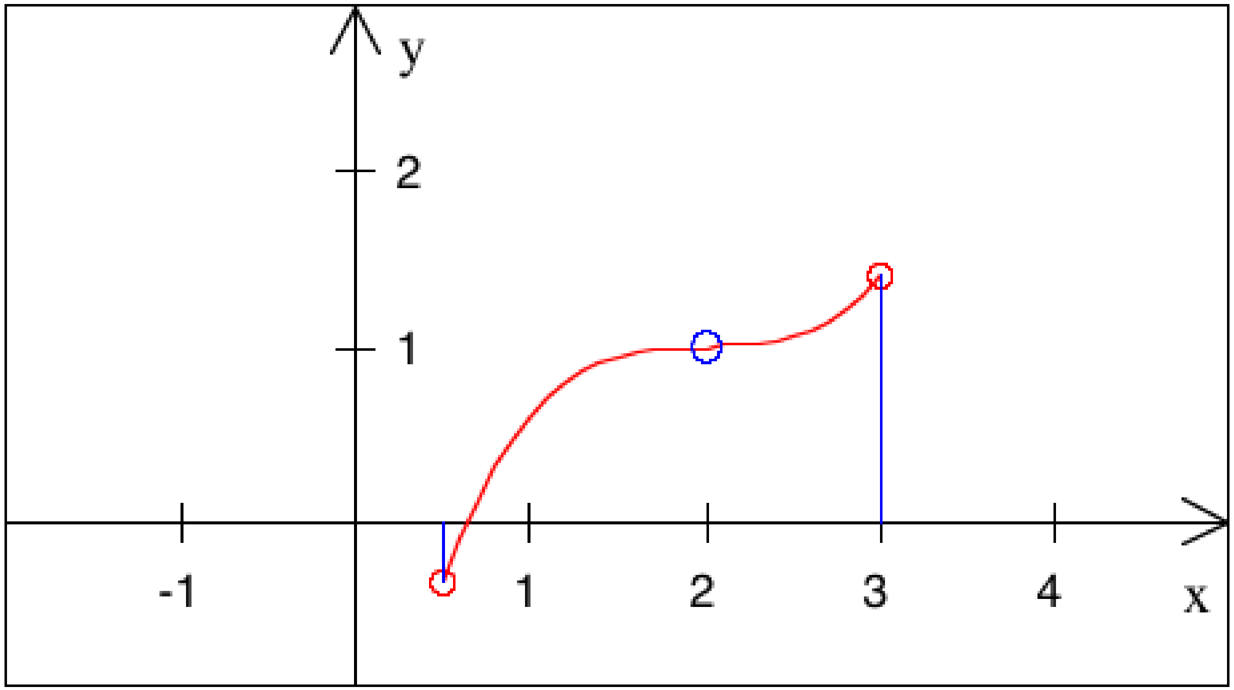
Sprungstelle, nicht differenzierbar

Auf der Suche nach Extremstellen:



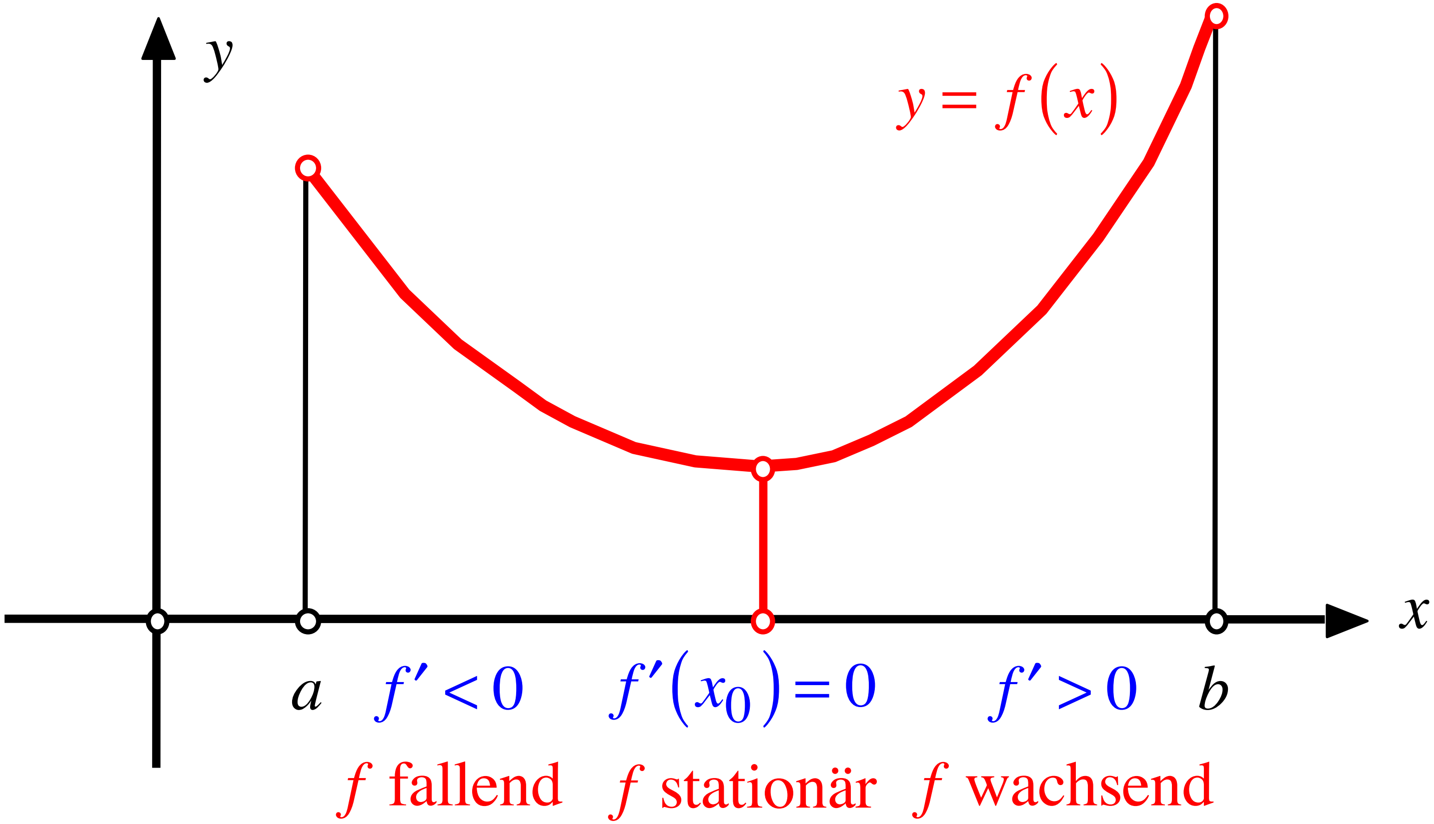
Klassischer Fall: Stationäre Stelle

Auf der Suche nach Extremstellen:

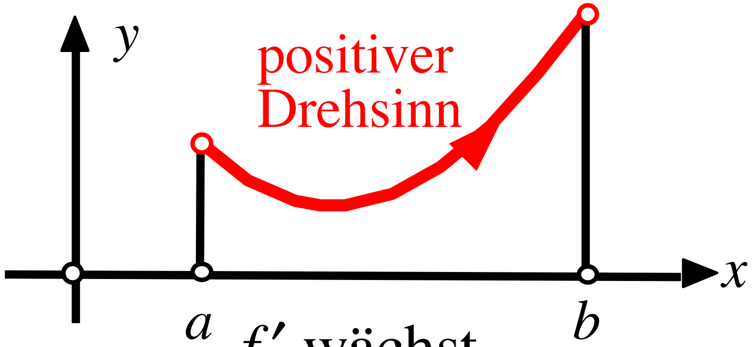


Vorsicht: Stationäre Stelle als Terrassenpunkt

Sichere Extremalstelle:
Stationäre Stelle mit
Vorzeichenwechsel der Ableitung

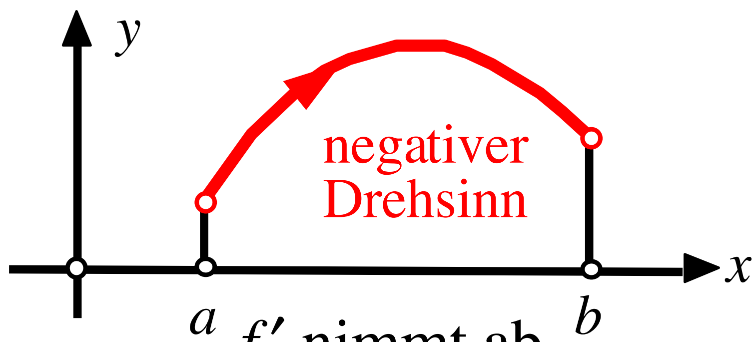


Zweite Ableitung



f' wächst,

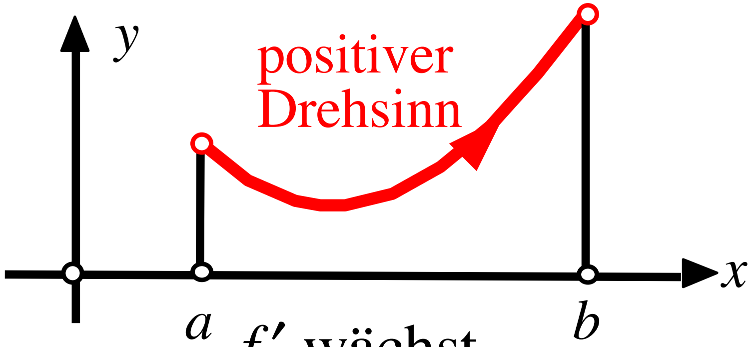
daher ist $f'' > 0$



f' nimmt ab,

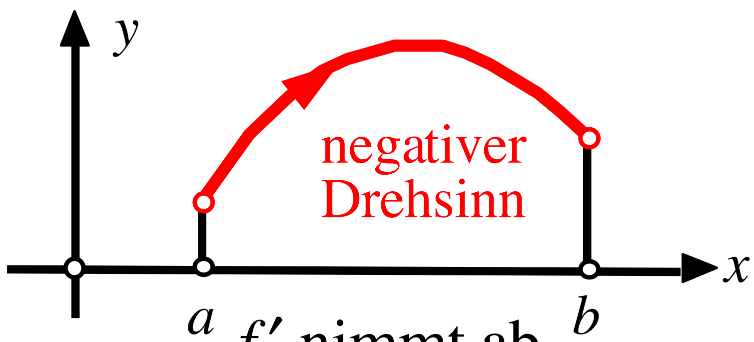
daher ist $f'' < 0$

Zweite Ableitung



f' wächst,

daher ist $f'' > 0$



f' nimmt ab,

daher ist $f'' < 0$

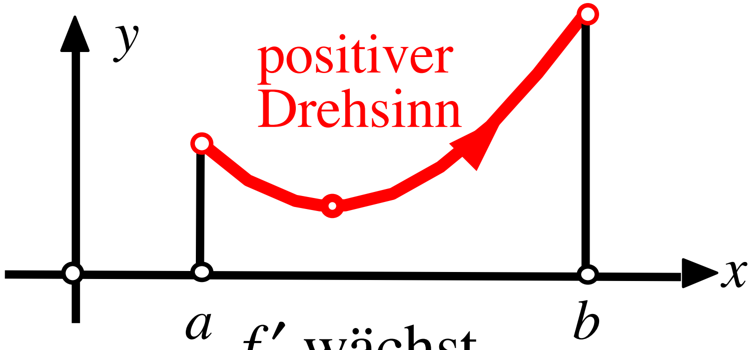
Zweite Ableitung

f' wächst,

daher ist $f'' > 0$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

Zweite Ableitung



f' wächst,

daher ist $f'' > 0$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

Umkehrung gilt nicht!

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

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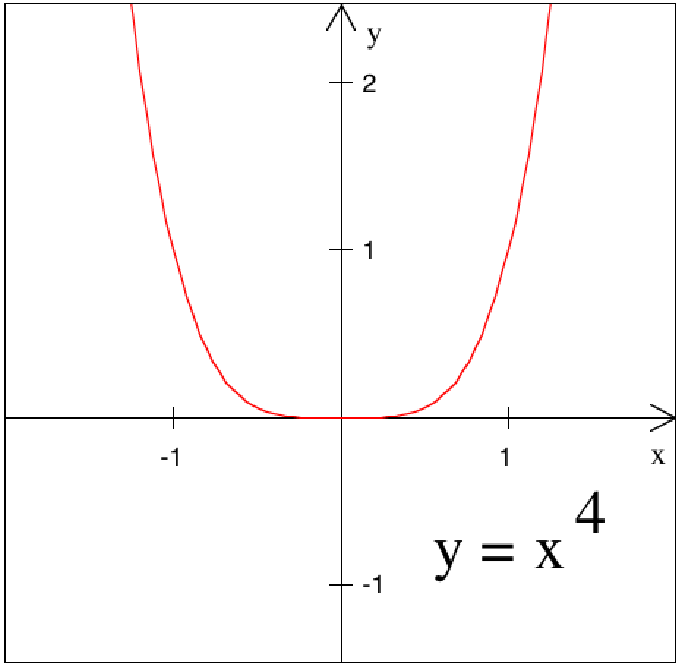
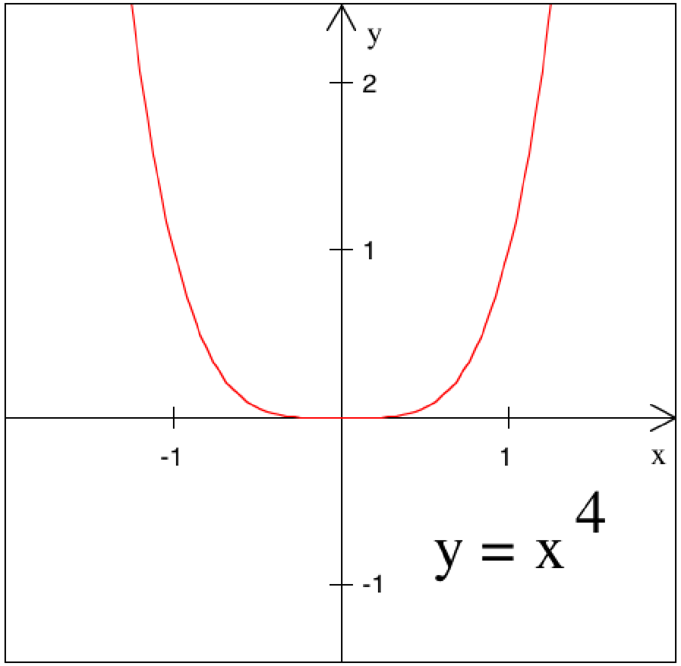
Umkehrung gilt nicht!



$$f(x) = x^4$$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

Umkehrung gilt nicht!

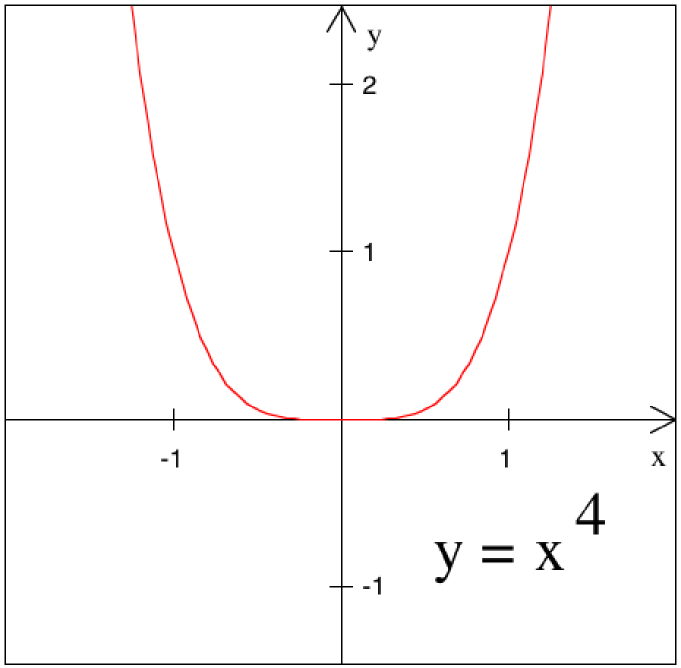


$$f(x) = x^4$$

$$f'(x) = 4x^3 \quad \Rightarrow \quad f'(0) = 0$$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

Umkehrung gilt nicht!



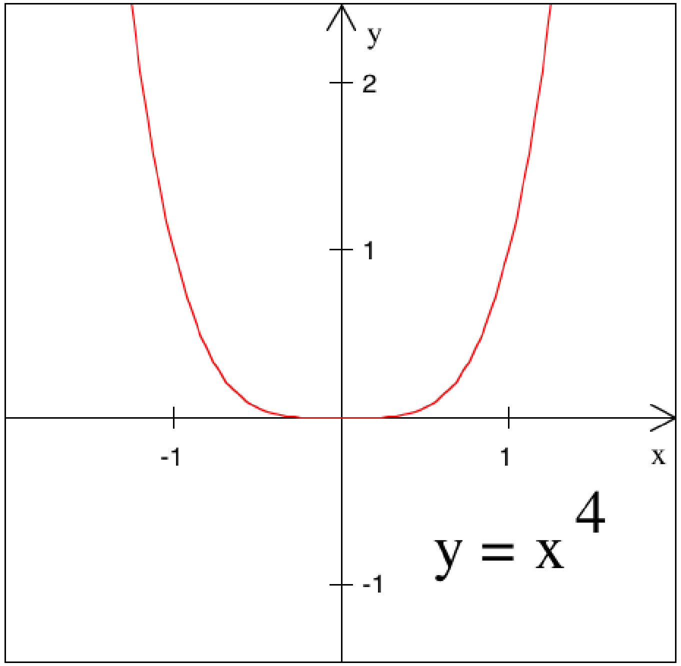
$$f(x) = x^4$$

$$f'(x) = 4x^3 \quad \Rightarrow \quad f'(0) = 0$$

$$f''(x) = 12x^2 \quad \Rightarrow \quad f''(0) = 0$$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

Umkehrung gilt nicht!



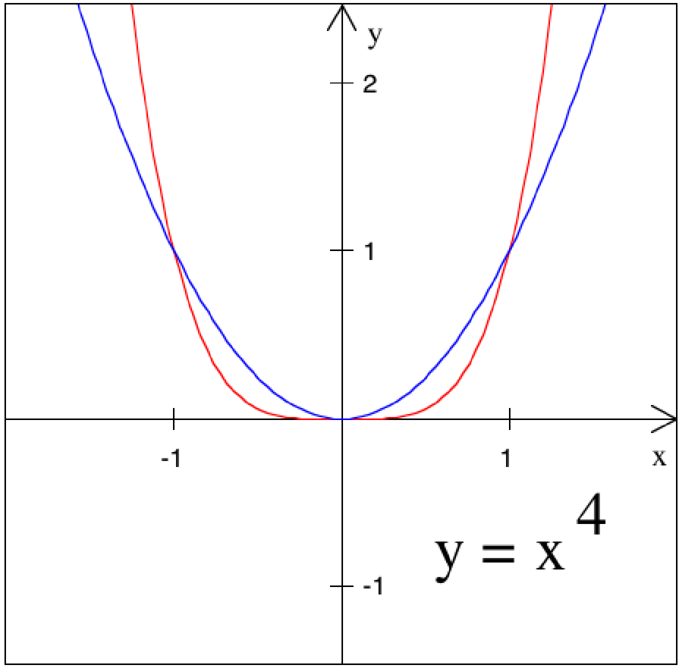
$$f(x) = x^4$$

$$f'(x) = 4x^3 \quad \Rightarrow \quad f'(0) = 0$$

$$f''(x) = 12x^2 \quad \Rightarrow \quad f''(0) = 0$$

Minimum bei $x = 0$, aber $f''(0) = 0$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

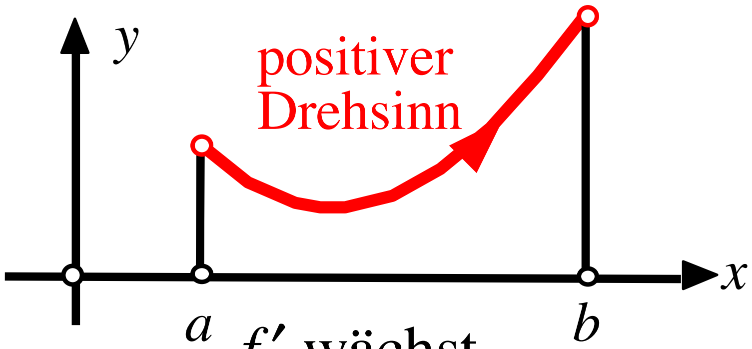


$f(x) = x^4$

Vergleich mit Standardparabel

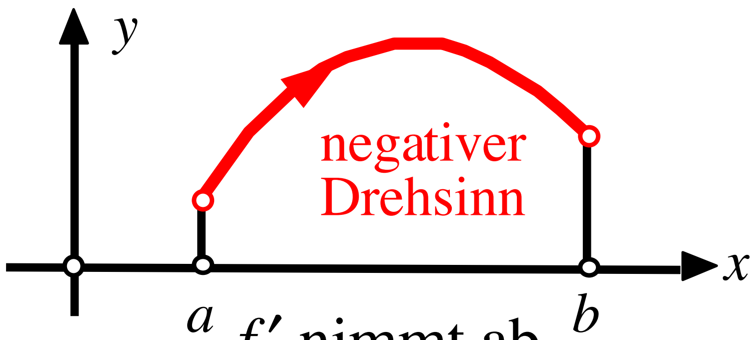
$y = x^2$

Zweite Ableitung



f' wächst,

daher ist $f'' > 0$

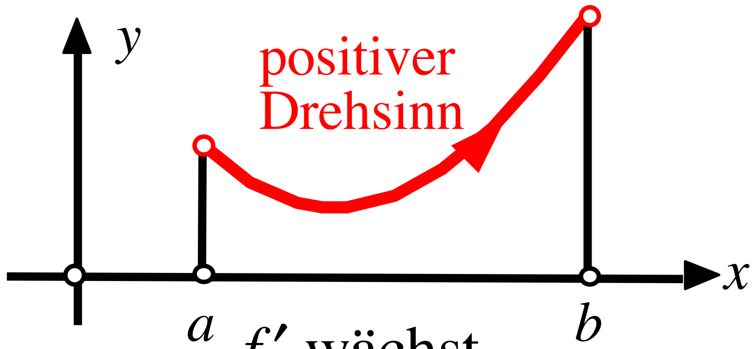


f' nimmt ab,

daher ist $f'' < 0$

$$\left.\begin{array}{c} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

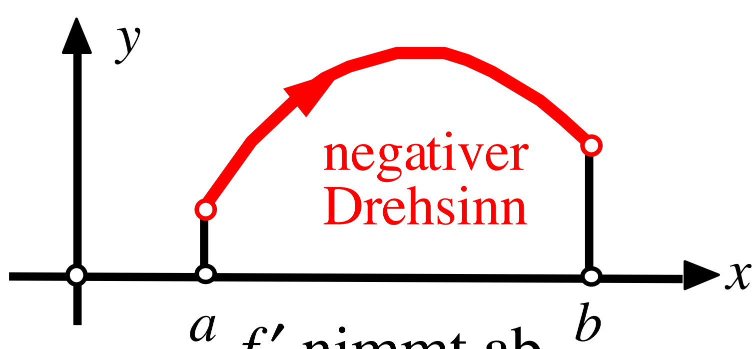
Zweite Ableitung



f' wächst,

daher ist $f'' > 0$

$$\left.\begin{matrix} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) > 0 \end{matrix}\right\} \Rightarrow \text{Minimum}$$



f' nimmt ab,

daher ist $f'' < 0$

$$\left.\begin{matrix} f'(x_0) = 0 \\ \text{UND} \\ f''(x_0) < 0 \end{matrix}\right\} \Rightarrow \text{Maximum}$$

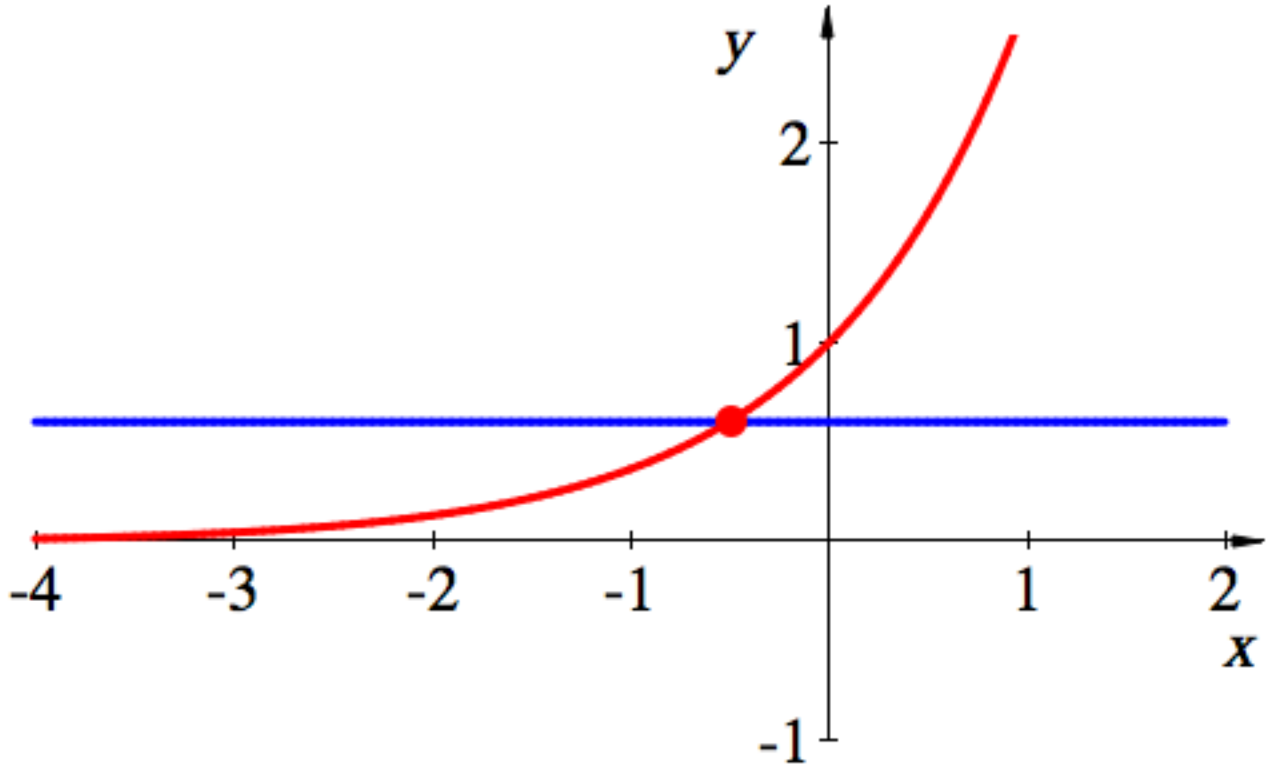
Taylorpolynome und Taylorreihen

Brook Taylor
1685 - 1731

Idee:
Lineare Approximation
verallgemeinern

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

Entwicklung an der Stelle $x_0 = -\frac{1}{2}$

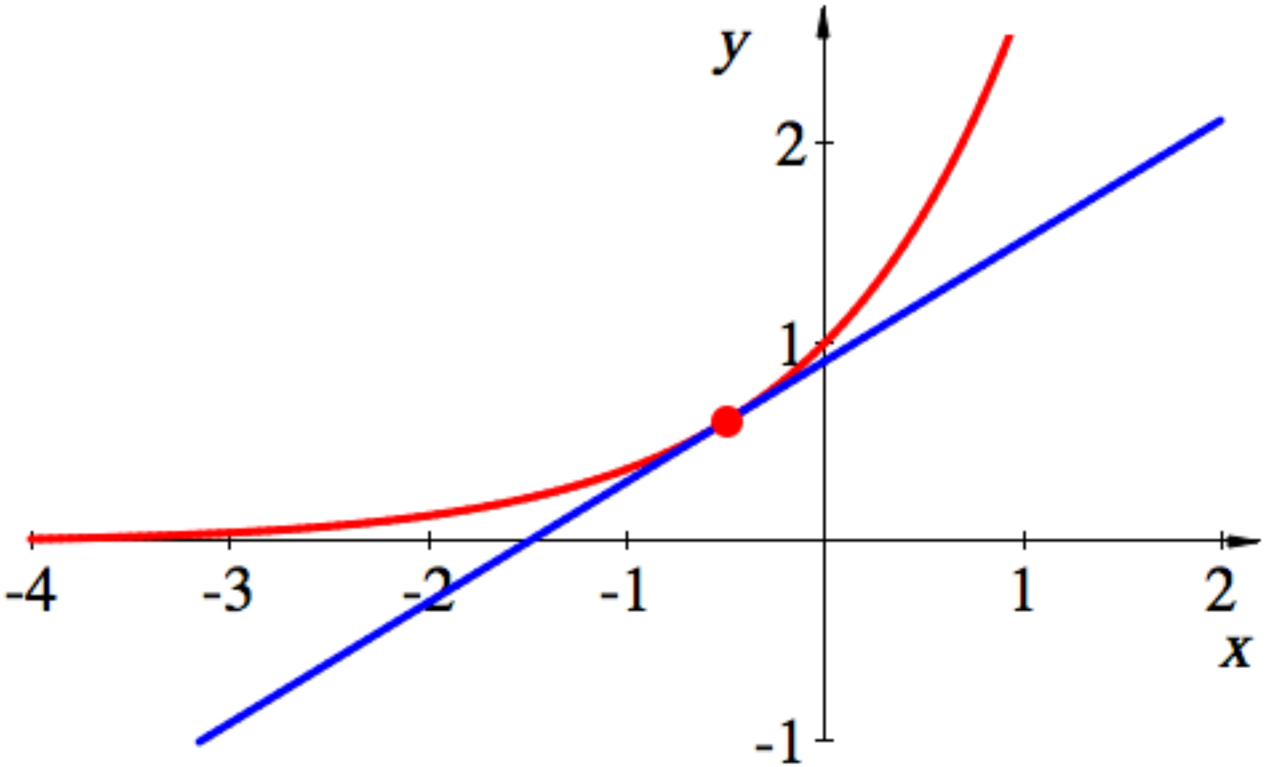


Approximation durch Konstante

One size fits all.

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

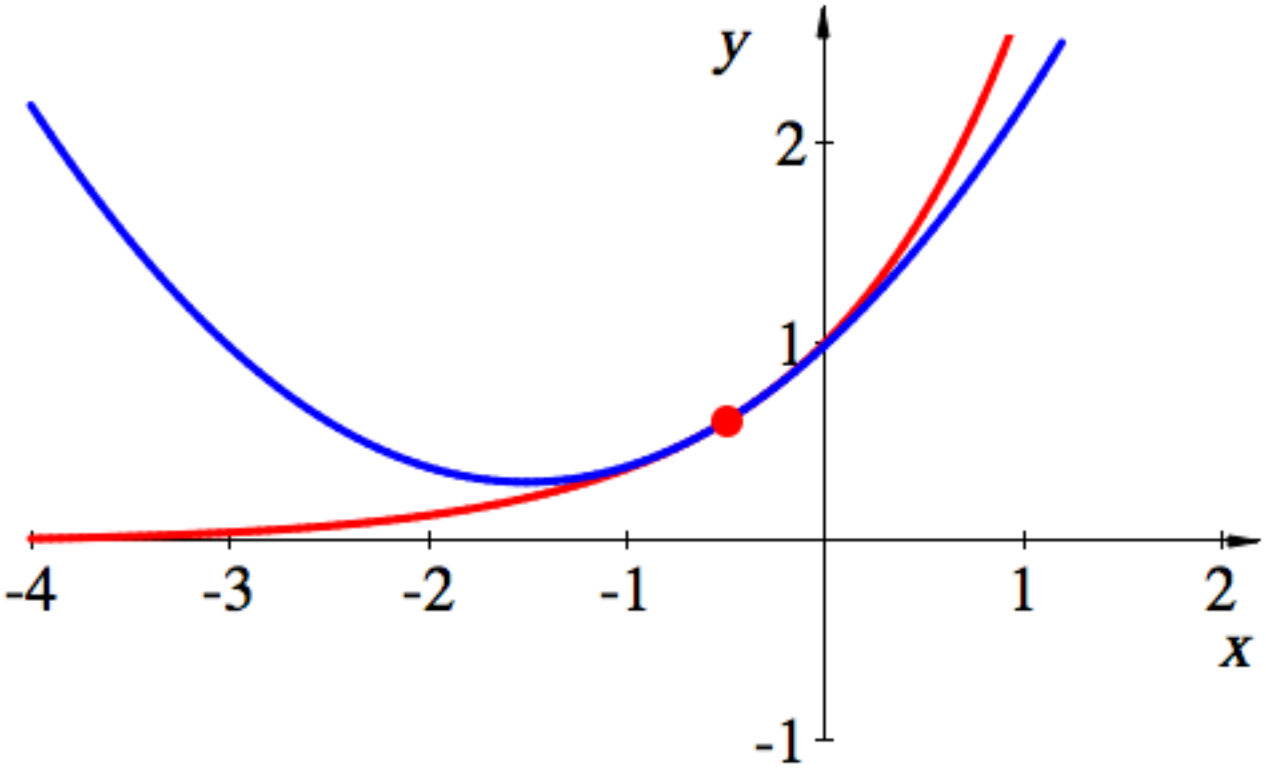
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Lineare Approximation (Tangente)

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

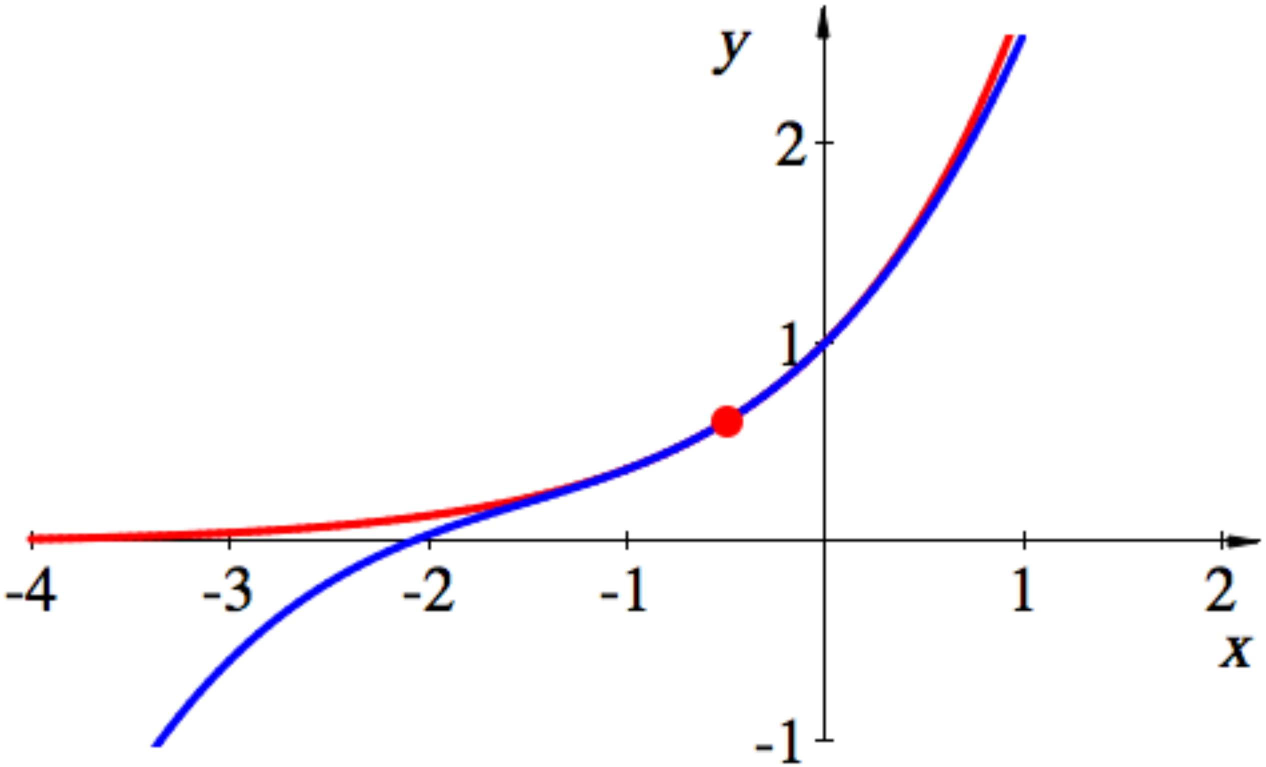
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Quadratische Approximation (tangentiale Parabel)

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

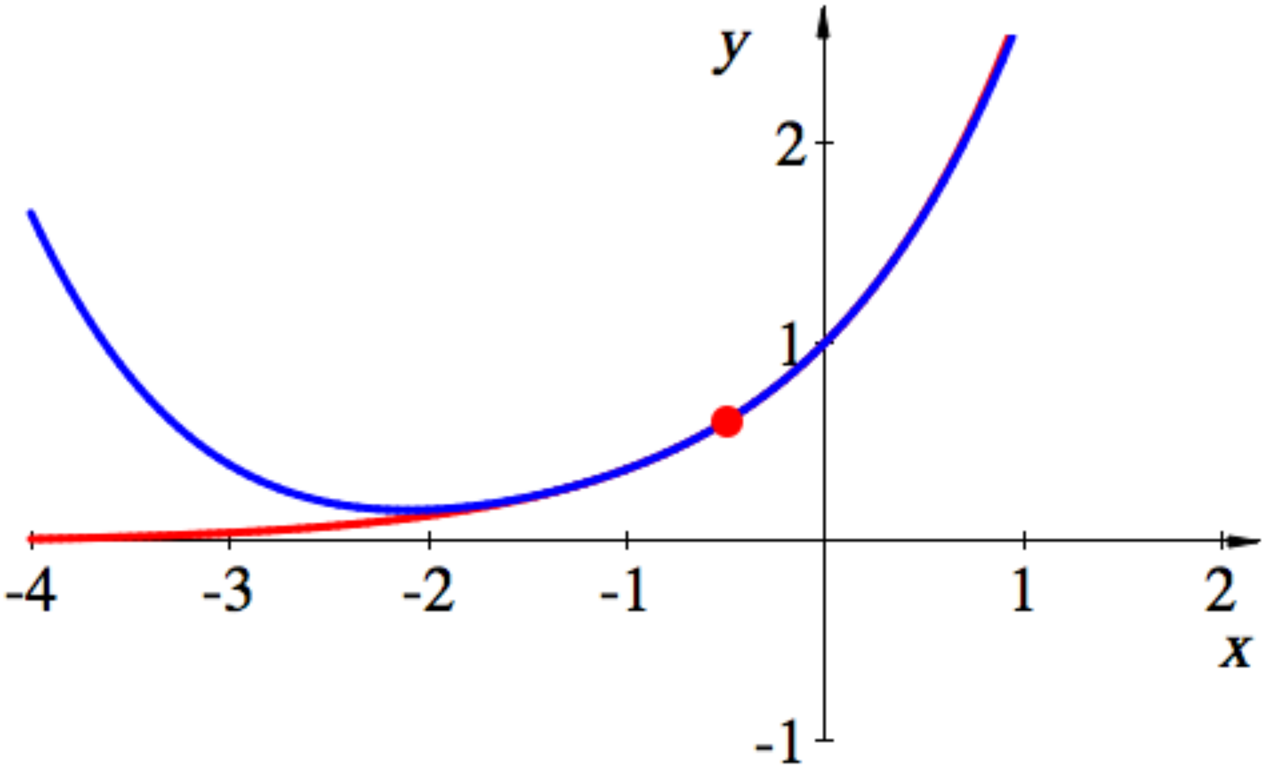
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation dritten Grades

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

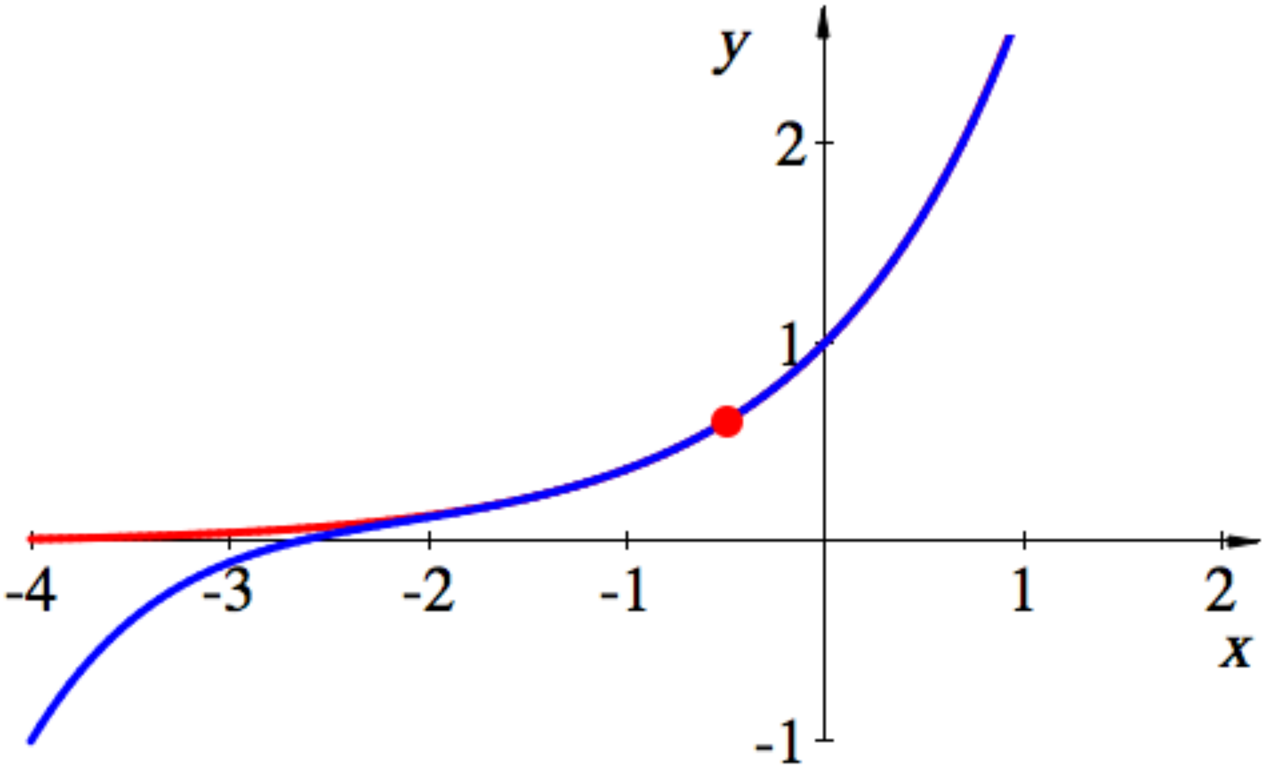
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation vierten Grades

Beispiel: Exponenzialfunktion $y = f(x) = e^x$

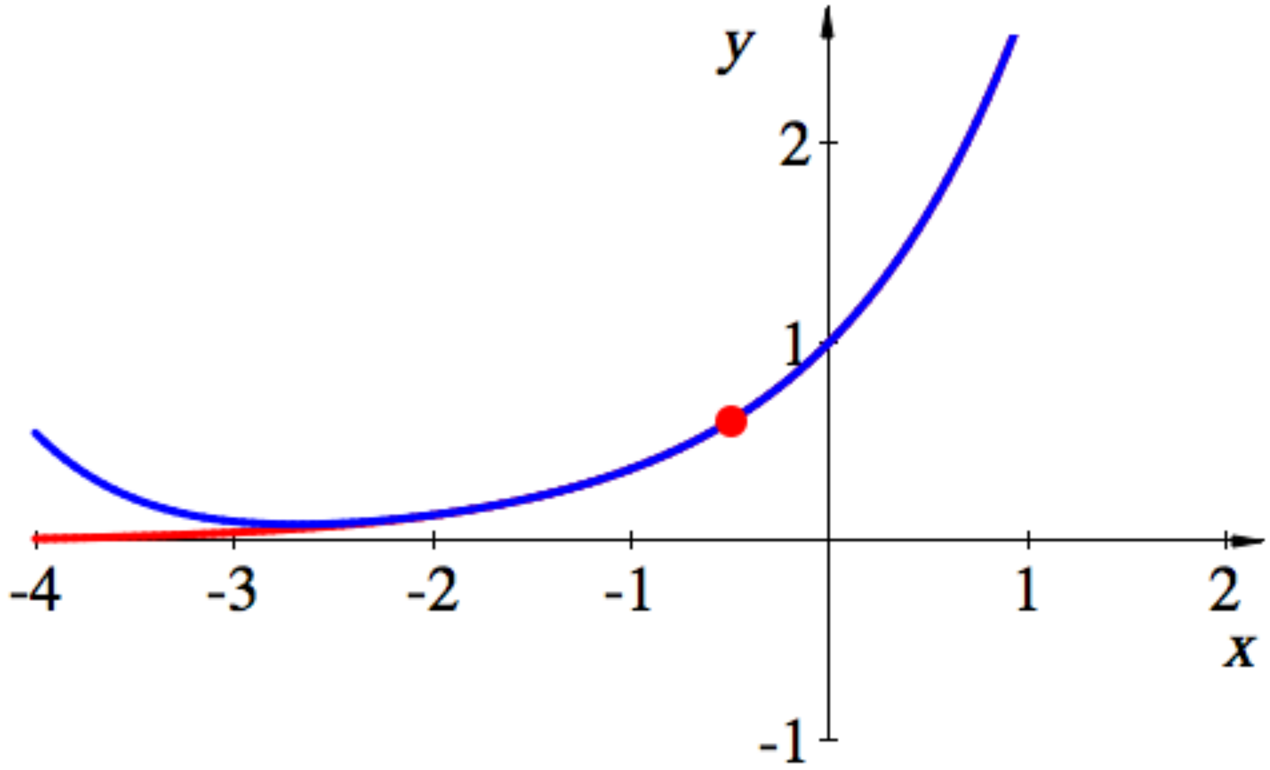
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation fünften Grades

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

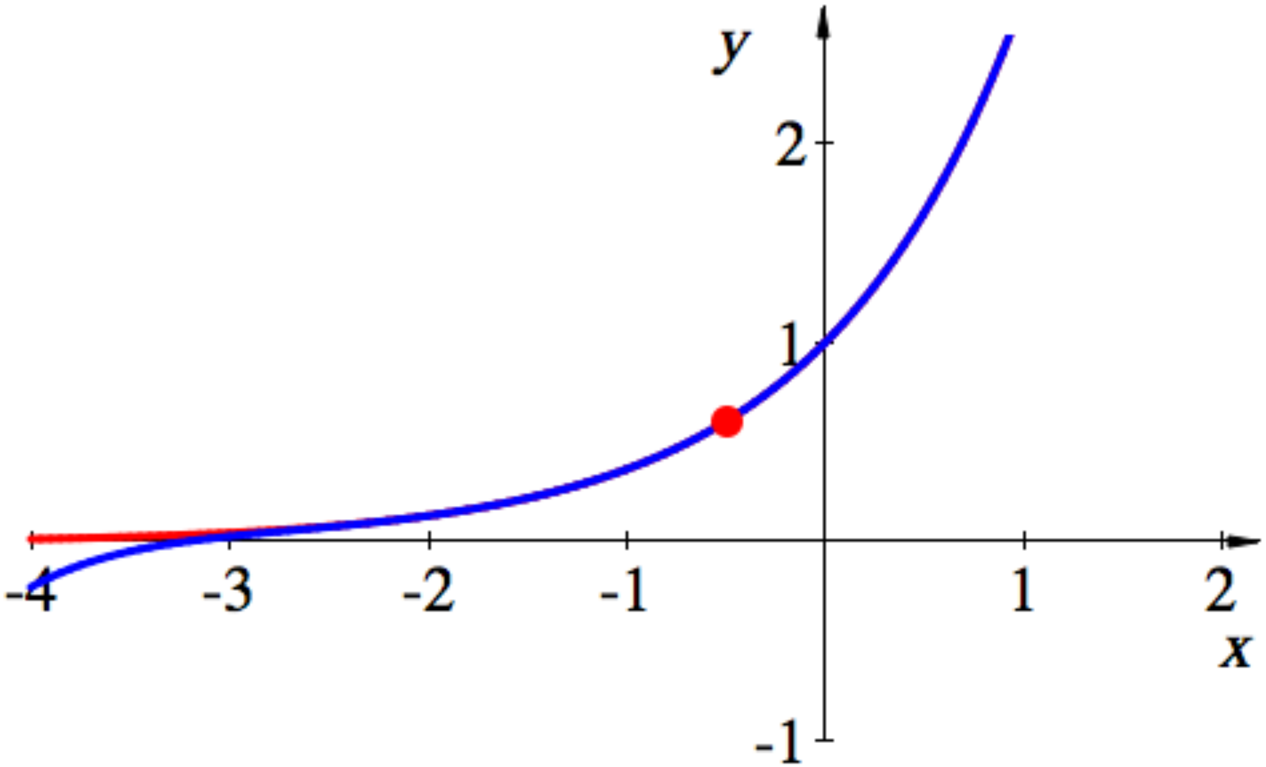
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation sechsten Grades

Beispiel: Exponenzialfunktion $y = f(x) = e^x$

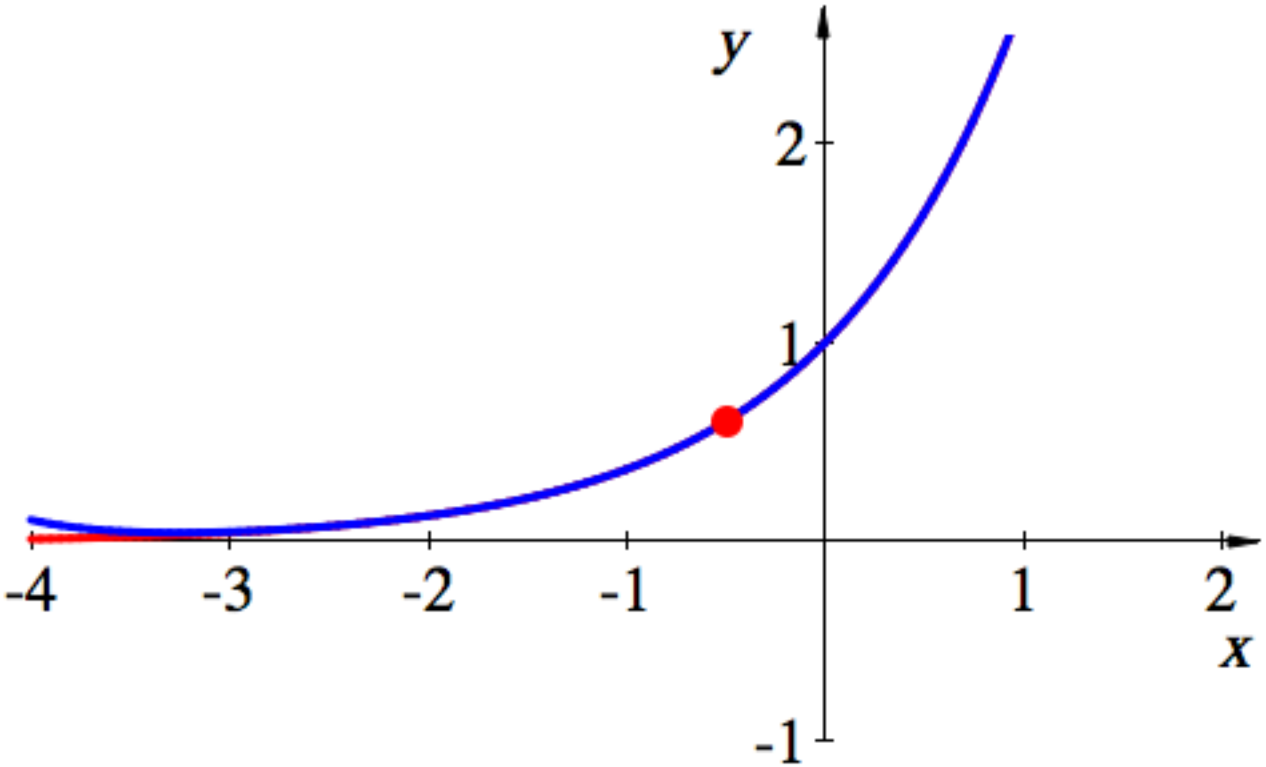
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation siebten Grades

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

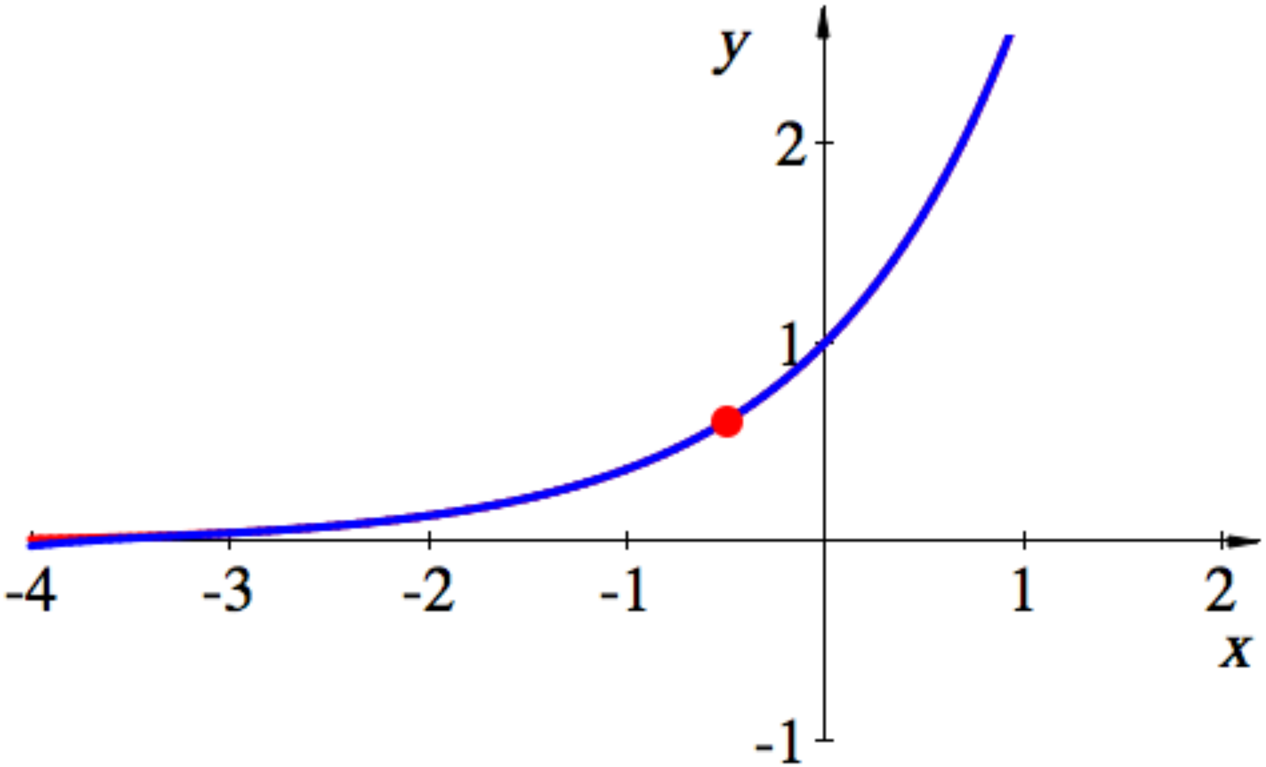
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation achten Grades

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^{x}$

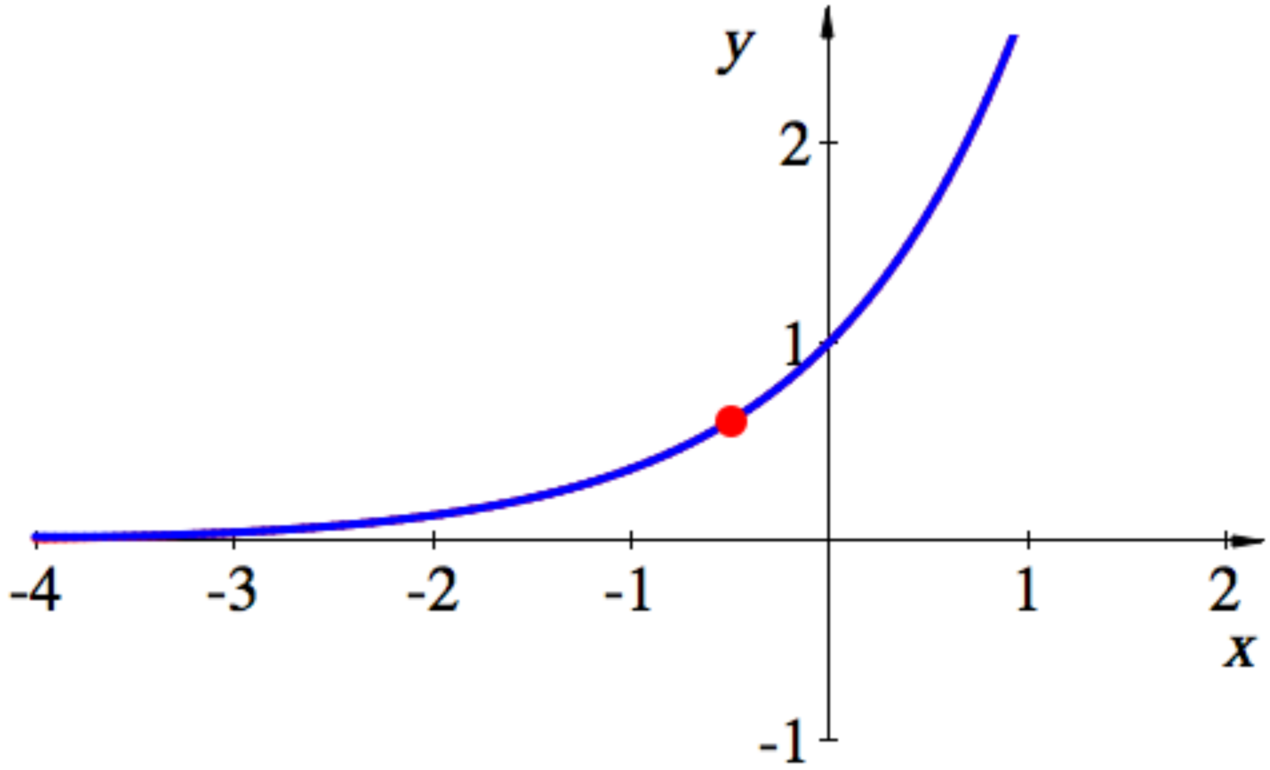
Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation neunten Grades

Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$

Entwicklung an der Stelle $x_0 = -\frac{1}{2}$



Approximation zehnten Grades

Formales

$f(x) \approx f(x_0)$ (Grobe) Approximation durch Konstante

Formales

$$f(x) \approx f(x_0)$$ (Grobe) Approximation durch Konstante

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$ Lineare Approximation

Formales

$$f(x) \approx f(x_0)$$ (Grobe) Approximation durch Konstante

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$ Lineare Approximation

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + (\quad ? \quad)(x - x_0)^2$$

Was passt hier?

$$p(x) = \underset{\substack{\uparrow \\ f(x_0)}}{a_0} + \underset{\substack{\uparrow \\ f'(x_0)}}{a_1}(x - x_0) + \underset{\substack{\uparrow \\ ?}}{a_2}(x - x_0)^2 + \underset{\substack{\uparrow \\ ?}}{a_3}(x - x_0)^3 + \cdots + \underset{\substack{\uparrow \\ ?}}{a_n}(x - x_0)^n$$

Was passt hier?

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p''(x_0) = 2a_2 \overset{!}{=} f''(x_0) \quad \text{(aus Systemgründen)}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p''(x_0) = 2a_2 \overset{!}{=} f''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_2 = \tfrac{1}{2} f''(x_0)$$

Vorsicht Faktor

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p''(x_0) = 2a_2 \stackrel{!}{=} f''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_2 = \frac{1}{2} f''(x_0)$$

$$p'''(x) = 2 \cdot 3a_3 + \cdots + (n-2)(n-1)na_n(x - x_0)^{n-3}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p''(x_0) = 2a_2 \overset{!}{=} f''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_2 = \tfrac{1}{2} f''(x_0)$$

$$p'''(x) = 2 \cdot 3a_3 + \cdots + (n-2)(n-1)na_n(x - x_0)^{n-3}$$

$$p'''(x_0) = 2 \cdot 3a_3 \overset{!}{=} f'''(x_0) \quad \text{(aus Systemgründen)}$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p''(x_0) = 2a_2 \overset{!}{=} f''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_2 = \tfrac{1}{2} f''(x_0)$$

$$p'''(x) = 2 \cdot 3a_3 + \cdots + (n-2)(n-1)na_n(x - x_0)^{n-3}$$

$$p'''(x_0) = 2 \cdot 3a_3 \overset{!}{=} f'''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_3 = \tfrac{1}{2 \cdot 3} f'''(x_0)$$

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$$p(x_0) = a_0 = f(x_0) \quad \text{(hatten wir schon)}$$

$$p'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$p'(x_0) = a_1 = f'(x_0) \quad \text{(hatten wir schon)}$$

$$p''(x) = 2a_2 + 2 \cdot 3a_3(x - x_0) + \cdots + (n-1)na_n(x - x_0)^{n-2}$$

$$p''(x_0) = 2a_2 \overset{!}{=} f''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_2 = \tfrac{1}{2} f''(x_0)$$

$$p'''(x) = 2 \cdot 3a_3 + \cdots + (n-2)(n-1)na_n(x - x_0)^{n-3}$$

$$p'''(x_0) = 2 \cdot 3a_3 \overset{!}{=} f'''(x_0) \quad \text{(aus Systemgründen)} \quad \Rightarrow \quad a_3 = \tfrac{1}{2 \cdot 3} f'''(x_0)$$

Allgemein: $a_k = \dfrac{1}{2 \cdot 3 \cdot 4 \cdot \cdots \cdot k} f^{(k)}(x_0)$

k-te Ableitung

$$a_k = \frac{1}{2 \cdot 3 \cdot 4 \cdot \dots \cdot k} f^{(k)}(x_0)$$

k-te Ableitung

$$a_k = \frac{1}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot k} f^{(k)}(x_0)$$

$$a_k = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot k} f^{(k)}(x_0)$$

k-te Ableitung

$$a_k = \frac{1}{2 \cdot 3 \cdot 4 \cdot \ldots \cdot k} f^{(k)}(x_0)$$

$$a_k = \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot k} f^{(k)}(x_0)$$

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{ - Fakultät"})$$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{ - Fakultät"})$$

$$2! = 1 \bullet 2 = 2$$

$$3! = 1 \bullet 2 \bullet 3 = 6$$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{-Fakultät"})$$

$2! = 1 \bullet 2 = 2$

$\bullet 3$

$3! = 1 \bullet 2 \bullet 3 = 6$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{-Fakultät"})$$

$2! = 1 \bullet 2 = 2$

$\bullet 3$

$3! = 1 \bullet 2 \bullet 3 = 6$

$\bullet 4$

$4! = 1 \bullet 2 \bullet 3 \bullet 4 = 24$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{ - Fakultät"})$$

$2! = 1 \bullet 2 = 2$

$\bullet 3$

$3! = 1 \bullet 2 \bullet 3 = 6$

$\bullet 4$

$4! = 1 \bullet 2 \bullet 3 \bullet 4 = 24$

$\bullet 5$

$5! = 120$

$\bullet 6$

$6! = 720$

$\bullet 7$

$7! = 5040$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{ - Fakultät"})$$

•1

•2

$2! = 1 \bullet 2 = 2$

•3

$3! = 1 \bullet 2 \bullet 3 = 6$

•4

$4! = 1 \bullet 2 \bullet 3 \bullet 4 = 24$

•5

$5! = 120$

•6

$6! = 720$

•7

$7! = 5040$

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot \quad \cdots \quad \cdot k = k! \qquad (\text{"}k\text{ - Fakultät"})$$

$0! = 1$

•1

$1! = 1$

•2

$2! = 1 \bullet 2 = 2$

•3

$3! = 1 \bullet 2 \bullet 3 = 6$

•4

$4! = 1 \bullet 2 \bullet 3 \bullet 4 = 24$

•5

$5! = 120$

•6

$6! = 720$

•7

$7! = 5040$

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 +$$

$$+ \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

Taylorpolynom n-ten Grades an der Stelle x_0

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

$$p(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \\ + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

$$f(x) \approx \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

Taylorreihe

Der kleine Unterschied

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$$

Examples, examples, examples

Examples, examples, examples

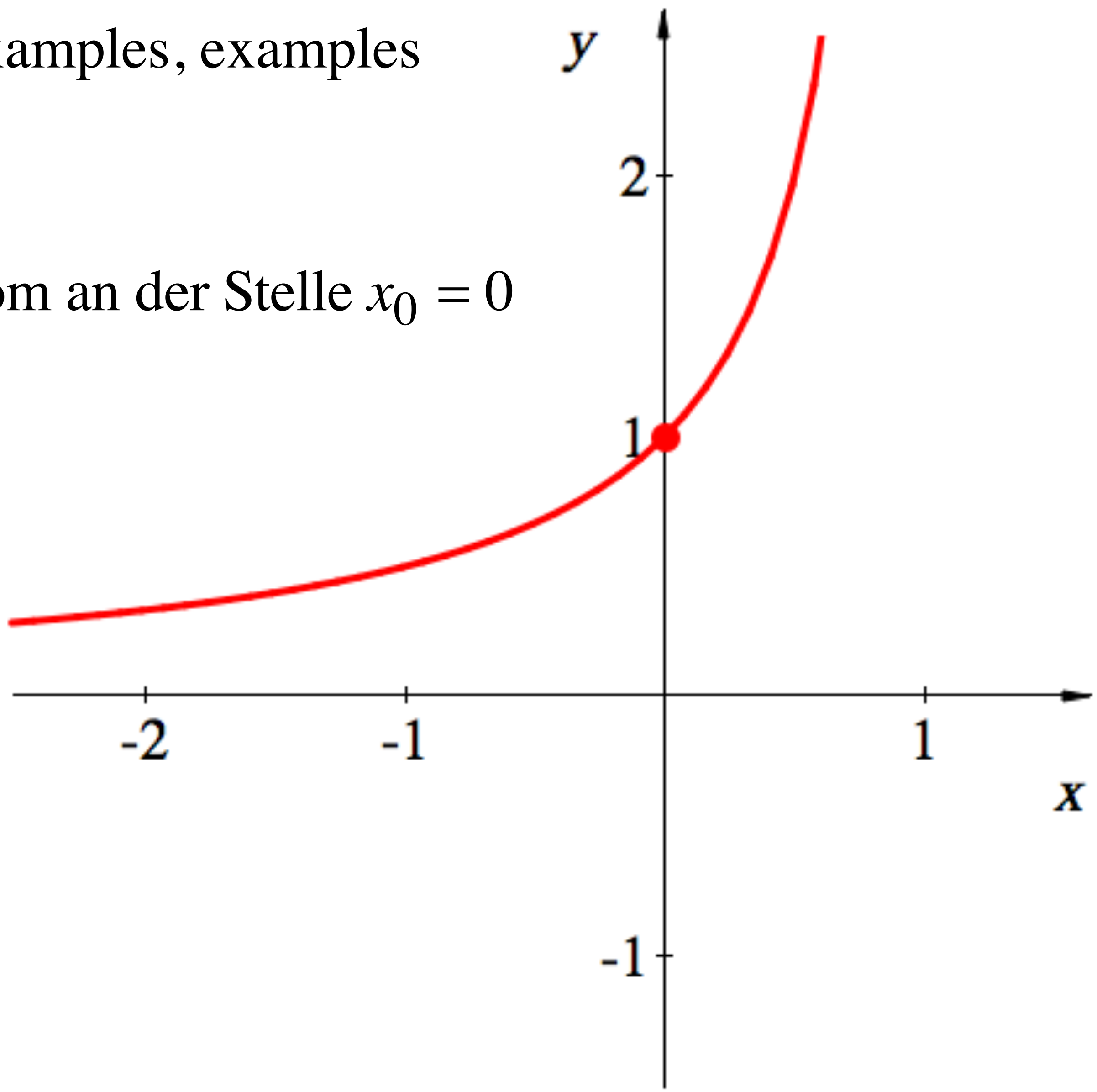
$f(x) = \frac{1}{1-x}$

Taylorpolynom an der Stelle $x_0 = 0$

Examples, examples, examples

$$f(x) = \frac{1}{1-x}$$

Taylorpolynom an der Stelle $x_0 = 0$



Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} =$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f(0) = 1$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

Innere Ableitung

$$f(0) = 1$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

Innere Ableitung

$$f(0) = 1$$

$$f'(0) = 1$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

Innere Ableitung

$$f(0) = 1$$

$$f'(0) = 1$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

Innere Ableitung

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -3 \cdot 2(1-x)^{-4}(-1) = 2 \cdot 3(1-x)^{-4}$$

Innere Ableitung

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -3 \cdot 2(1-x)^{-4}(-1) = 2 \cdot 3(1-x)^{-4}$$

Innere Ableitung

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2$$

$$f'''(0) = 2 \cdot 3$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -3 \cdot 2(1-x)^{-4}(-1) = 2 \cdot 3(1-x)^{-4}$$

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2$$

$$f'''(0) = 2 \cdot 3$$

allgemein

$$f^{(k)}(x) = k!(1-x)^{-k-1}$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -3 \cdot 2(1-x)^{-4}(-1) = 2 \cdot 3(1-x)^{-4}$$

$$f(0) = 1$$

$$f'(0) = 1$$

$$f''(0) = 2$$

$$f'''(0) = 2 \cdot 3$$

allgemein

$$f^{(k)}(x) = k!(1-x)^{-k-1}$$

$$f^{(k)}(0) = k!$$

$$f^{(k)}(x) = k!(1-x)^{-k-1}$$

$$\boxed{f^{(k)}(0) = k!}$$

$$f^{(k)}(x) = k!(1-x)^{-k-1}$$

$$f^{(k)}(0) = k!$$

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

$$f^{(k)}(x) = k!(1-x)^{-k-1}$$

$$f^{(k)}(0) = k!$$

$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k = \sum_{k=0}^{n} \frac{1}{k!} k!(x-0)^k$$

$$f^{(k)}(x) = k!(1-x)^{-k-1}$$

$$f^{(k)}(0) = k!$$

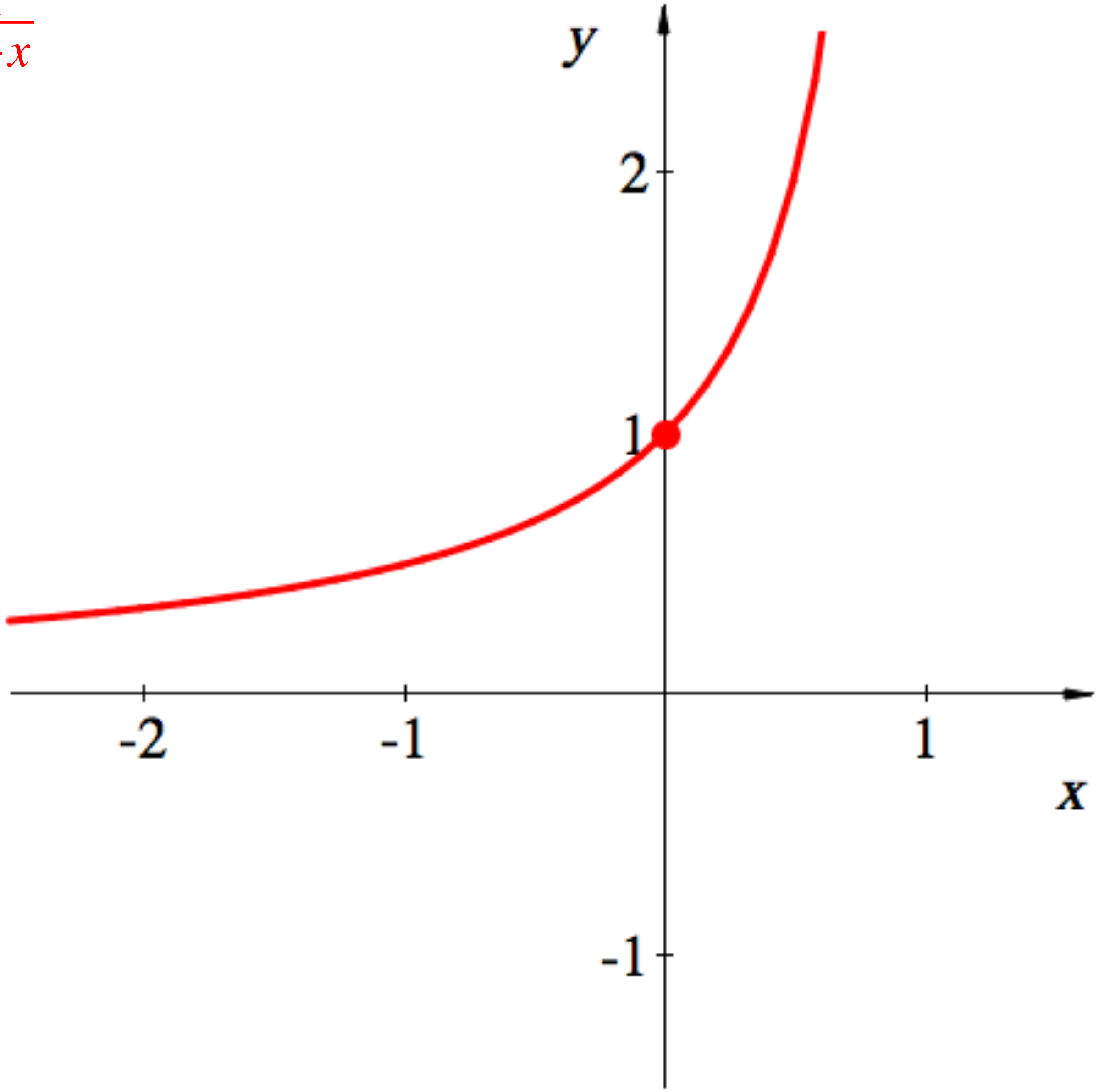
$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k = \sum_{k=0}^{n} \frac{1}{k!} k!(x-0)^k = \sum_{k=0}^{n} x^k$$

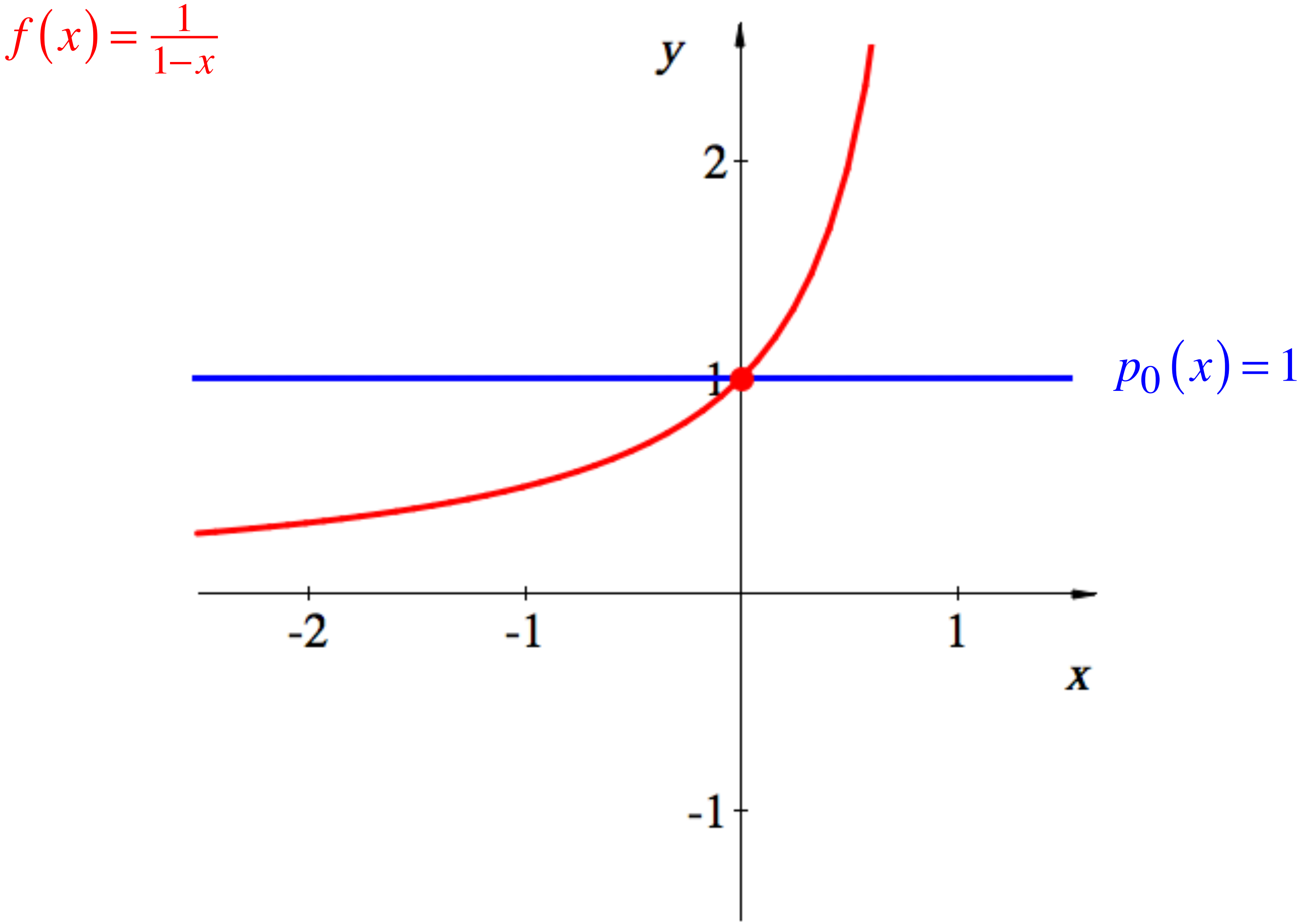
$$f^{(k)}(x) = k!(1-x)^{-k-1} \qquad f^{(k)}(0) = k!$$

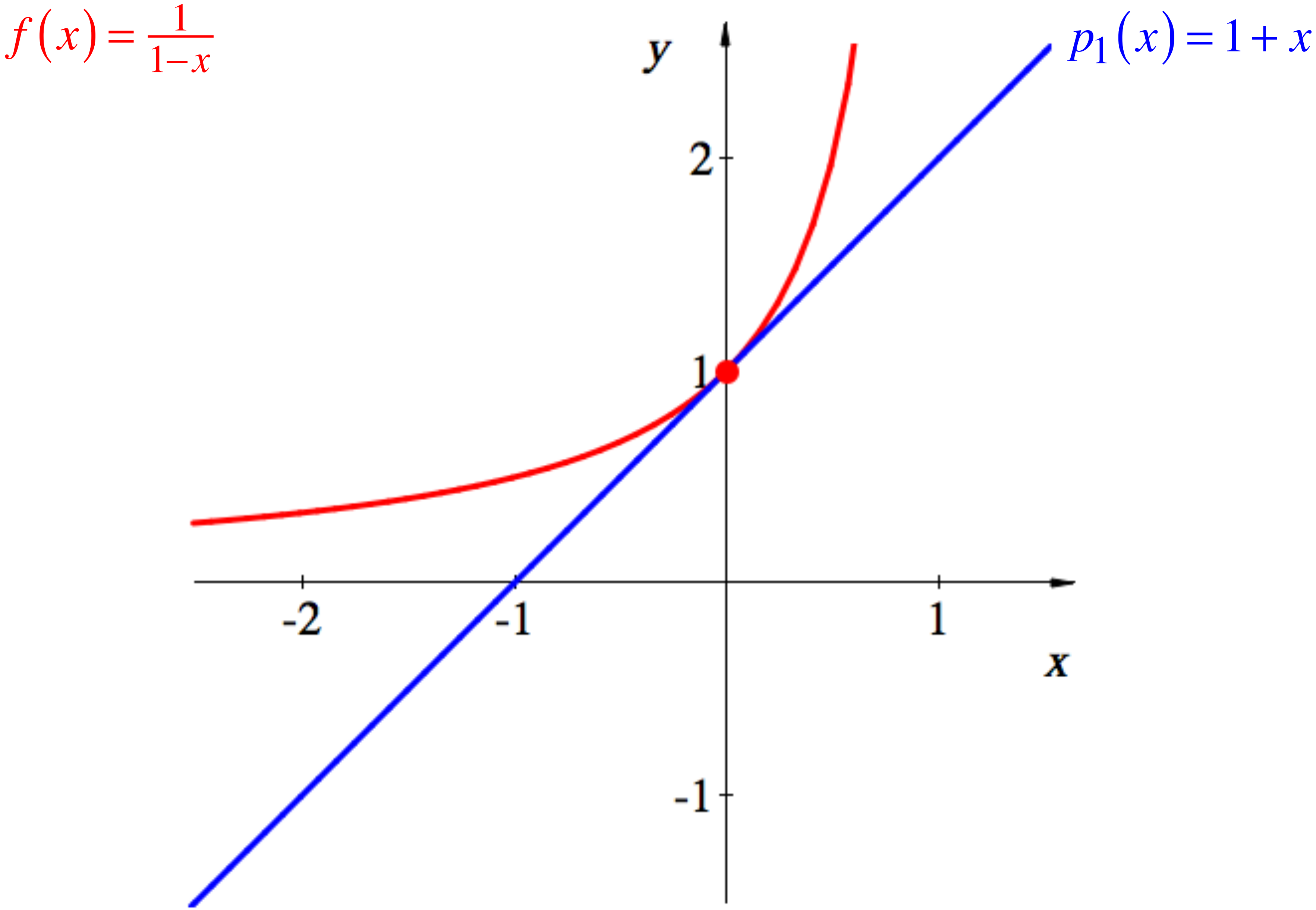
$$p(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k = \sum_{k=0}^{n} \frac{1}{k!} k!(x-0)^k = \sum_{k=0}^{n} x^k$$

$$p(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n$$

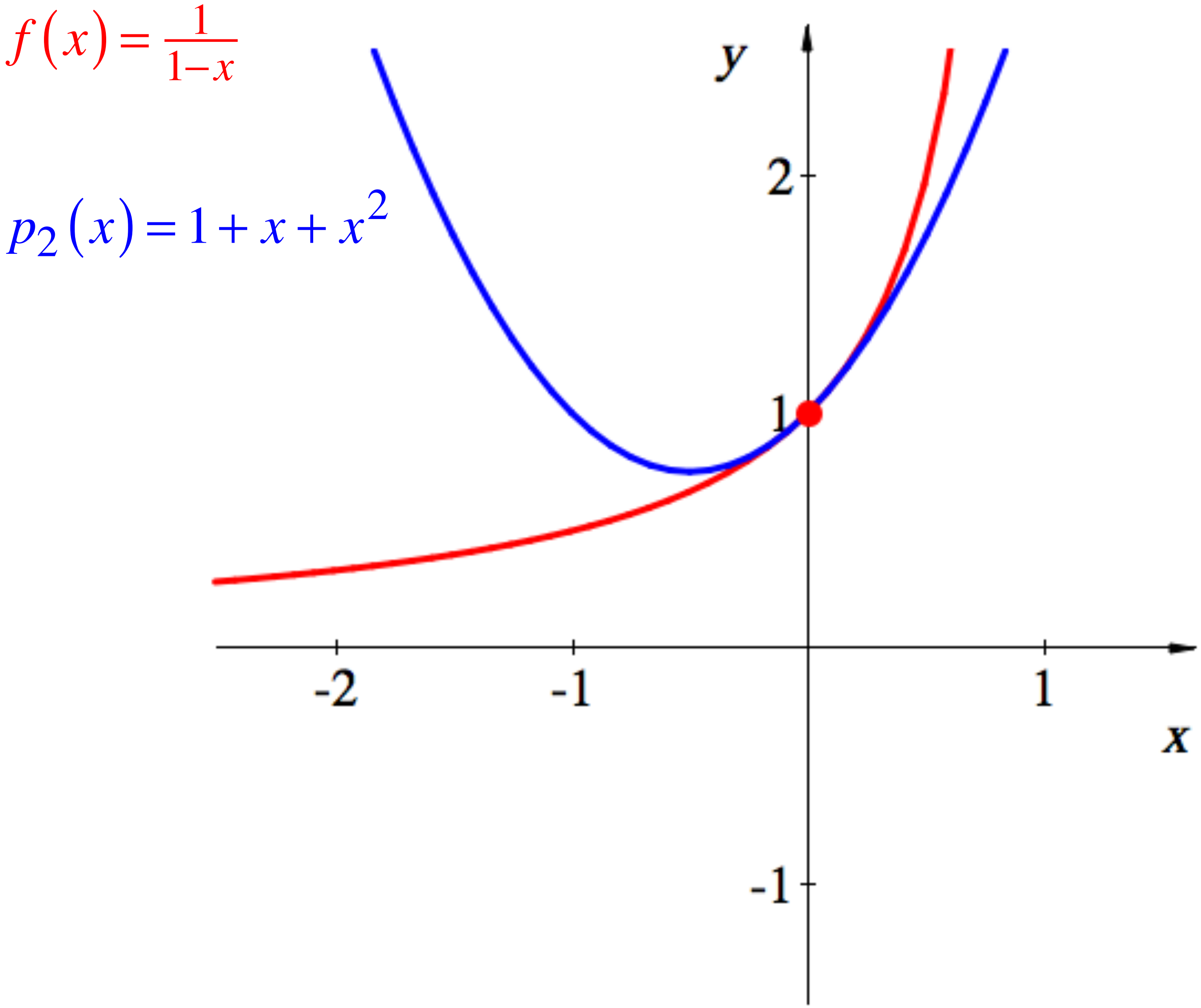
$$f(x) = \frac{1}{1-x}$$



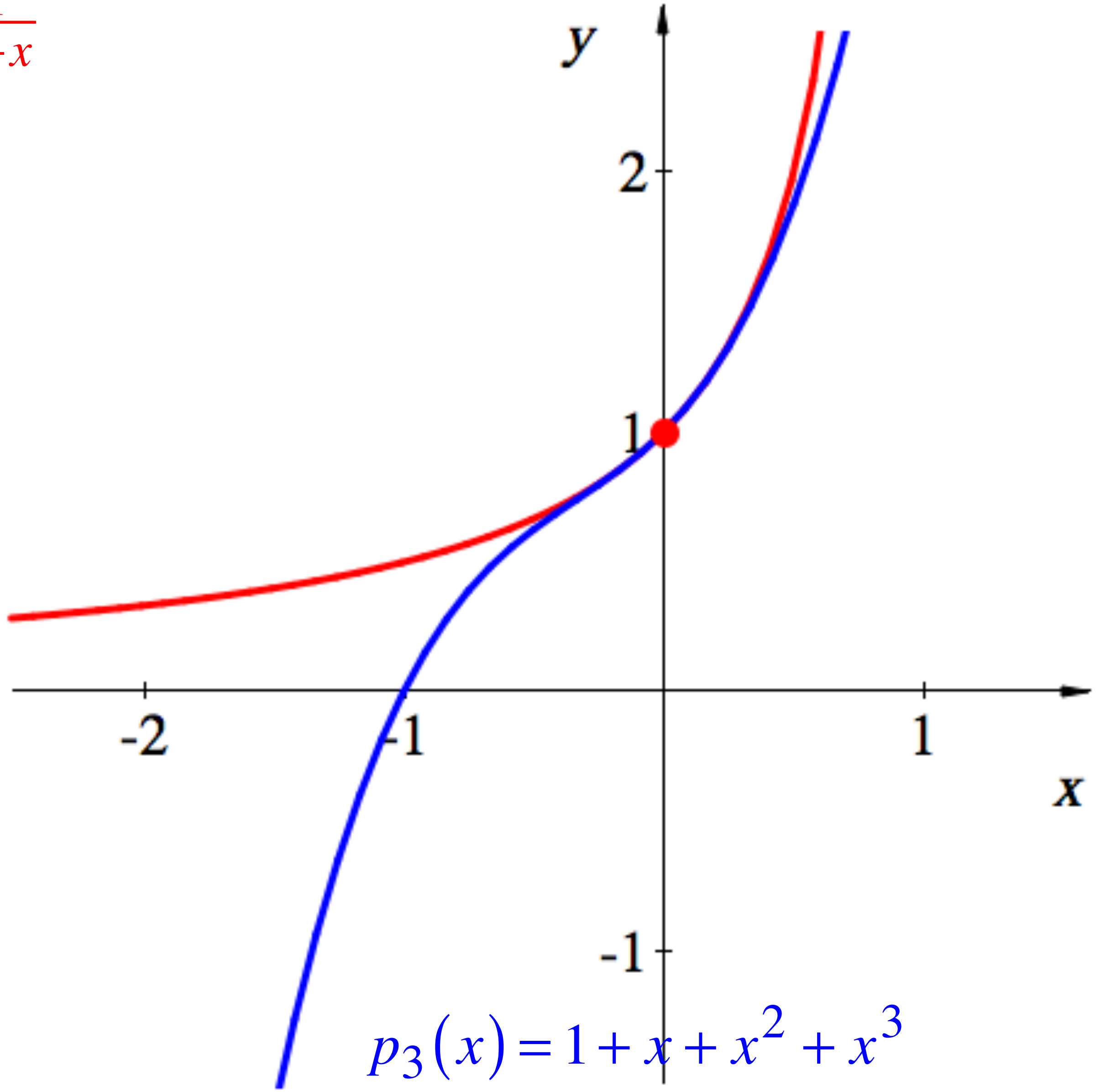

$f(x) = \frac{1}{1-x}$
y
2
1
$p_0(x) = 1$
-2
-1
1
x
-1



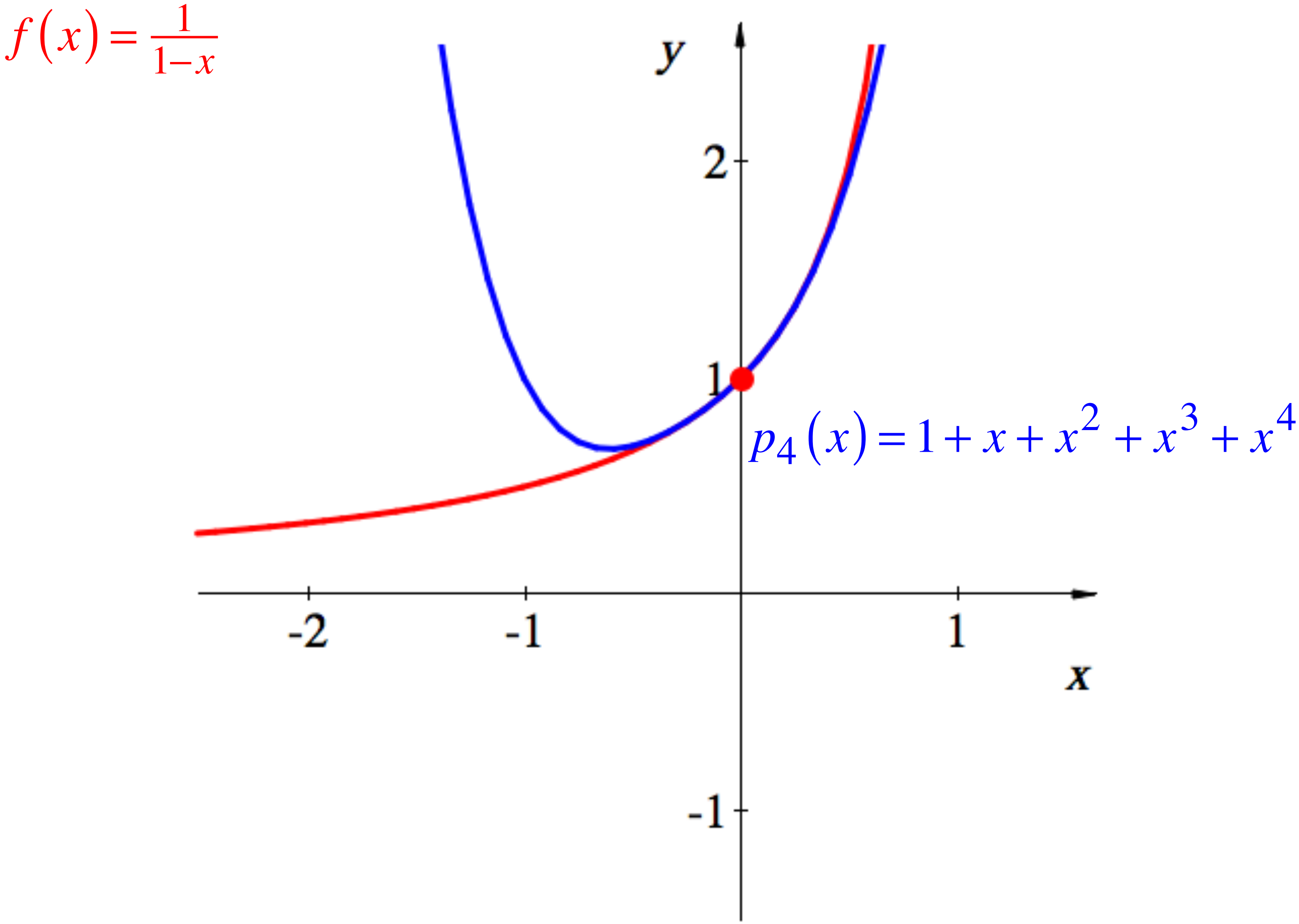
$f(x) = \frac{1}{1-x}$
$p_1(x) = 1 + x$
y
2
1
-1
-2
-1
1
x


$f(x) = \frac{1}{1-x}$
$p_2(x) = 1 + x + x^2$
y
2
1
-1
-2
-1
1
x

$$f(x)=\frac{1}{1-x}$$



$$p_3(x)=1+x+x^2+x^3$$



$f(x) = \frac{1}{1-x}$
y
2
1
-1
-2
-1
1
x
$p_4(x) = 1 + x + x^2 + x^3 + x^4$

$$f(x) = \frac{1}{1-x}$$

$$p(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \cdots + x^n$$

$$f(x) = \frac{1}{1-x} \approx 1 + x + x^2 + x^3 + \cdots + x^n$$

Erinnerung: $\quad 1 + q + q^2 + \cdots = \frac{1}{1-q}$

Examples, examples, examples

$$f(x) = \ln(x)$$

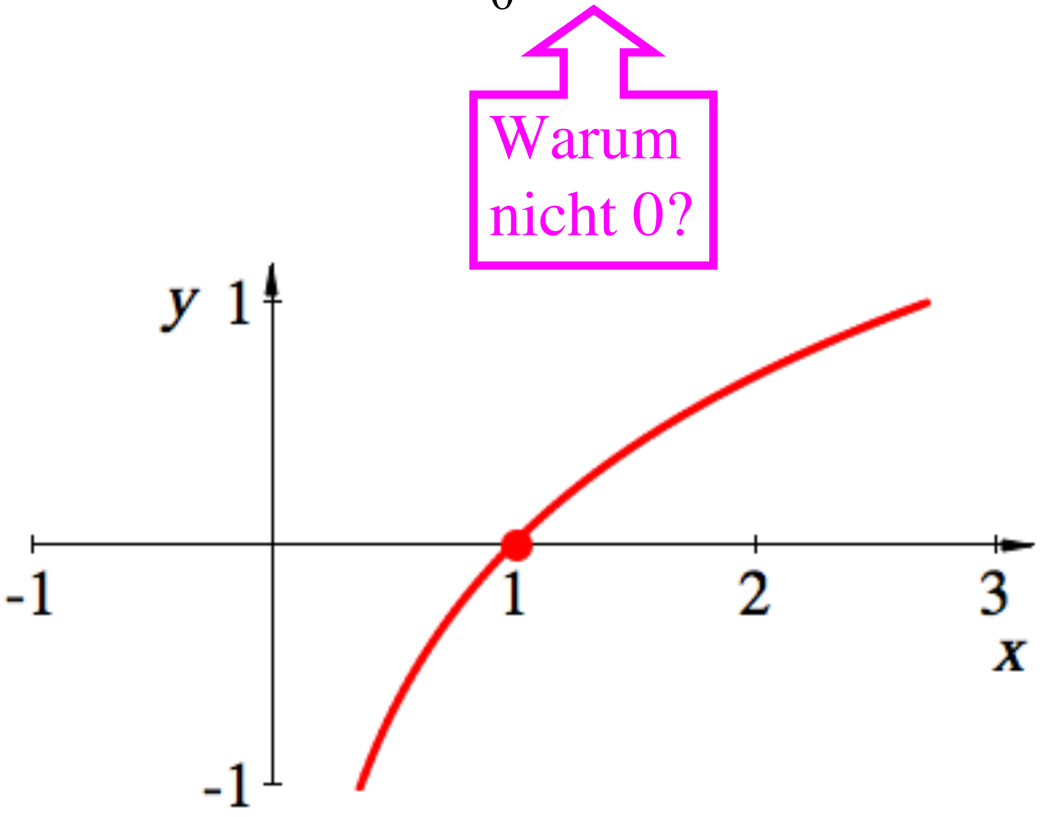
Taylorreihe an der Stelle $x_0 = 1$

Warum nicht 0?

Examples, examples, examples

$f(x) = \ln(x)$

Taylorreihe an der Stelle $x_0 = 1$



Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x)$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = (-1)x^{-2}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = (-1)x^{-2}$$

$$f'''(x) = (-1)(-2)x^{-3}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x)$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = (-1)x^{-2}$$

$$f'''(x) = (-1)(-2)x^{-3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x) \qquad \Rightarrow \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1}$$

$$f''(x) = (-1)x^{-2}$$

$$f'''(x) = (-1)(-2)x^{-3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x) \quad \Rightarrow \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = (-1)x^{-2}$$

$$f'''(x) = (-1)(-2)x^{-3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x) \quad \Rightarrow \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(1) = -1$$

$$f'''(x) = (-1)(-2)x^{-3}$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x) \qquad \Rightarrow \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \qquad \Rightarrow \qquad f'(1) = 1$$

$$f''(x) = (-1)x^{-2} \qquad \Rightarrow \qquad f''(1) = -1$$

$$f'''(x) = (-1)(-2)x^{-3} \qquad \Rightarrow \qquad f'''(1) = (-1)(-2) = 2!$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4}$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x) \quad \Rightarrow \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(1) = -1$$

$$f'''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad f'''(1) = (-1)(-2) = 2!$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4} \quad \Rightarrow \quad f^{(4)}(1) = (-1)(-2)(-3) = -3!$$

Entwicklung an der Stelle $x_0 = 1$

$$f(x) = \ln(x) \quad \Rightarrow \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} = x^{-1} \quad \Rightarrow \quad f'(1) = 1$$

$$f''(x) = (-1)x^{-2} \quad \Rightarrow \quad f''(1) = -1$$

$$f'''(x) = (-1)(-2)x^{-3} \quad \Rightarrow \quad f'''(1) = (-1)(-2) = 2!$$

$$f^{(4)}(x) = (-1)(-2)(-3)x^{-4} \quad \Rightarrow \quad f^{(4)}(1) = (-1)(-2)(-3) = -3!$$

allgemein: $$f^{(k)}(1) = \underbrace{(-1)^{k-1}}_{\text{alternierendes Vorzeichen}} (k-1)!$$

Entwicklung an der Stelle $x_0 = 1$

$$f^{(k)}(1) = (-1)^{k-1}(k-1)!$$

Entwicklung an der Stelle $x_0 = 1$

$$f^{(k)}(1) = (-1)^{k-1}(k-1)!$$

$$a_k = \frac{1}{k!} f^{(k)}(x_0)$$

Entwicklung an der Stelle $x_0 = 1$

$$\left.\begin{aligned} f^{(k)}(1) &= (-1)^{k-1}(k-1)! \\ a_k &= \frac{1}{k!} f^{(k)}(x_0) \end{aligned}\right\} \Rightarrow \qquad a_k = \frac{1}{k}(-1)^{k-1}$$

Entwicklung an der Stelle $x_0 = 1$

$$\left.\begin{aligned} & f^{(k)}(1) = (-1)^{k-1}(k-1)! \\ & a_k = \frac{1}{k!} f^{(k)}(x_0) \end{aligned}\right\} \Rightarrow \qquad a_k = \frac{1}{k}(-1)^{k-1} \quad \text{für } k > 0$$

Entwicklung an der Stelle $x_0 = 1$

$$\left.\begin{array}{l} f^{(k)}(1) = (-1)^{k-1}(k-1)! \\ a_k = \frac{1}{k!} f^{(k)}(x_0) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_k = \frac{1}{k}(-1)^{k-1} \quad \text{für } k > 0 \\ a_0 = 0 \quad \text{weil } f(1) = \ln(1) = 0 \end{array}\right.$$

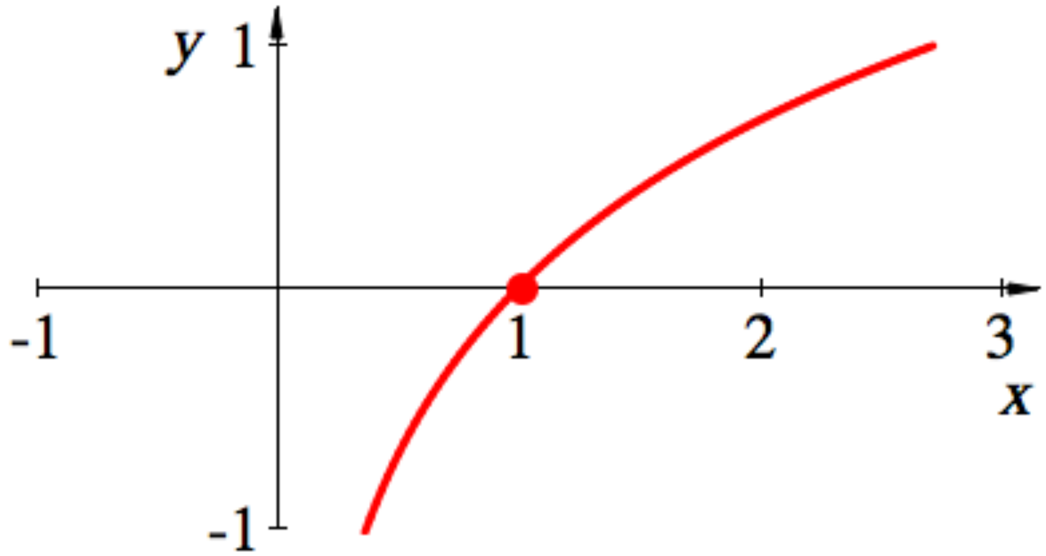
Entwicklung an der Stelle $\quad x_0 = 1$

$$\left.\begin{array}{l} f^{(k)}(1) = (-1)^{k-1}(k-1)! \\ a_k = \frac{1}{k!} f^{(k)}(x_0) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} a_k = \frac{1}{k}(-1)^{k-1} \quad \text{für } k > 0 \\ a_0 = 0 \quad \text{weil } f(1) = \ln(1) = 0 \end{array}\right.$$

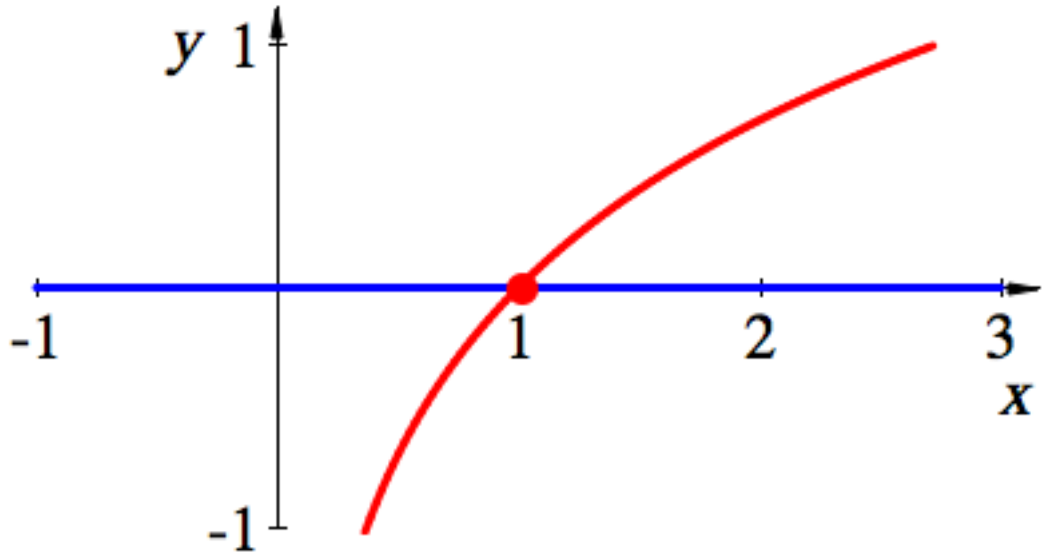
$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 \pm \cdots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k}(x-1)^k$$

Entwicklung an der Stelle $x_0 = 1$

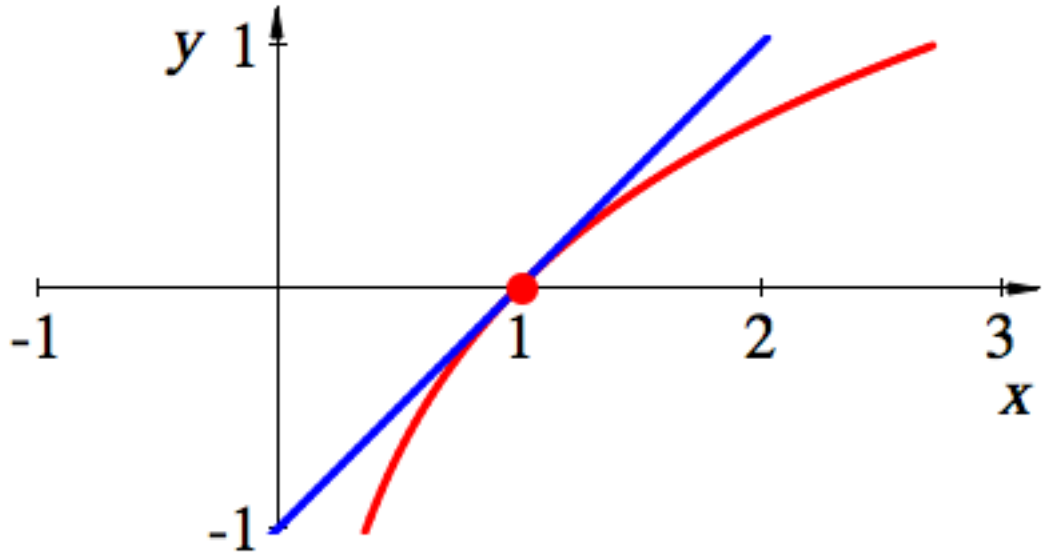
$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



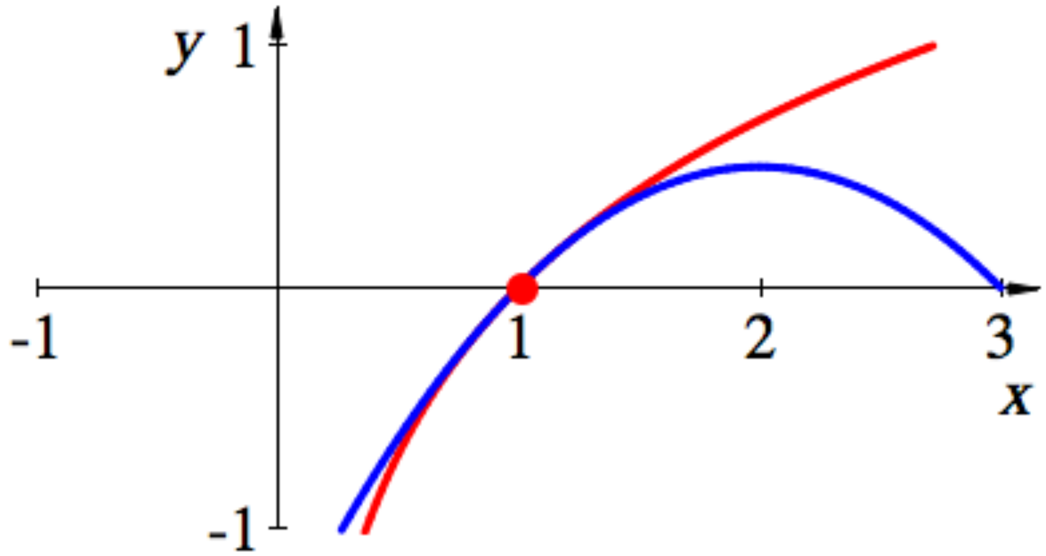
$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



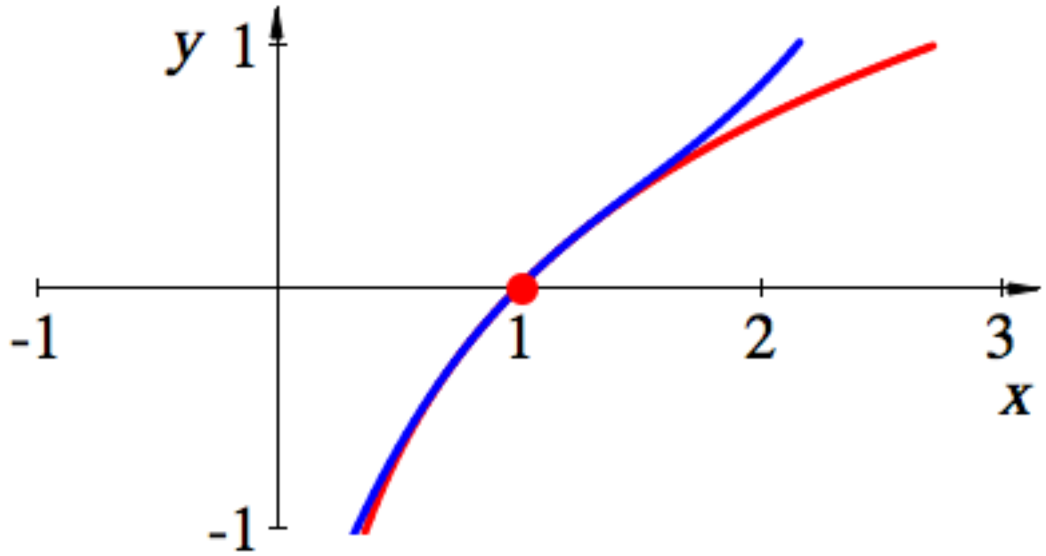
$$\ln(x) \approx (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \tfrac{1}{5}(x-1)^5 - \tfrac{1}{6}(x-1)^6 \pm \cdots$$



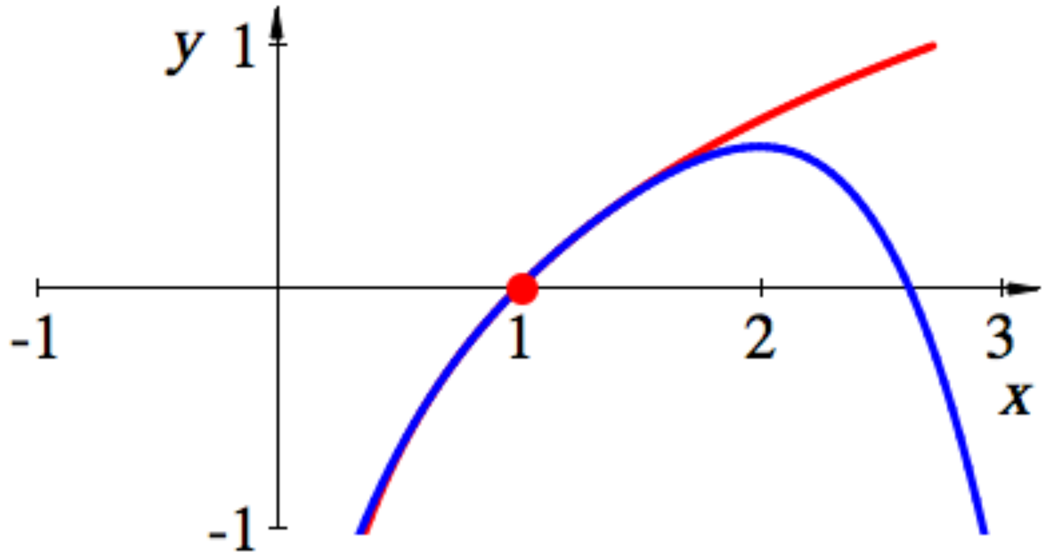
$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



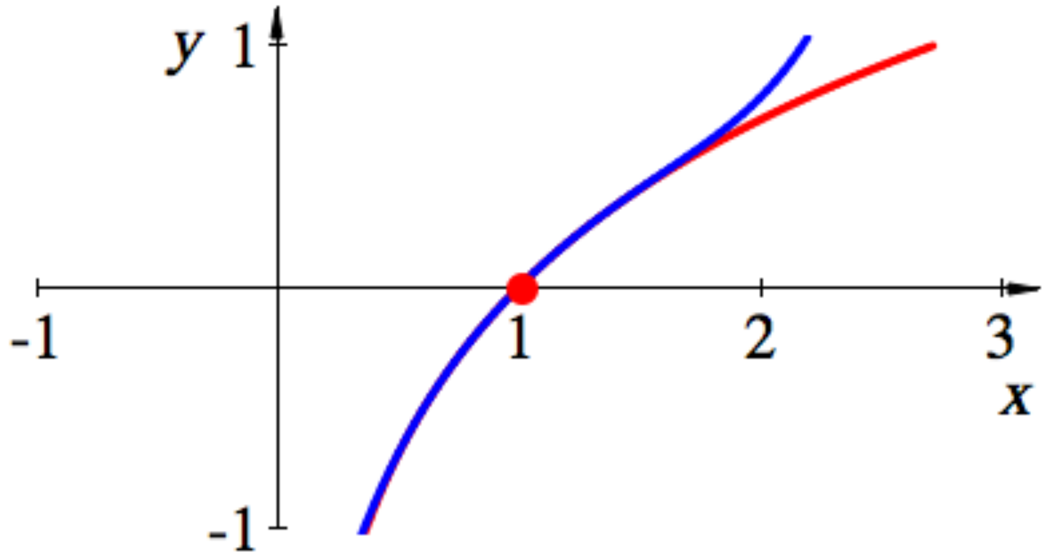
$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



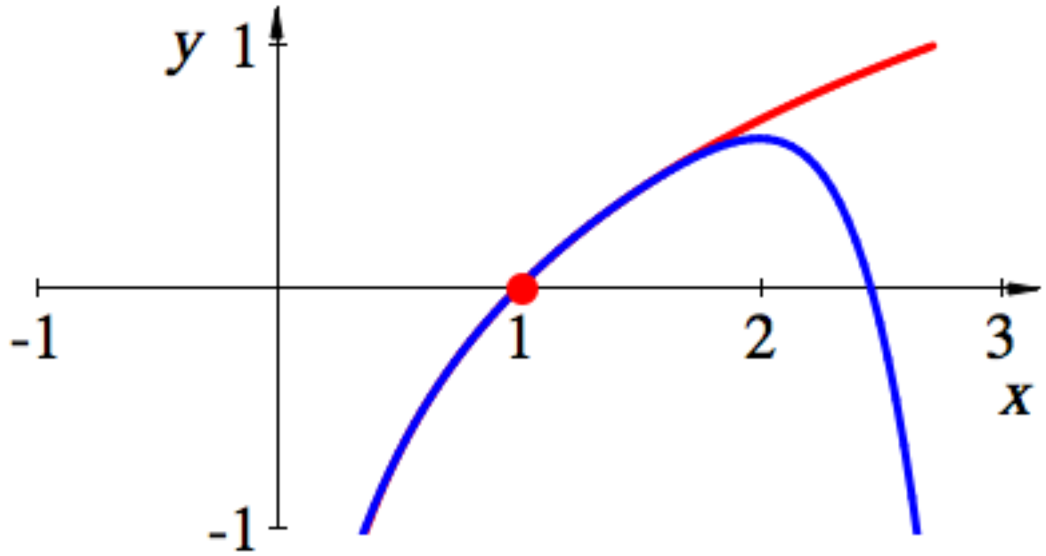
$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



$$\ln(x) \approx (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 \pm \cdots$$



$$\ln(x) \approx (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \tfrac{1}{5}(x-1)^5 - \tfrac{1}{6}(x-1)^6 \pm \cdots$$

CAS

```
p(x):=taylor( ln(x),   x = 1,   7);
```

$$p(x) := x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}$$

$$(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 +$$

$$O((x-1)^7)$$

$$\ln(x) \approx (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4 + \tfrac{1}{5}(x-1)^5 - \tfrac{1}{6}(x-1)^6 \pm \cdots$$

Entwicklung an der Stelle $x_0 = 1$

CAS

```
p(x):=taylor( ln(x),  x = 1,  7);
```

$$p(x) := x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \frac{1}{6}(x-1)^6 + O((x-1)^7)$$

Examples, examples, examples

$f(x) = e^x$

Taylorpolynom an der Stelle $x_0 = 0$

Entwicklung an der Stelle $x_0 = 0$

$f(x) = \mathrm{e}^x$ $\qquad$ $f(0) = 1$

Entwicklung an der Stelle $x_0 = 0$

$f(x) = \mathrm{e}^x \qquad f(0) = 1$

$f'(x) = \mathrm{e}^x \qquad f'(0) = 1$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \mathrm{e}^x \qquad f(0) = 1$$

$$f'(x) = \mathrm{e}^x \qquad f'(0) = 1$$

$$f''(x) = \mathrm{e}^x \qquad f''(0) = 1$$

$$f'''(x) = \mathrm{e}^x \qquad f'''(0) = 1$$

allgemein

$$f^{(k)}(x) = \mathrm{e}^x \qquad f^{(k)}(0) = 1$$

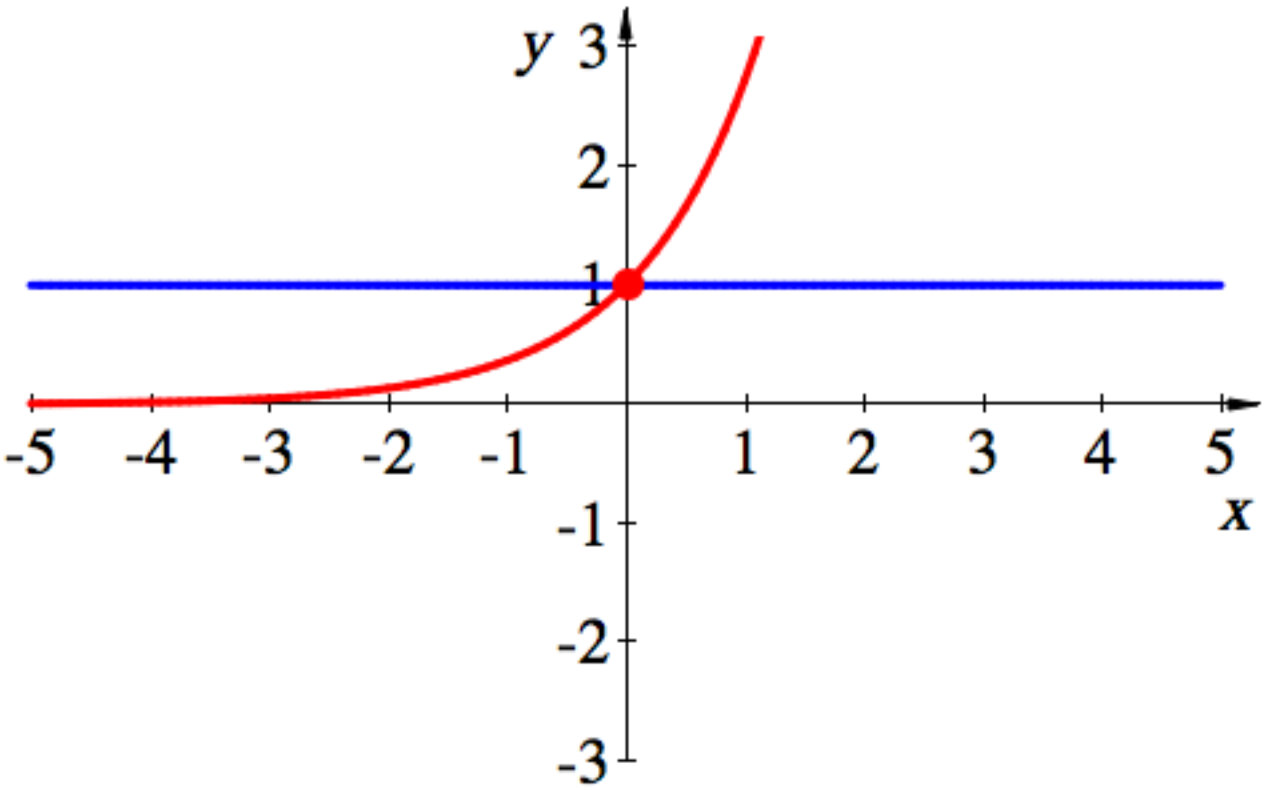
$$f^{(k)}(x) = \mathrm{e}^{x} \qquad\qquad f^{(k)}(0) = 1$$

$$p(x) = \sum_{k=0}^{n} \tfrac{1}{k!} f^{(k)}(x_0)(x - x_0)^k = \sum_{k=0}^{n} \tfrac{1}{k!}(x - 0)^k = \sum_{k=0}^{n} \tfrac{1}{k!} x^k$$

$$p(x) = \sum_{k=0}^{n} \tfrac{1}{k!} x^k = 1 + x + \tfrac{1}{2} x^2 + \tfrac{1}{6} x^3 + \cdots + \tfrac{1}{n!} x^n$$

$$\mathrm{e}^{x} \approx 1 + x + \tfrac{1}{2} x^2 + \tfrac{1}{6} x^3 + \cdots + \tfrac{1}{n!} x^n$$

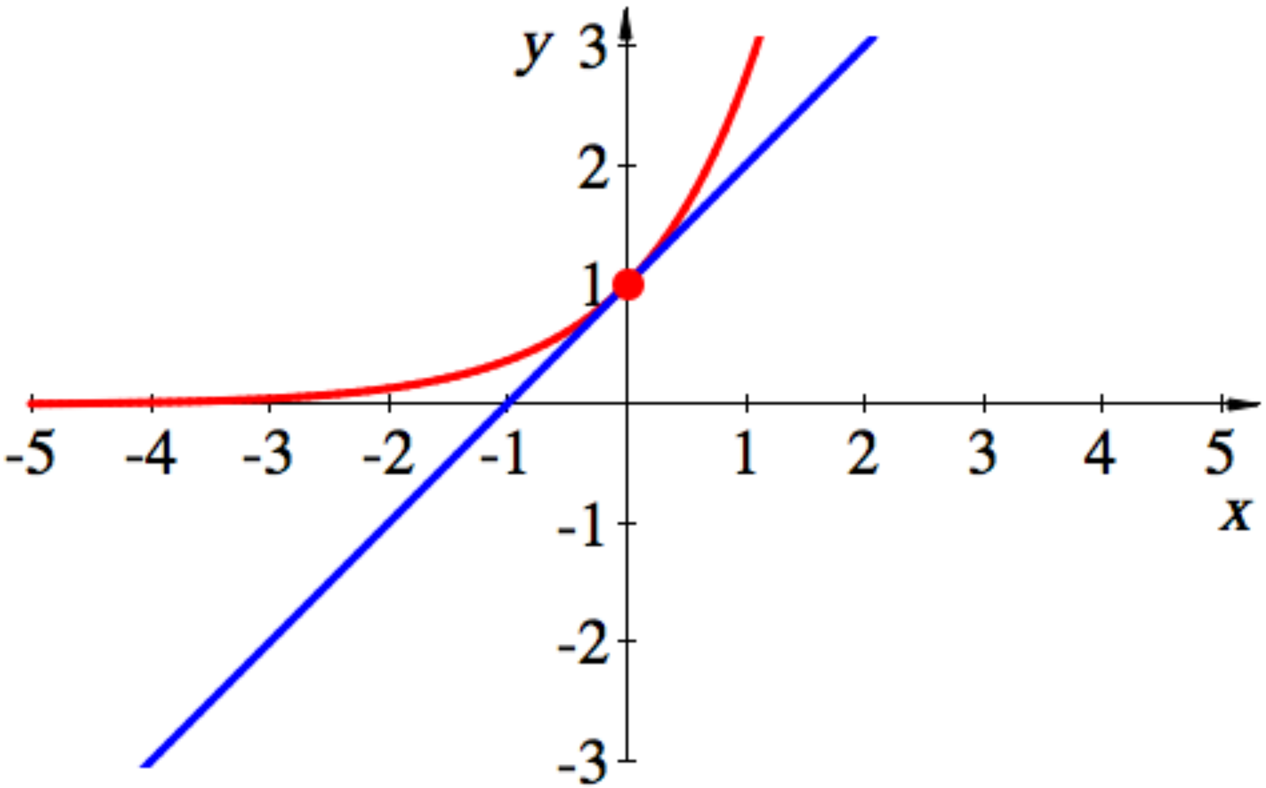
Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$



Approximation durch Konstante

$$\mathrm{e}^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n$$

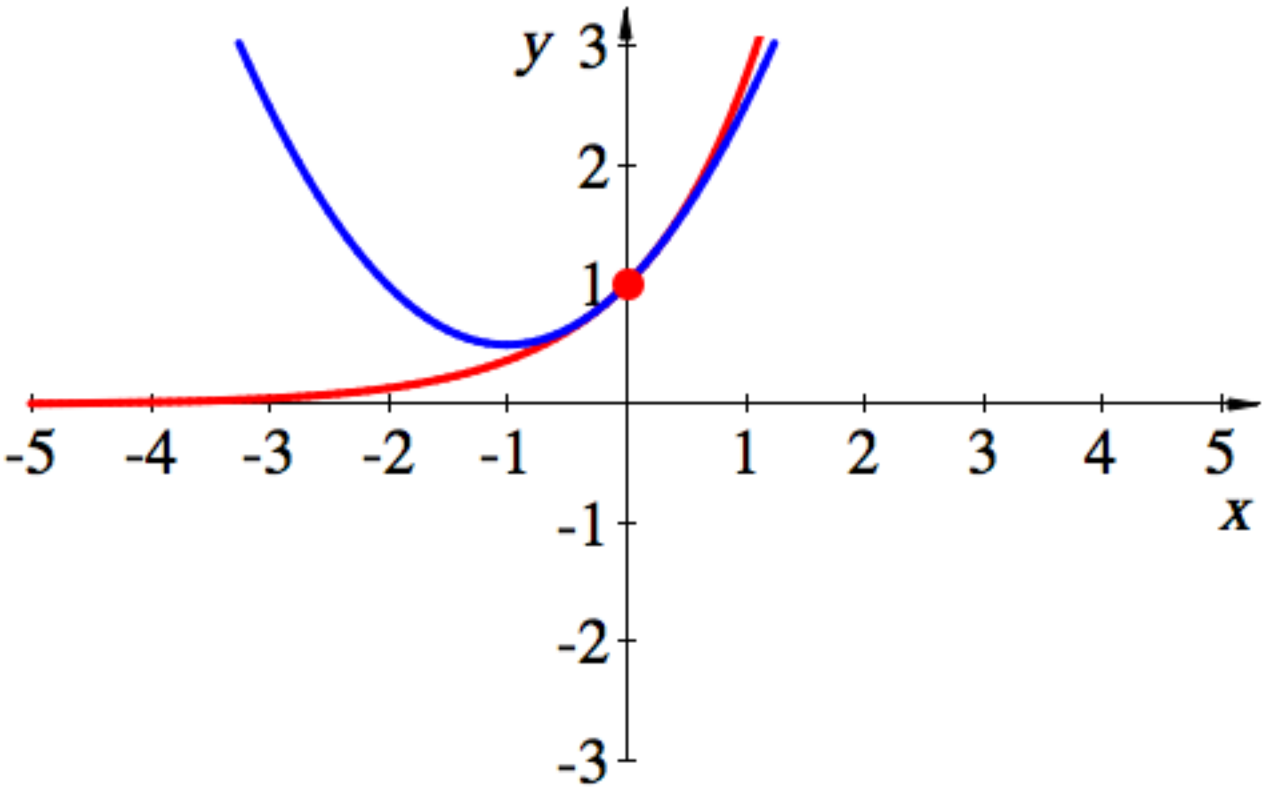
Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$



Lineare Approximation (Tangente)

$$\mathrm{e}^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n$$

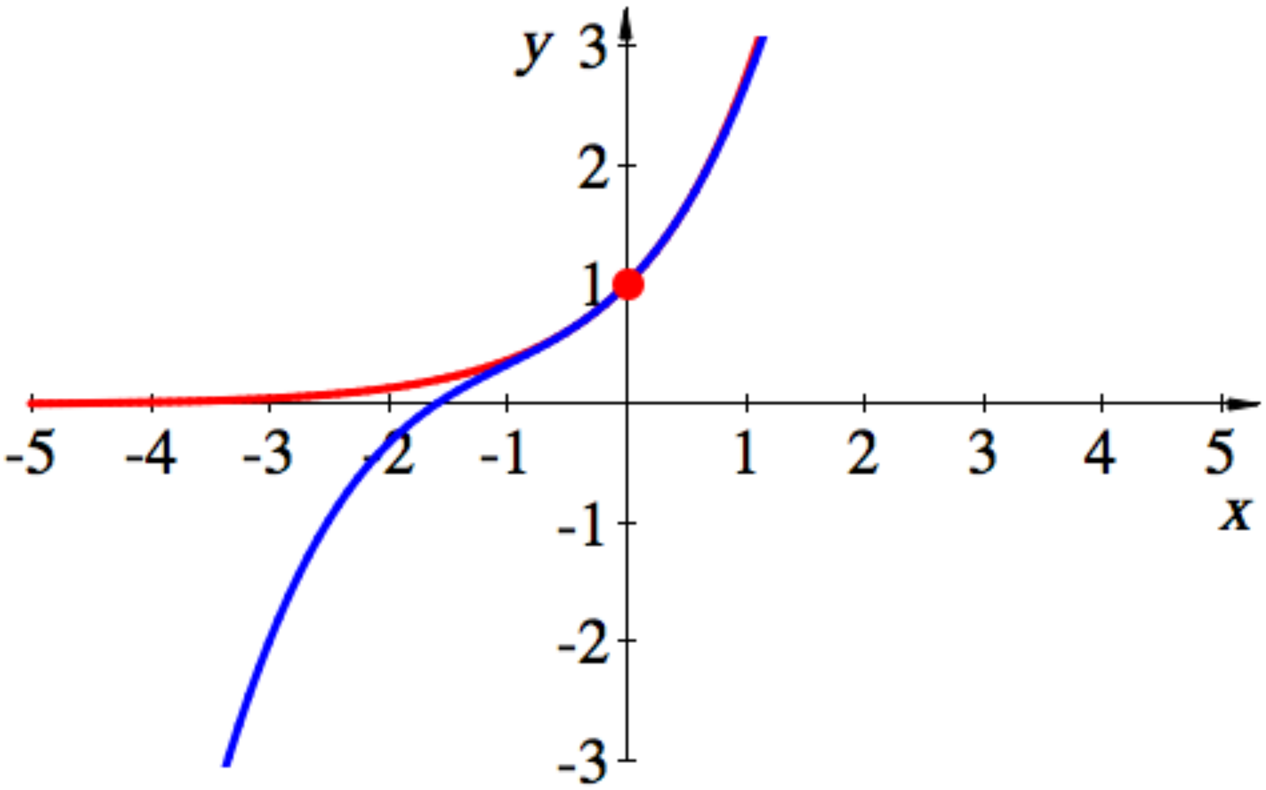
Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^x$



Quadratische Approximation (tangentiale Parabel)

$$\mathrm{e}^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n$$

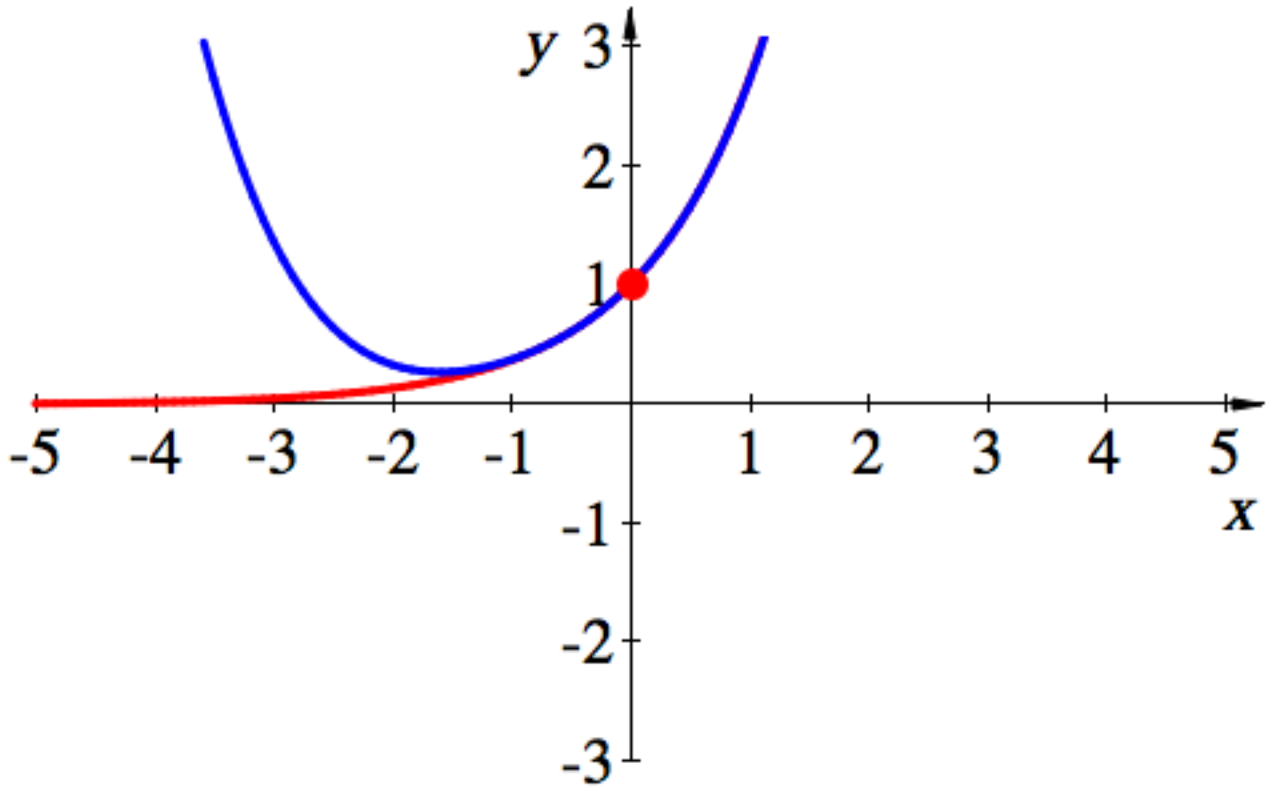
Beispiel: Exponenzialfunktion $y = f(x) = \mathrm{e}^{x}$



Approximation dritten Grades

$$\mathrm{e}^{x} \approx 1 + x + \frac{1}{2}x^{2} + \frac{1}{6}x^{3} + \cdots + \frac{1}{n!}x^{n}$$

Beispiel: Exponenzialfunktion $y = f(x) = e^x$



Approximation vierten Grades

$$e^x \approx 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \cdots + \frac{1}{n!}x^n$$

Taylor-Reihe

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

Taylor-Reihe

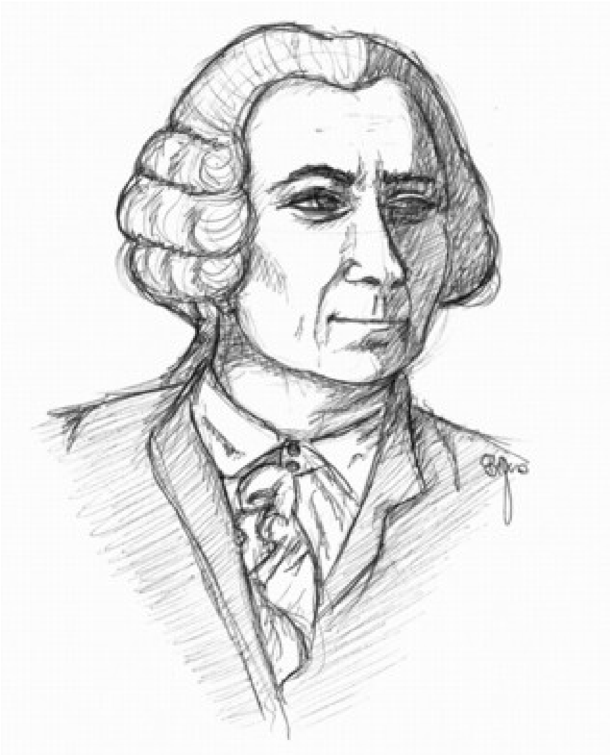
$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

Für $x = 1$ folgt:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Formel von Euler



Leonhard Euler
1707 - 1783

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

Formel von Euler $e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}$

k	***k* !**	**1/*k* !**	**Summe**
0	1	1	1.0000000000
1	1	1	2.0000000000
2	2	0.5	2.5000000000
3	6	0.166666667	2.6666666667
4	24	0.041666667	2.7083333333
5	120	0.008333333	2.7166666667
6	720	0.001388889	2.7180555556
7	5040	0.000198413	2.7182539683
8	40320	2.48016E-05	2.7182787698
9	362880	2.75573E-06	2.7182815256
10	3628800	2.75573E-07	2.7182818011
11	39916800	2.50521E-08	2.7182818262
12	479001600	2.08768E-09	2.7182818283
13	6227020800	1.6059E-10	2.7182818284
14	87178291200	1.14707E-11	2.7182818285

Examples, examples, examples

$f(x) = \sin(x)$

Taylorreihe an der Stelle $x_0 = 0$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

Jetzt fängt es
wieder von vorne an.

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x) \qquad \Rightarrow \qquad f(0) = 0$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x) \quad \Rightarrow \quad f(0) = 0$$

$$f'(x) = \cos(x) \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x) \quad \Rightarrow \quad f(0) = 0$$

$$f'(x) = \cos(x) \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = -\sin(x) \quad \Rightarrow \quad f''(0) = 0$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x) \quad \Rightarrow \quad f(0) = 0$$

$$f'(x) = \cos(x) \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = -\sin(x) \quad \Rightarrow \quad f''(0) = 0$$

$$f'''(x) = -\cos(x) \quad \Rightarrow \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x)$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x) \quad \Rightarrow \quad f(0) = 0$$

$$f'(x) = \cos(x) \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = -\sin(x) \quad \Rightarrow \quad f''(0) = 0$$

$$f'''(x) = -\cos(x) \quad \Rightarrow \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x) \quad \Rightarrow \quad f^{(4)}(0) = 0$$

allgemein:

$$f^{(k)}(0) = \begin{cases} 0 \text{ falls } k \text{ gerade} \\ 1 \text{ falls } k:4 \text{ Rest } 1 \text{ ergibt} \\ -1 \text{ falls } k:4 \text{ Rest } 3 \text{ ergibt} \end{cases}$$

Entwicklung an der Stelle $x_0 = 0$

$$f(x) = \sin(x) \quad \Rightarrow \quad f(0) = 0$$

$$f'(x) = \cos(x) \quad \Rightarrow \quad f'(0) = 1$$

$$f''(x) = -\sin(x) \quad \Rightarrow \quad f''(0) = 0$$

$$f'''(x) = -\cos(x) \quad \Rightarrow \quad f'''(0) = -1$$

$$f^{(4)}(x) = \sin(x) \quad \Rightarrow \quad f^{(4)}(0) = 0$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots = \sum_{k=0}^{\infty} \; ?$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots = \sum_{k=0}^{\infty} (-1)^k \;?$$

Hinkendes Vorzeichen

Alternierendes Vorzeichen

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}$$

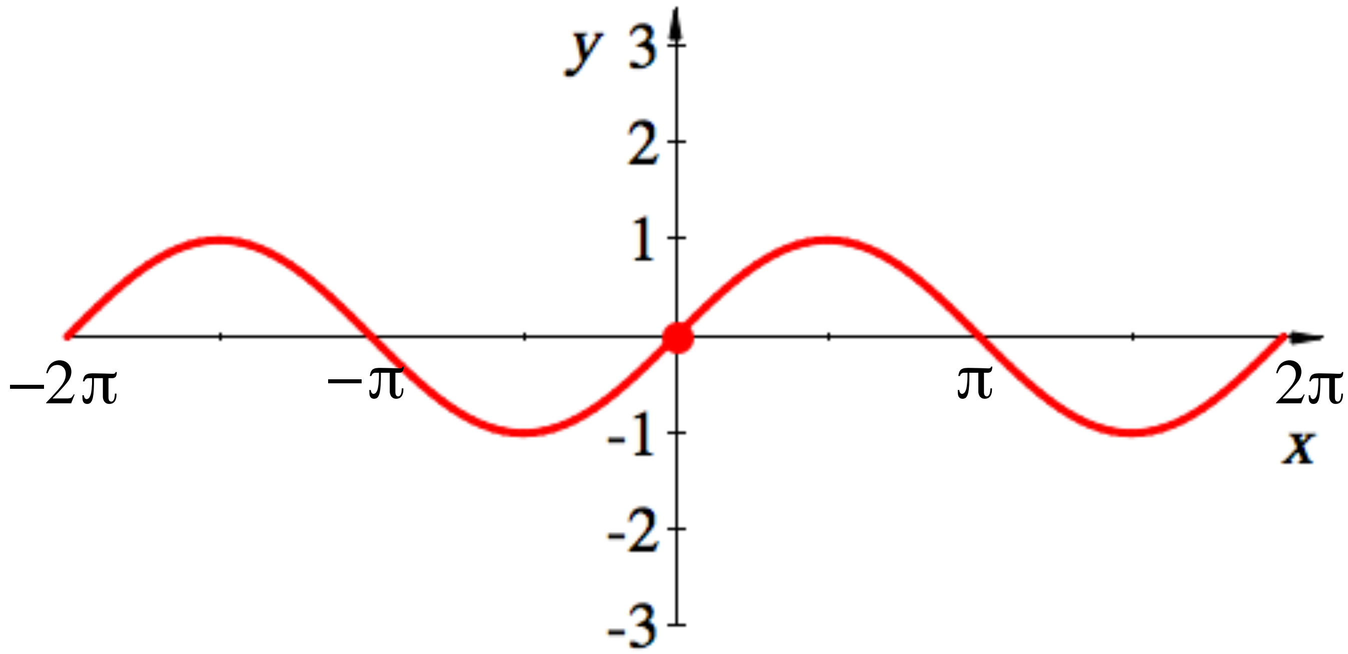
Nur ungerade Exponenten
Fakultäten von ungeraden Zahlen

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \pm \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}$$

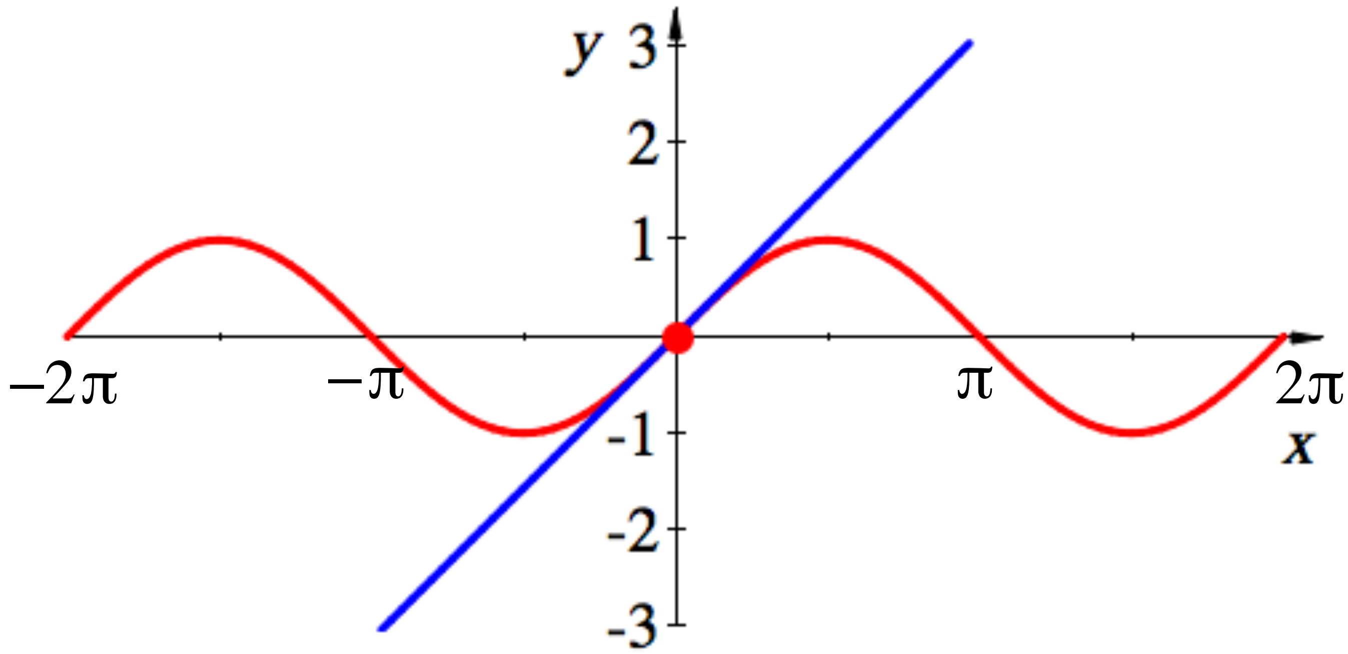
Nur ungerade Exponenten

Sinus ist eine ungerade Funktion

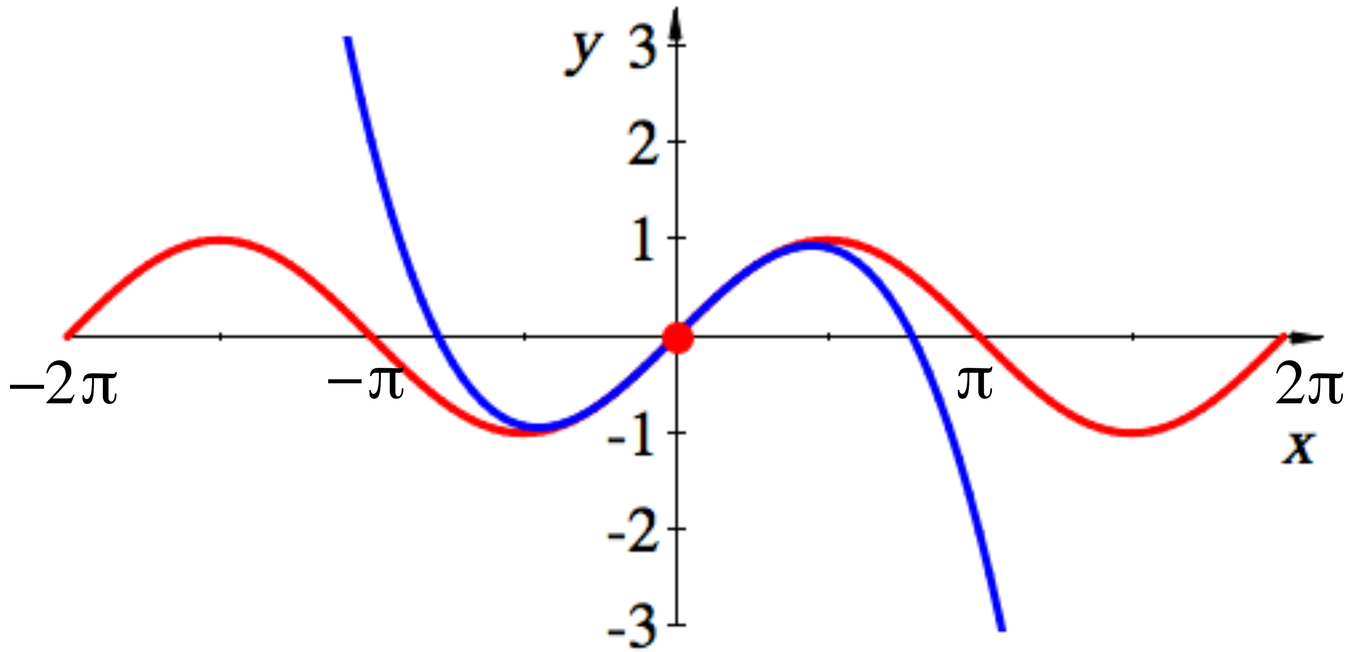
$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



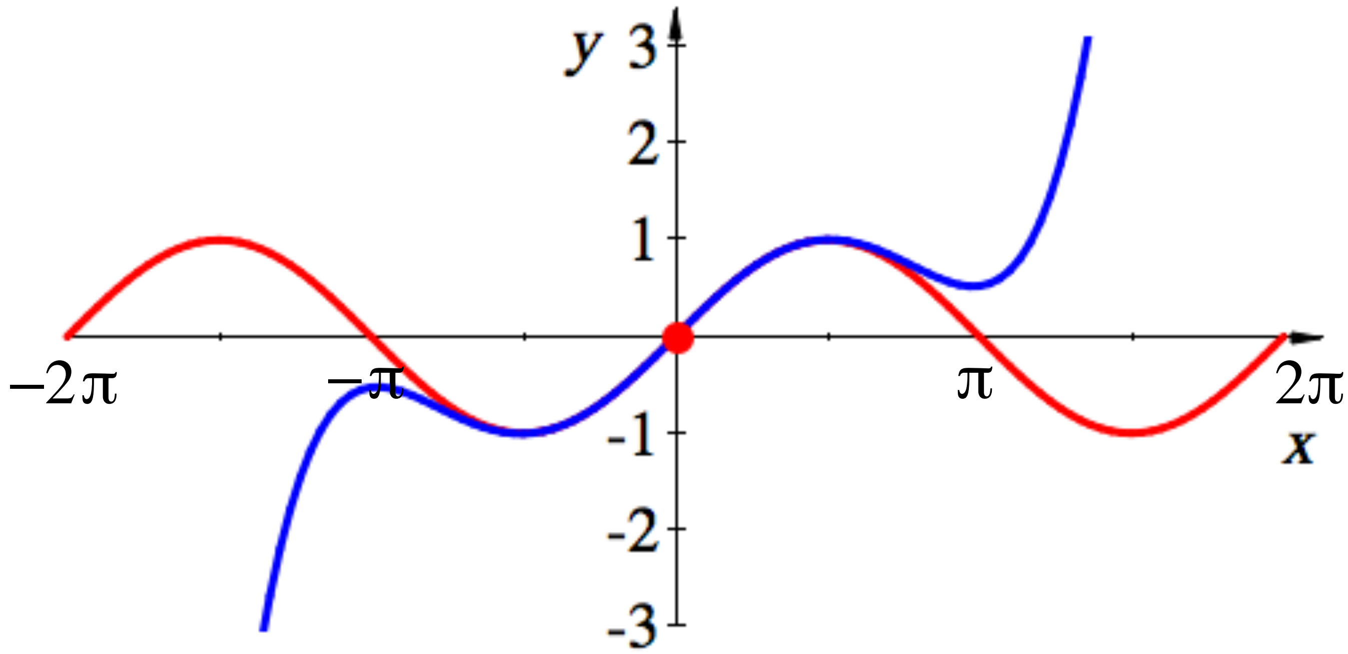
$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



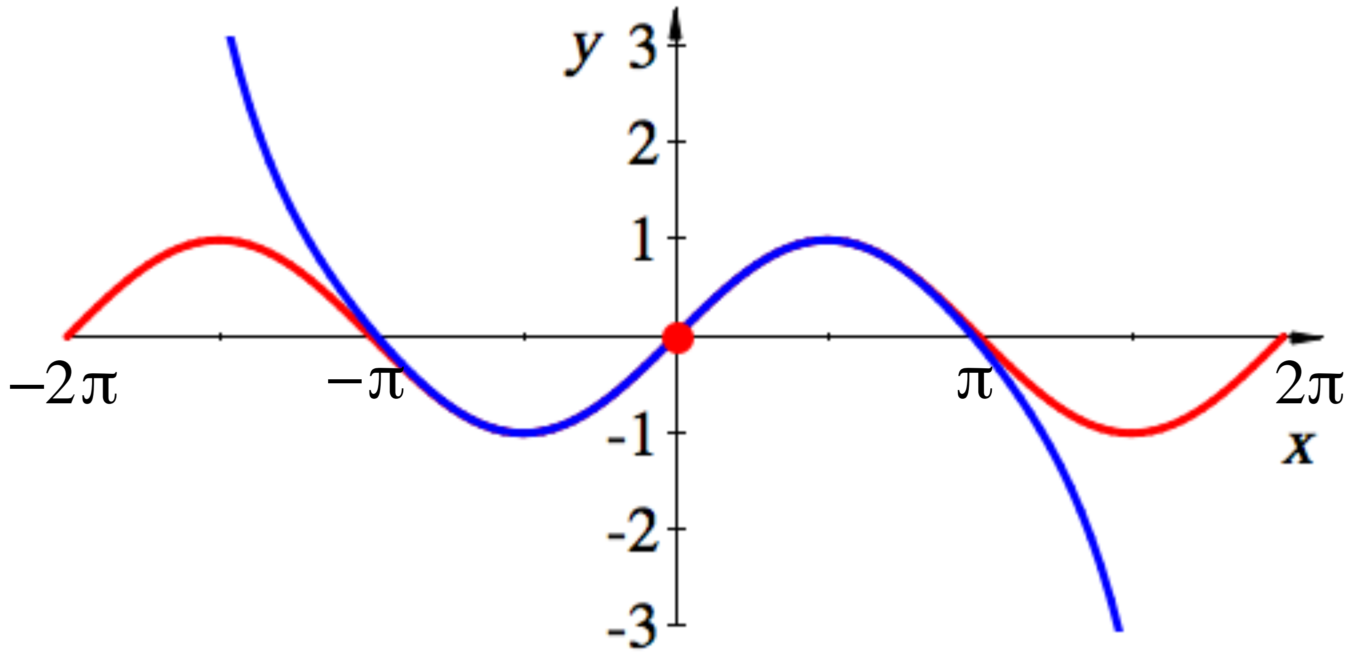
$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



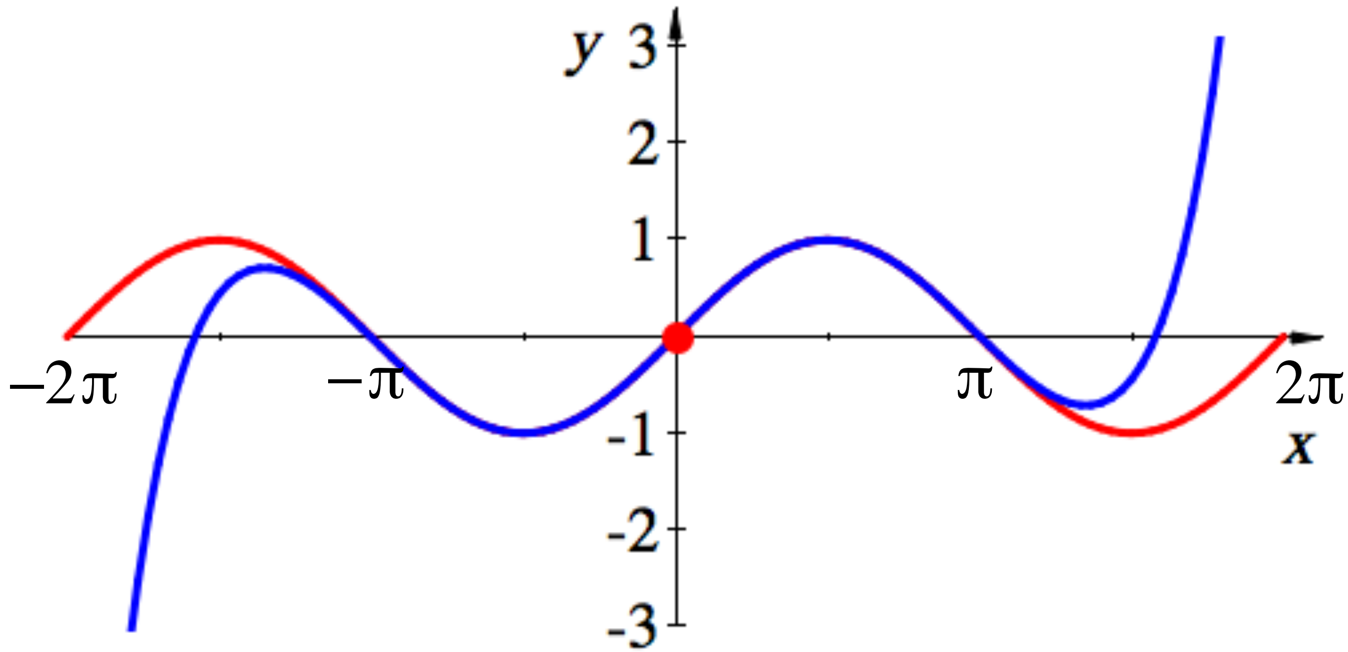
$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



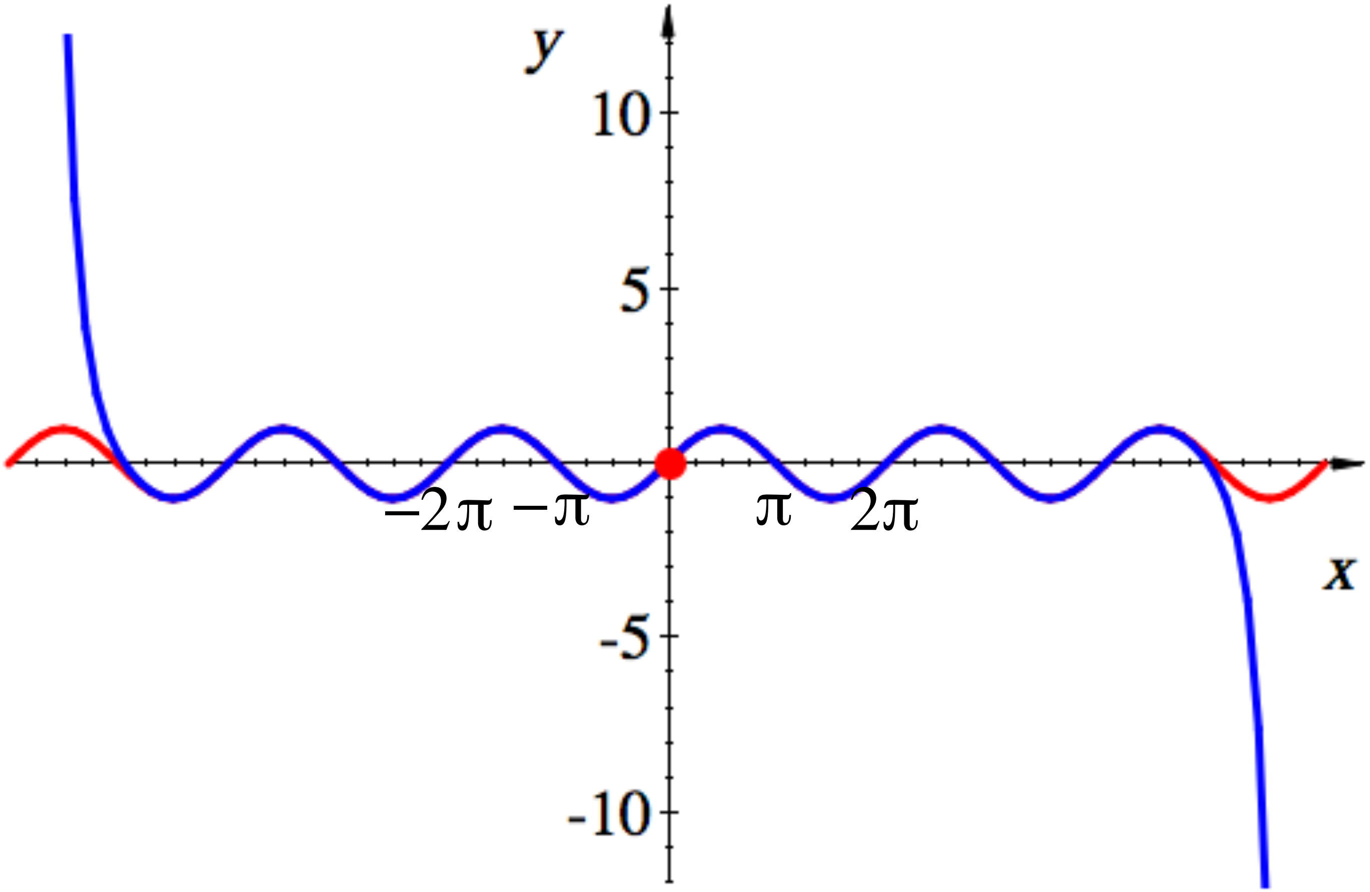
$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$



Entwicklung bis zum Grad 39

$$\sin(x) \approx x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 \mp \cdots$$

CAS

```
> p(x):=taylor( sin(x),   x = 0,   17);
```

$$p(x) := x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \frac{1}{362880}x^9 - \frac{1}{39916800}x^{11} + \frac{1}{6227020800}x^{13} - \frac{1}{1307674368000}x^{15} + O(x^{17})$$

Entwicklung bis zum Grad 15

Examples, examples, examples

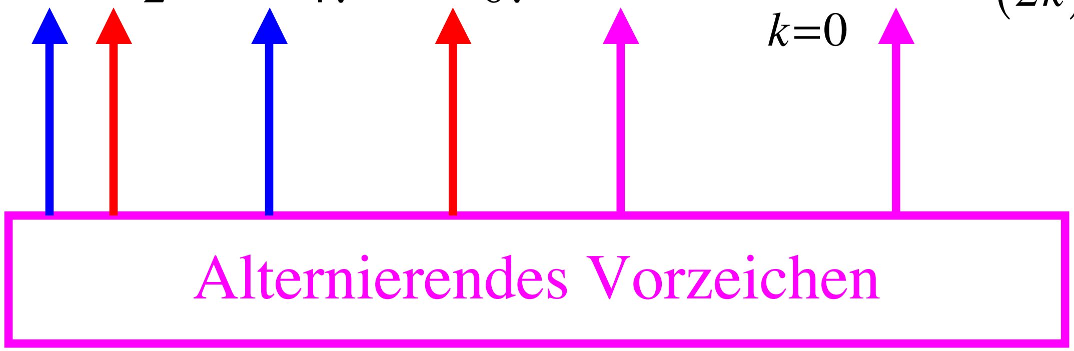
$f(x) = \cos(x)$

Taylorreihe an der Stelle $x_0 = 0$

Analog $\sin(x)$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}$$



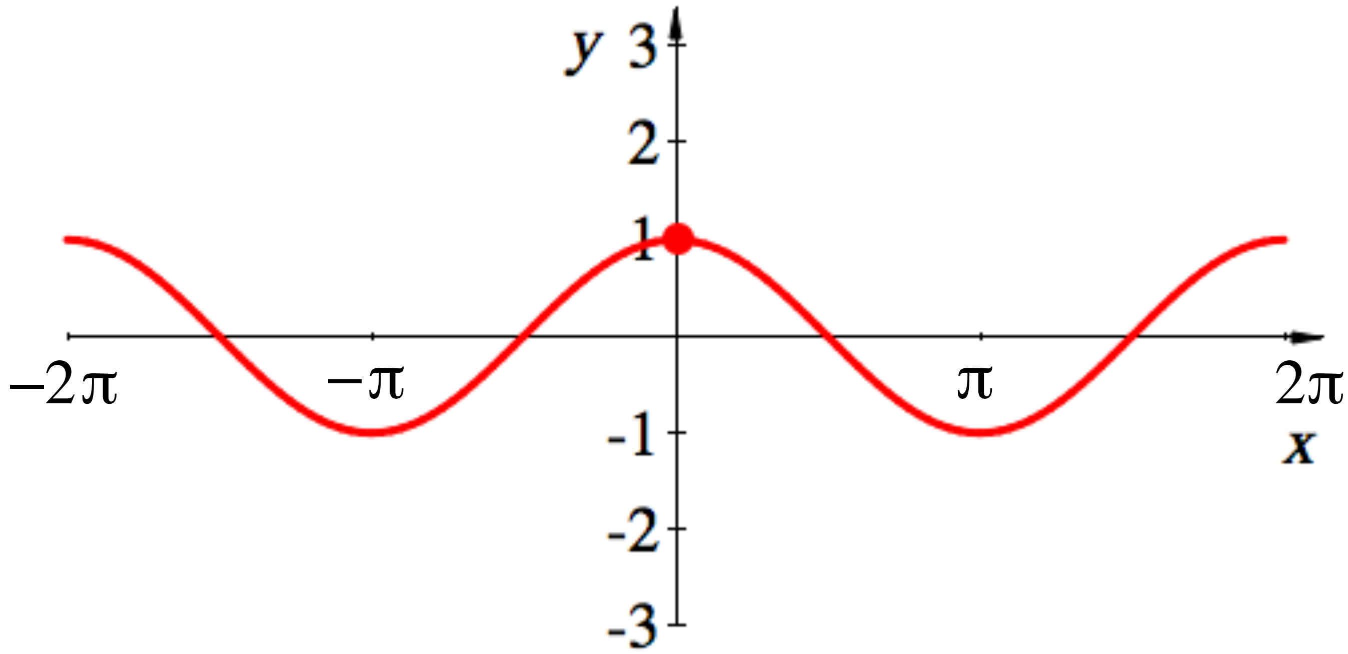
$1 = x^0$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k}$$

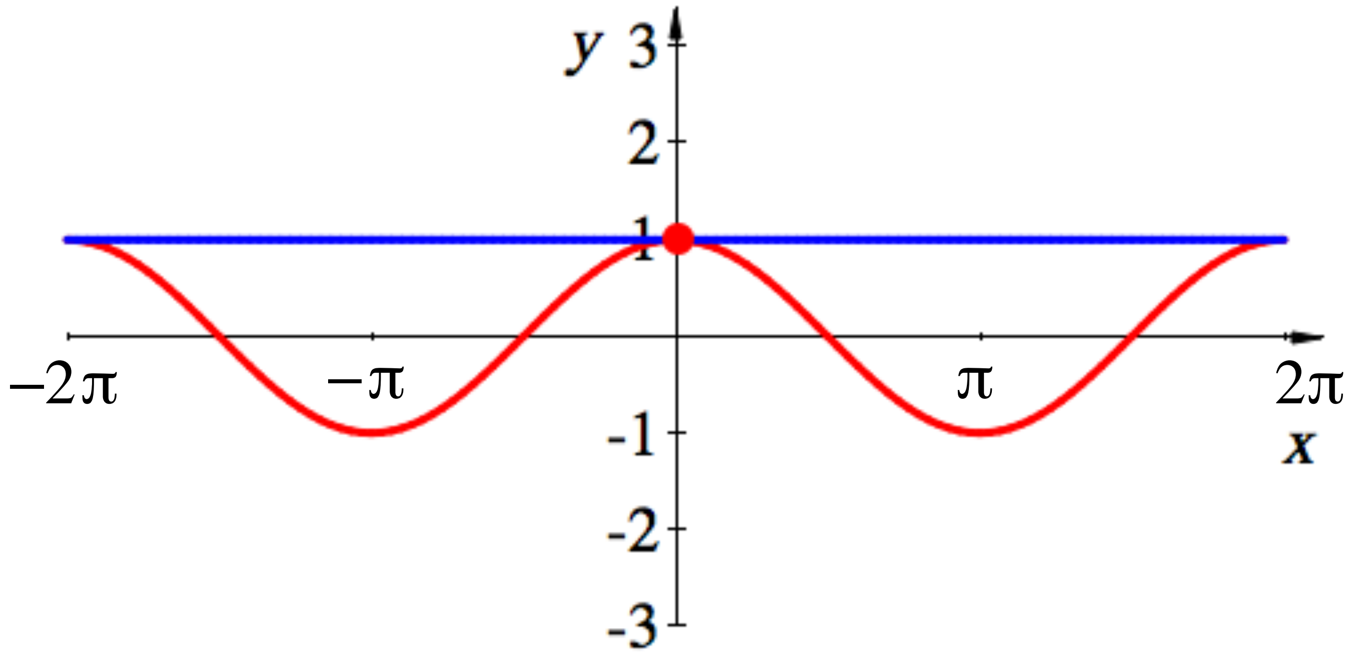
Nur gerade Exponenten

Kosinus ist eine gerade Funktion

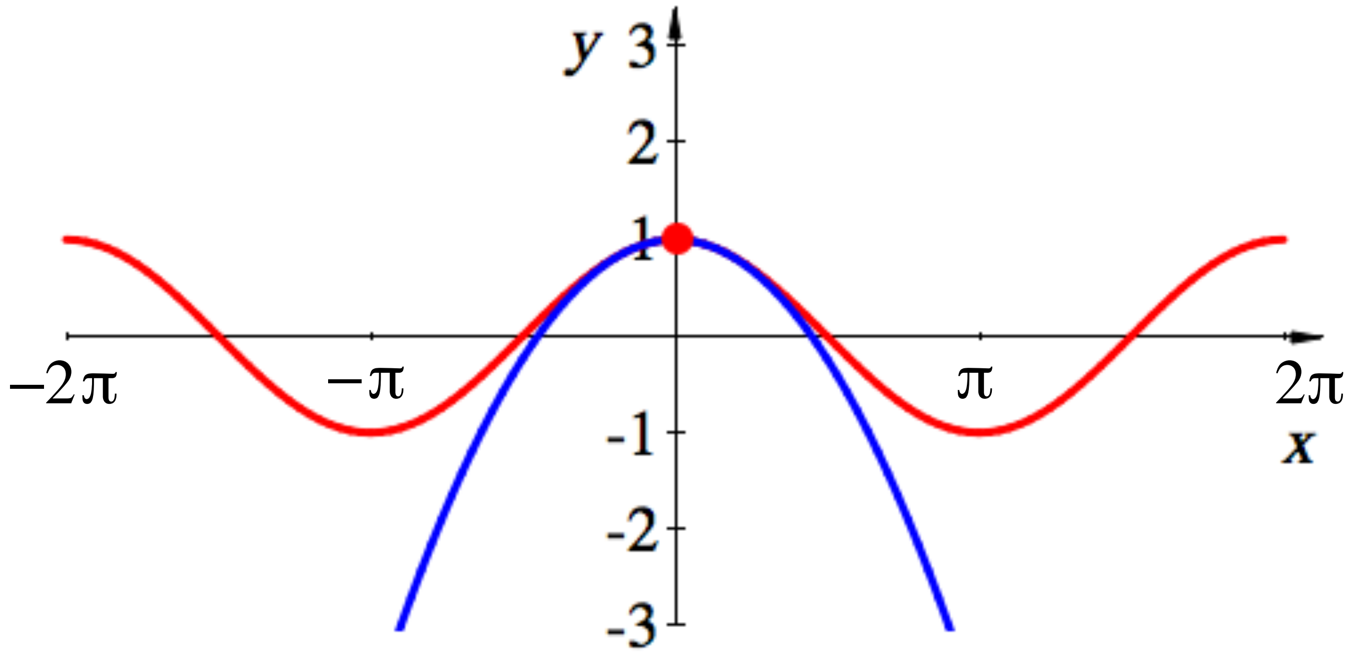
$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \mp \cdots$$



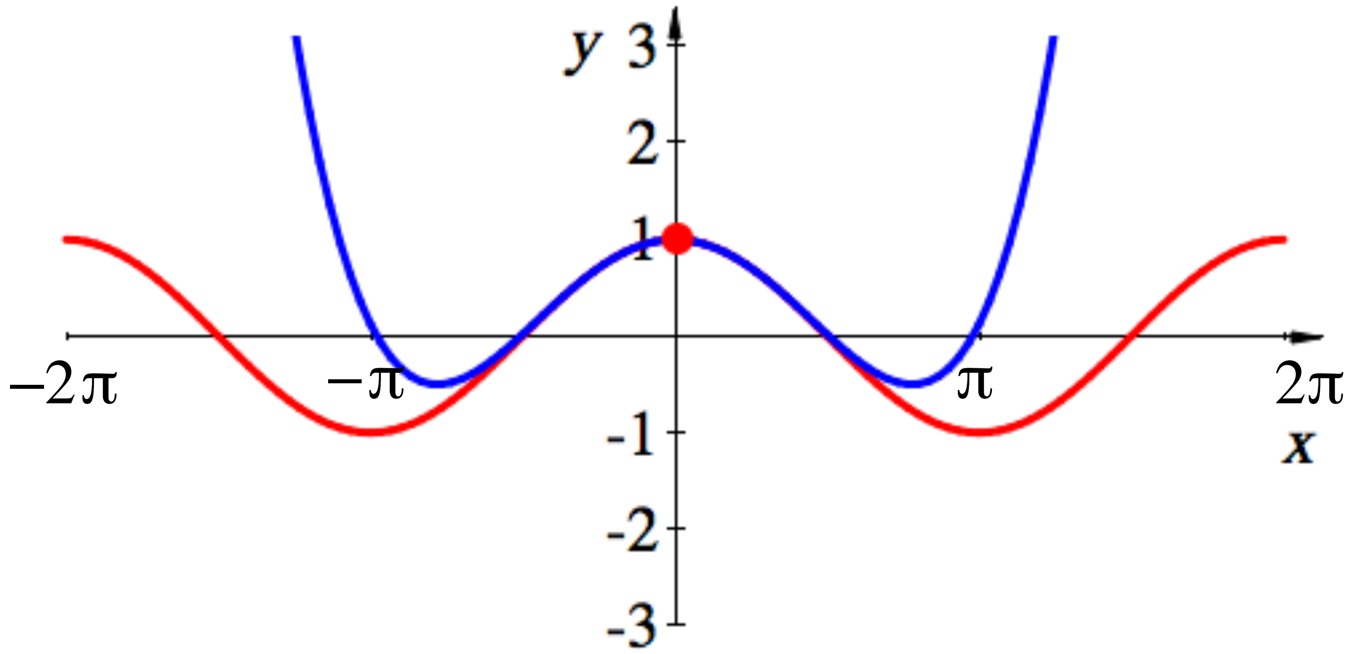
$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \mp \cdots$$



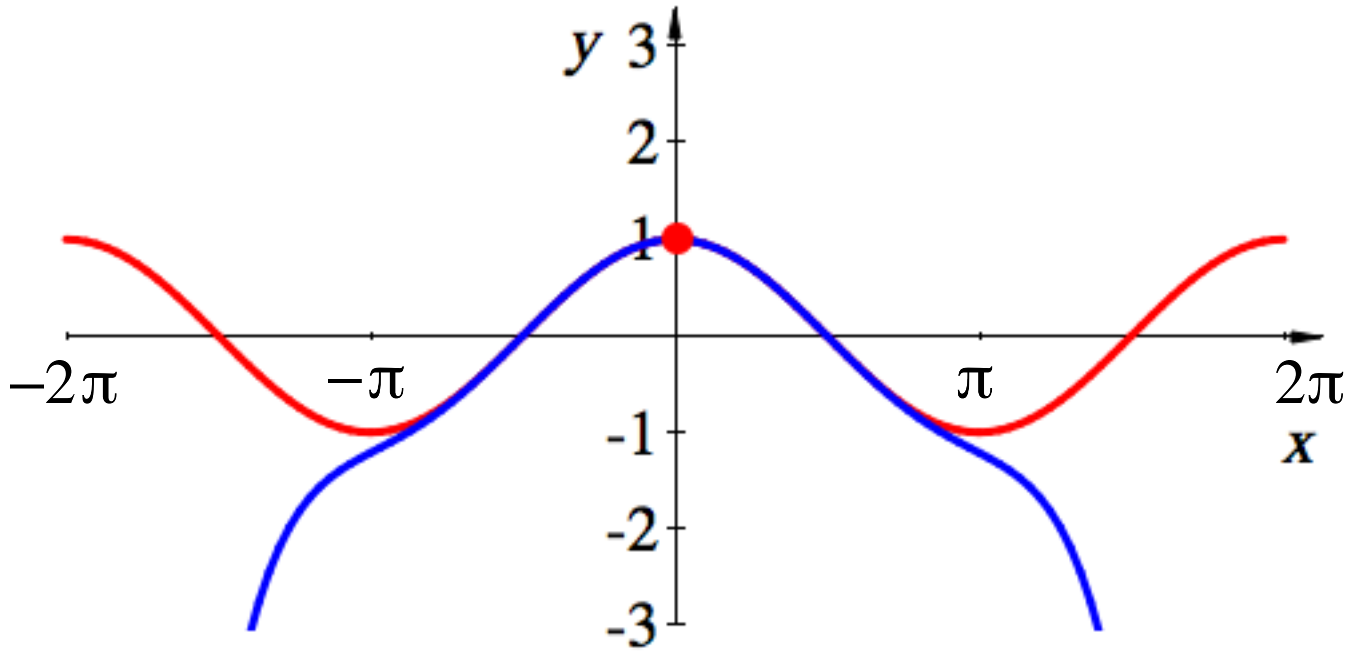
$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \mp \cdots$$



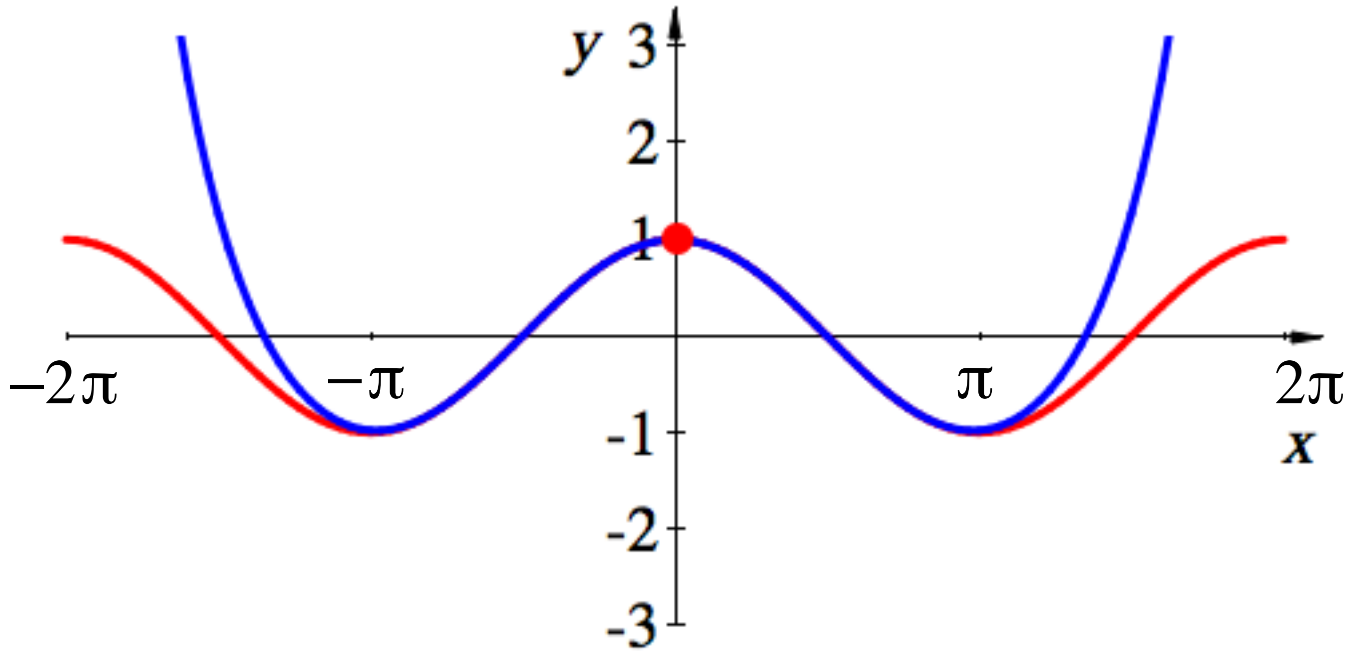
$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \mp \cdots$$



$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \mp \cdots$$



$$\cos(x) \approx 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 \mp \cdots$$



Vergleich

$$\mathrm{e}^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots$$

$$\cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \pm \cdots$$

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \mp \cdots$$

Was ist der Link?

Problem mit Minuszeichen

Erinnerung

$$\mathbb{C}$$

Erinnerung

i so, dass $i^2 = -1$

Menge der komplexen Zahlen

$$\mathbb{C} = \left\{ a + ib \,\middle|\, a, b \in \mathbb{R},\ i^2 = -1 \right\}$$

Potenzen von i

$$i^0 = 1$$

$$i^1 = i$$

Potenzen von i

$$
\begin{aligned}
i^0 &= 1 \\
i^1 &= i \\
i^2 &= -1
\end{aligned}
$$

⇐ Gemäß Definition

Potenzen von i

$$i^0 = 1$$

$$i^1 = i$$

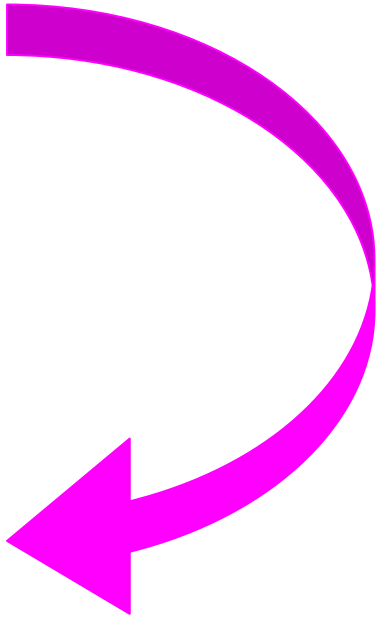
$$i^2 = -1$$

Gemäß Definition

$$i^3 = -i$$

Potenzen von i

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}$$



Periodisches Verhalten
Periodenlänge 4

Potenzen von i

$$i^0 = 1$$

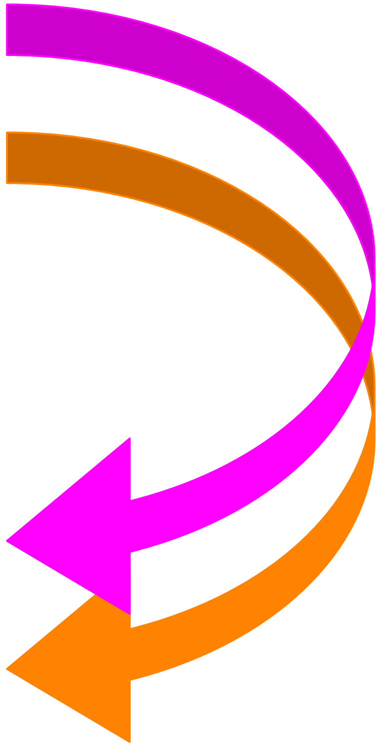
$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

$$i^5 = i$$



Vergleich

$$e^{\mathrm{i}x} = 1 + \mathrm{i}x + \tfrac{1}{2}(\mathrm{i}x)^2 + \tfrac{1}{3!}(\mathrm{i}x)^3 + \tfrac{1}{4!}(\mathrm{i}x)^4 + \tfrac{1}{5!}(\mathrm{i}x)^5 + \tfrac{1}{6!}(\mathrm{i}x)^6 + \cdots$$

Vergleich

$$
\begin{array}{lcccccccccccccccccc}
\hline
e^{ix} & = & 1 & + & ix & + & \frac{1}{2}(ix)^2 & + & \frac{1}{3!}(ix)^3 & + & \frac{1}{4!}(ix)^4 & + & \frac{1}{5!}(ix)^5 & + & \frac{1}{6!}(ix)^6 & + & \cdots \\
\hline
e^{ix} & = & 1 & + & ix & - & \frac{1}{2}x^2 & - & i\frac{1}{3!}x^3 & + & \frac{1}{4!}x^4 & + & i\frac{1}{5!}x^5 & - & \frac{1}{6!}x^6 & \mp & \\
\hline
\end{array}
$$

Vergleich

$$
\begin{array}{lcccccccccccccccccc}
e^{ix} & = & 1 & + & ix & + & \frac{1}{2}(ix)^2 & + & \frac{1}{3!}(ix)^3 & + & \frac{1}{4!}(ix)^4 & + & \frac{1}{5!}(ix)^5 & + & \frac{1}{6!}(ix)^6 & + & \cdots \\
\hline
e^{ix} & = & 1 & + & ix & - & \frac{1}{2}x^2 & - & i\frac{1}{3!}x^3 & + & \frac{1}{4!}x^4 & + & i\frac{1}{5!}x^5 & - & \frac{1}{6!}x^6 & \mp & \\
\hline
\cos(x) & = & 1 & & & - & \frac{1}{2}x^2 & & & + & \frac{1}{4!}x^4 & & & - & \frac{1}{6!}x^6 & \pm & \cdots
\end{array}
$$

Hurra hurra
die Vorzeichen sind da!

Vergleich

$$
\begin{array}{lcccccccccccccccc}
e^{ix} & = & 1 & + & ix & + & \frac{1}{2}(ix)^2 & + & \frac{1}{3!}(ix)^3 & + & \frac{1}{4!}(ix)^4 & + & \frac{1}{5!}(ix)^5 & + & \frac{1}{6!}(ix)^6 & + & \cdots \\
\hline
e^{ix} & = & 1 & + & ix & - & \frac{1}{2}x^2 & - & i\frac{1}{3!}x^3 & + & \frac{1}{4!}x^4 & + & i\frac{1}{5!}x^5 & - & \frac{1}{6!}x^6 & \mp & \\
\hline
\cos(x) & = & 1 & & & - & \frac{1}{2}x^2 & & & + & \frac{1}{4!}x^4 & & & - & \frac{1}{6!}x^6 & \pm & \cdots \\
i\sin(x) & = & & & ix & & & - & i\frac{1}{3!}x^3 & & & + & i\frac{1}{5!}x^5 & & & \mp & \cdots
\end{array}
$$

Faktor i dazu tun

Faktor i dazu tun

Faktor i dazu tun

Faktor i dazu tun

Vergleich

$$\begin{array}{lcccccccccccccccccc}
e^{ix} & = & 1 & + & ix & + & \frac{1}{2}(ix)^2 & + & \frac{1}{3!}(ix)^3 & + & \frac{1}{4!}(ix)^4 & + & \frac{1}{5!}(ix)^5 & + & \frac{1}{6!}(ix)^6 & + & \cdots \\
\hline
e^{ix} & = & 1 & + & ix & - & \frac{1}{2}x^2 & - & i\frac{1}{3!}x^3 & + & \frac{1}{4!}x^4 & + & i\frac{1}{5!}x^5 & - & \frac{1}{6!}x^6 & \mp & \\
\hline
\cos(x) & = & 1 & & & - & \frac{1}{2}x^2 & & & + & \frac{1}{4!}x^4 & & & - & \frac{1}{6!}x^6 & \pm & \cdots \\
i\sin(x) & = & & & ix & & & - & i\frac{1}{3!}x^3 & & & + & i\frac{1}{5!}x^5 & & & \mp & \cdots
\end{array}$$

$$\boxed{e^{ix} = \cos(x) + i\ \sin(x)}$$

Formel von Euler

$$e^{ix} = \cos(x) + i \sin(x)$$

Formel von Euler

Sonderfall: $x = 2\pi$

$$e^{ix} = \cos(x) + i\ \sin(x)$$

Formel von Euler

Sonderfall: $x = 2\pi$

$$e^{2\pi i} = \underbrace{\cos(2\pi)}_{1} + i\ \underbrace{\sin(2\pi)}_{0}$$

$$e^{ix} = \cos(x) + i\ \sin(x)$$

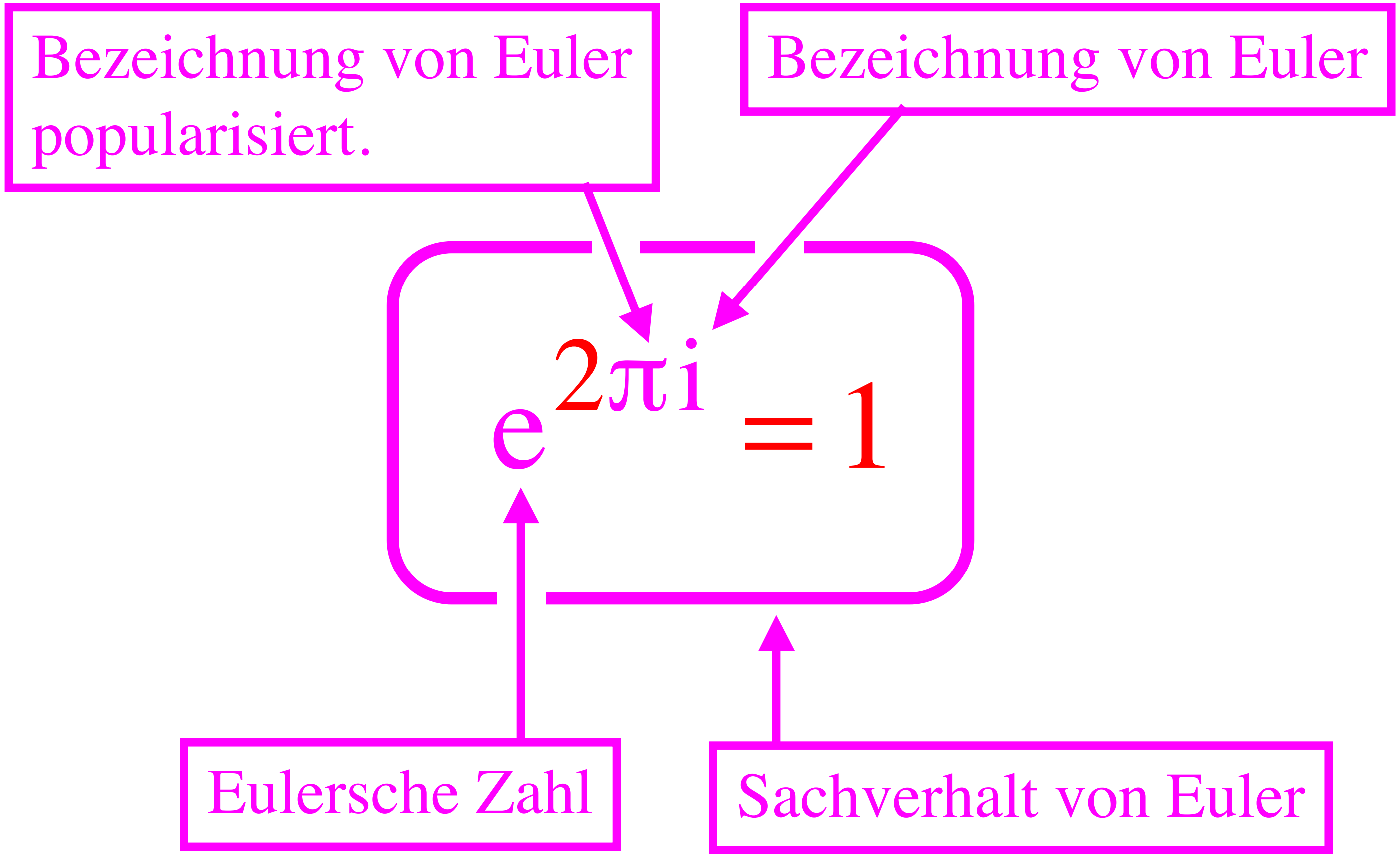
Formel von Euler

Sonderfall: $x = 2\pi$

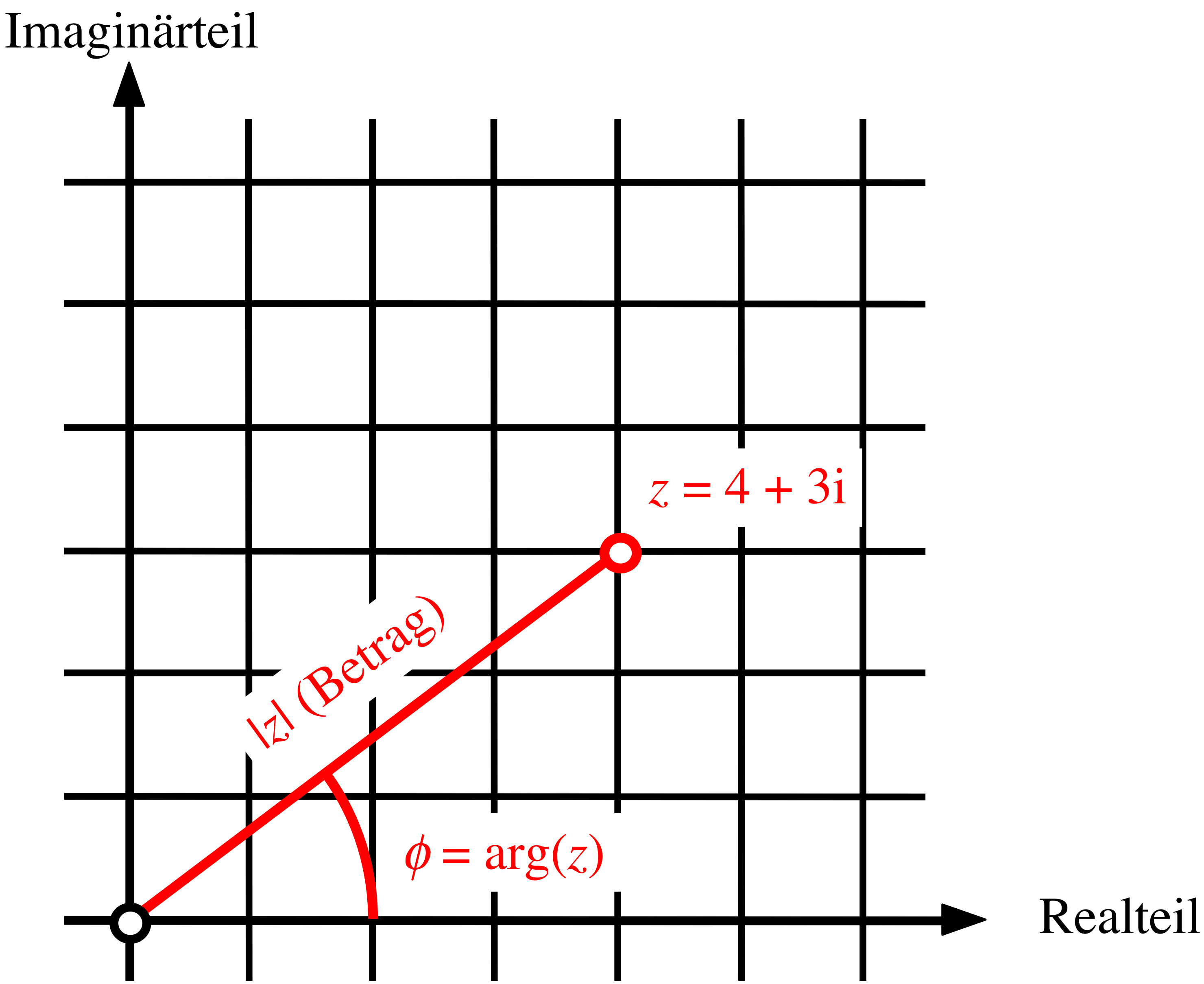
$$e^{2\pi i} = \underbrace{\cos(2\pi)}_{1} + i\ \underbrace{\sin(2\pi)}_{0}$$

$$e^{2\pi i} = 1$$

Die schönste Formel



Die Geometrie der Sache


Imaginärteil
$z = 4 + 3i$
$|z|$ (Betrag)
$\phi = \arg(z)$
Realteil

Argument und Betrag
„Polarkoordinaten“

$$\tan\left(\arg(z)\right) = \tan(\phi) = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$$

$$|z| = \sqrt{\left(\mathrm{Re}(z)\right)^2 + \left(\mathrm{Im}(z)\right)^2} = \sqrt{z\,\bar{z}}$$

$$\mathrm{Re}(z) = |z|\cos(\phi)$$
$$\mathrm{Im}(z) = |z|\sin(\phi)$$

$$\mathrm{Re}(z) = |z|\cos(\phi) \qquad \mathrm{Im}(z) = |z|\sin(\phi)$$

$$\mathrm{Re}(z) = |z|\cos(\phi) \qquad \mathrm{Im}(z) = |z|\sin(\phi)$$

$$z = \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z) = |z|\big(\cos(\phi) + \mathrm{i}\,\sin(\phi)\big)$$

$$\mathrm{Re}(z) = |z|\cos(\phi) \qquad \mathrm{Im}(z) = |z|\sin(\phi)$$

$$z = \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z) = |z|\left(\cos(\phi) + \mathrm{i}\sin(\phi)\right)$$

Euler: $\mathrm{e}^{\mathrm{i}\phi} = \cos(\phi) + \mathrm{i}\sin(\phi)$

$$\mathrm{Re}(z) = |z|\cos(\phi) \qquad \mathrm{Im}(z) = |z|\sin(\phi)$$

$$z = \mathrm{Re}(z) + \mathrm{i}\ \mathrm{Im}(z) = |z|\left(\cos(\phi) + \mathrm{i}\ \sin(\phi)\right)$$

Euler: $\mathrm{e}^{\mathrm{i}\phi} = \cos(\phi) + \mathrm{i}\ \sin(\phi)$

$$z = \mathrm{Re}(z) + \mathrm{i}\ \mathrm{Im}(z) = |z|\left(\cos(\phi) + \mathrm{i}\ \sin(\phi)\right) = |z|\mathrm{e}^{\mathrm{i}\phi}$$

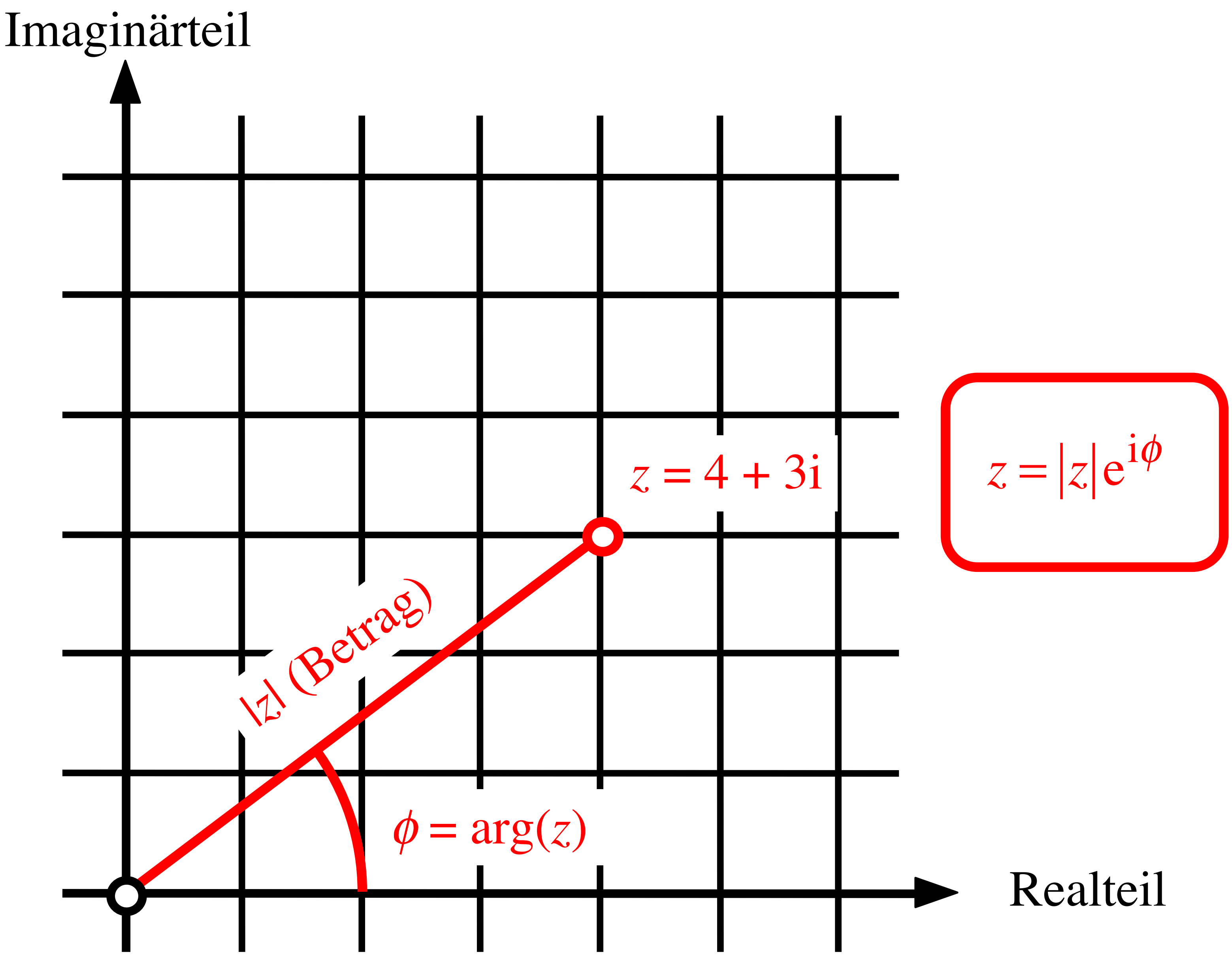
$$\mathrm{Re}(z) = |z|\cos(\phi) \qquad \mathrm{Im}(z) = |z|\sin(\phi)$$

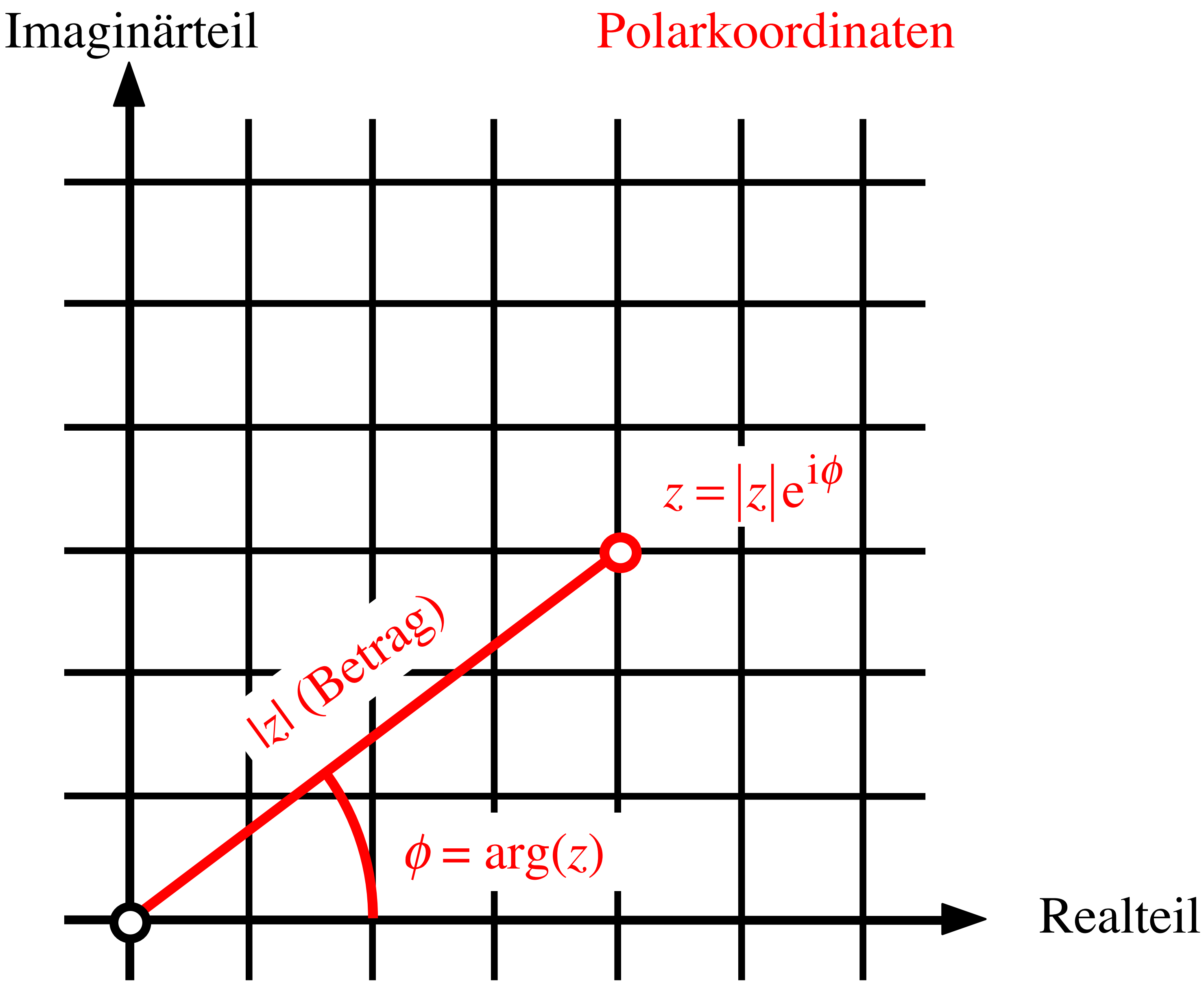
$$z = \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z) = |z|\big(\cos(\phi) + \mathrm{i}\sin(\phi)\big)$$

Euler: $\mathrm{e}^{\mathrm{i}\phi} = \cos(\phi) + \mathrm{i}\sin(\phi)$

$$z = \mathrm{Re}(z) + \mathrm{i}\,\mathrm{Im}(z) = |z|\big(\cos(\phi) + \mathrm{i}\sin(\phi)\big) = |z|\mathrm{e}^{\mathrm{i}\phi}$$

$$\boxed{z = |z|\mathrm{e}^{\mathrm{i}\phi}}$$


Imaginärteil
Realteil
$z = 4 + 3\mathrm{i}$
$|z|$ (Betrag)
$\phi = \arg(z)$
$z = |z|\mathrm{e}^{\mathrm{i}\phi}$


Imaginärteil
Polarkoordinaten
$z = |z|e^{i\phi}$
|z| (Betrag)
$\phi = \arg(z)$
Realteil

Multiplikation

$$z = |z| e^{i\phi} \qquad w = |w| e^{i\psi}$$

Multiplikation

$$z = |z|\mathrm{e}^{\mathrm{i}\phi} \qquad w = |w|\mathrm{e}^{\mathrm{i}\psi}$$

$$zw = |z|\mathrm{e}^{\mathrm{i}\phi}\,|w|\mathrm{e}^{\mathrm{i}\psi} = |z||w|\mathrm{e}^{\mathrm{i}(\phi+\psi)}$$

Multiplikation

$$z = |z| \mathrm{e}^{\mathrm{i}\phi} \qquad w = |w| \mathrm{e}^{\mathrm{i}\psi}$$

$$zw = |z| \mathrm{e}^{\mathrm{i}\phi} |w| \mathrm{e}^{\mathrm{i}\psi} = |z||w| \mathrm{e}^{\mathrm{i}(\phi+\psi)}$$

Andererseits:

$$zw = |zw| \mathrm{e}^{\mathrm{i}\,\arg(zw)}$$

Multiplikation

$$z = |z|\mathrm{e}^{\mathrm{i}\phi} \qquad w = |w|\mathrm{e}^{\mathrm{i}\psi}$$

$$zw = |z|\mathrm{e}^{\mathrm{i}\phi}|w|\mathrm{e}^{\mathrm{i}\psi} = |z||w|\mathrm{e}^{\mathrm{i}(\phi+\psi)}$$

Andererseits:

$$zw = |zw|\mathrm{e}^{\mathrm{i}\arg(zw)}$$

Vergleich:

$$|zw| = |z||w|$$

$$\arg(zw) = \arg(z) + \arg(w)$$

Multiplikation

$\lvert zw \rvert = \lvert z \rvert \lvert w \rvert$	Der Betrag des Produktes ist gleich dem Produkt der Beträge der Faktoren.
$\arg(zw) = \arg(z) + \arg(w)$	Das Argument des Produktes ist gleich der *Summe* der Argumente der Faktoren

Division

$$\left| \frac{z}{w} \right| = \frac{\lvert z \rvert}{\lvert w \rvert}$$

$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$$

Multiplikation

$$|zw| = |z||w|$$

Der Betrag des Produktes ist gleich dem Produkt der Beträge der Faktoren.

$$\arg(zw) = \arg(z) + \arg(w)$$

Das Argument des Produktes ist gleich der *Summe* der Argumente der Faktoren

Beispiel: $-1 = e^{i\pi}$

Multiplikation

$$|zw| = |z||w|$$

Der Betrag des Produktes ist gleich dem Produkt der Beträge der Faktoren.

$$\arg(zw) = \arg(z) + \arg(w)$$

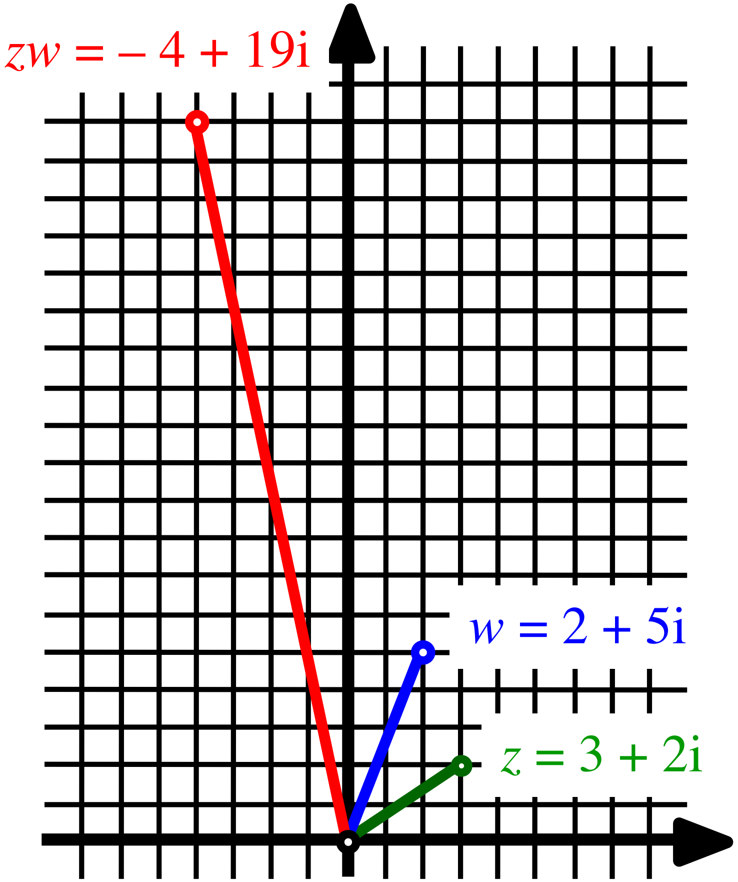
Das Argument des Produktes ist gleich der *Summe* der Argumente der Faktoren

Beispiel:

$$-1 = \mathrm{e}^{\mathrm{i}\pi}$$

$$(-1)(-1) = \mathrm{e}^{\mathrm{i}\pi}\mathrm{e}^{\mathrm{i}\pi} = \mathrm{e}^{\mathrm{i}\pi+\mathrm{i}\pi} = \mathrm{e}^{2\pi\mathrm{i}} = 1$$

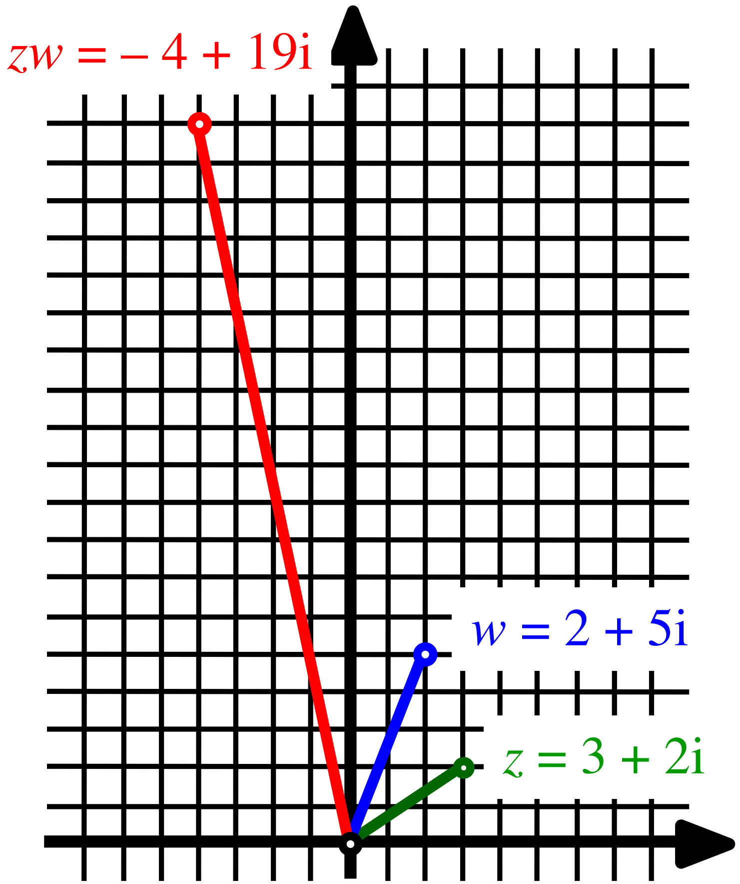
Beispiel:



$z = 3 + 2i \qquad w = 2 + 5i$

$\Rightarrow \quad zw = -4 + 19i$

Beispiel:



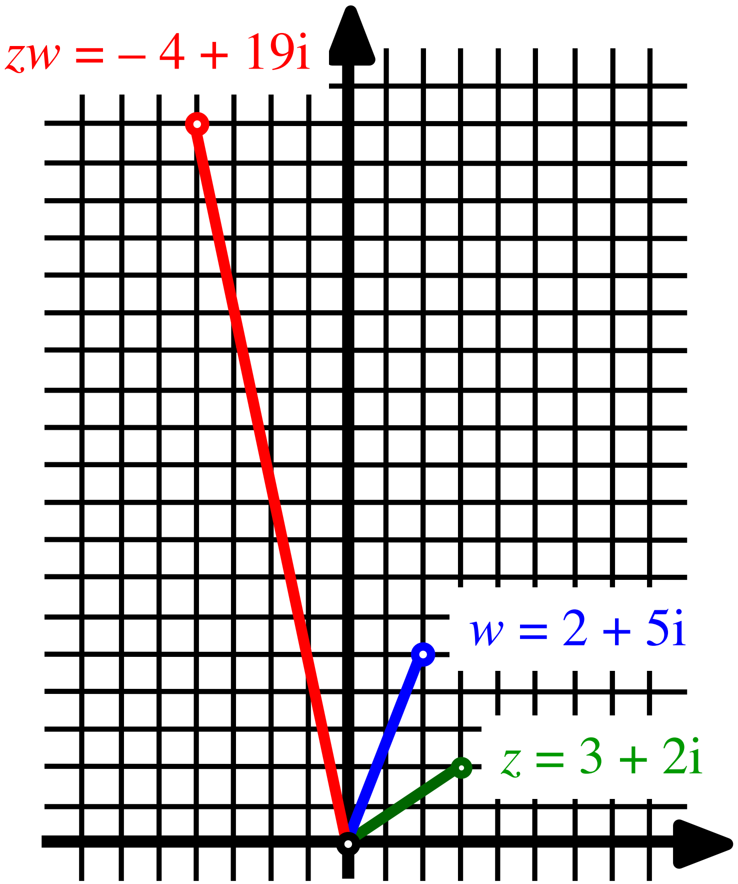
$$z = 3 + 2\mathrm{i} \qquad w = 2 + 5\mathrm{i}$$

$$\Rightarrow \quad zw = -4 + 19\mathrm{i}$$

$$|z| = \sqrt{13}, \quad |w| = \sqrt{29}$$

und $\quad |zw| = \sqrt{377} = \sqrt{13}\sqrt{29}$

Beispiel:



$z = 3 + 2i \qquad w = 2 + 5i$

$\Rightarrow \quad zw = -4 + 19i$

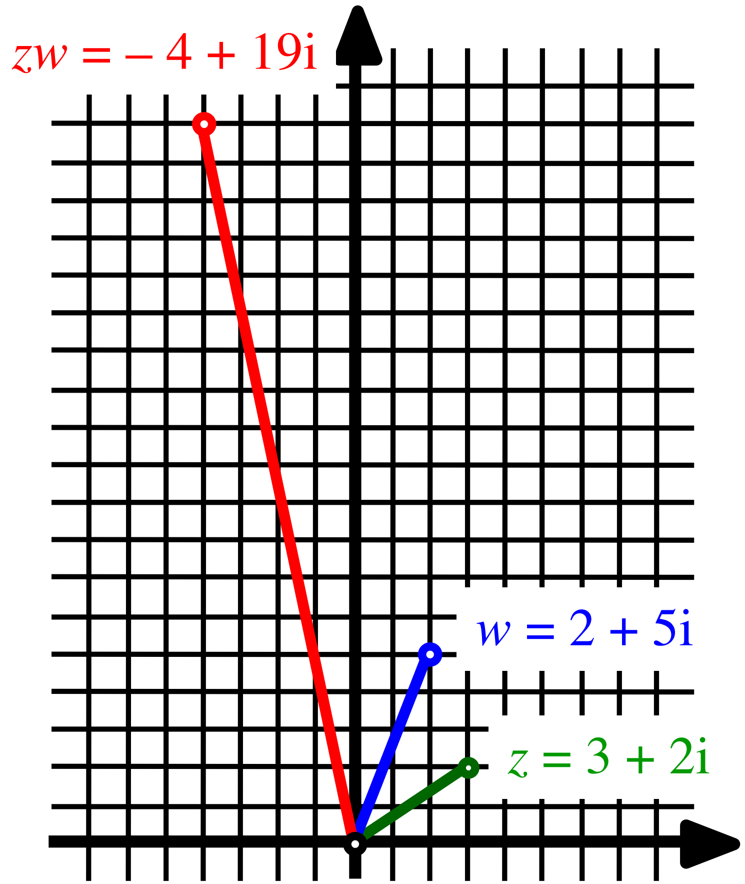
$|z| = \sqrt{13}, \quad |w| = \sqrt{29}$

und $\quad |zw| = \sqrt{377} = \sqrt{13}\sqrt{29}$

$\arg(z) = \arctan\left(\frac{2}{3}\right) \approx 0.588 \approx 33.69°$

$\arg(w) = \arctan\left(\frac{5}{2}\right) \approx 1.190 \approx 68.20°$

Beispiel:



$$z = 3 + 2i \qquad w = 2 + 5i$$

$$\Rightarrow \quad zw = -4 + 19i$$

$$|z| = \sqrt{13}, \quad |w| = \sqrt{29}$$

und $|zw| = \sqrt{377} = \sqrt{13}\sqrt{29}$

$$\arg(z) = \arctan\left(\frac{2}{3}\right) \approx 0.588 \approx 33.69°$$

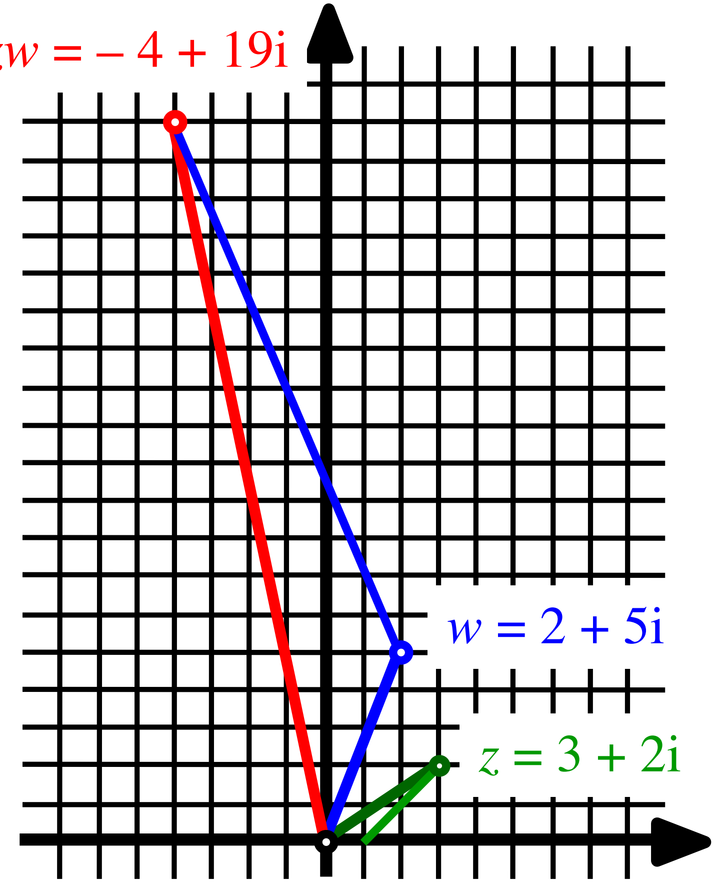
$$\arg(w) = \arctan\left(\frac{5}{2}\right) \approx 1.190 \approx 68.20°$$

und

$$\arg(zw) = \arctan\left(\frac{19}{-4}\right) + \pi \approx 1.778 \approx 101.89°$$

Beispiel:

Drehstreckung mit $w = 2 + 5\mathrm{i}$



Faktor:

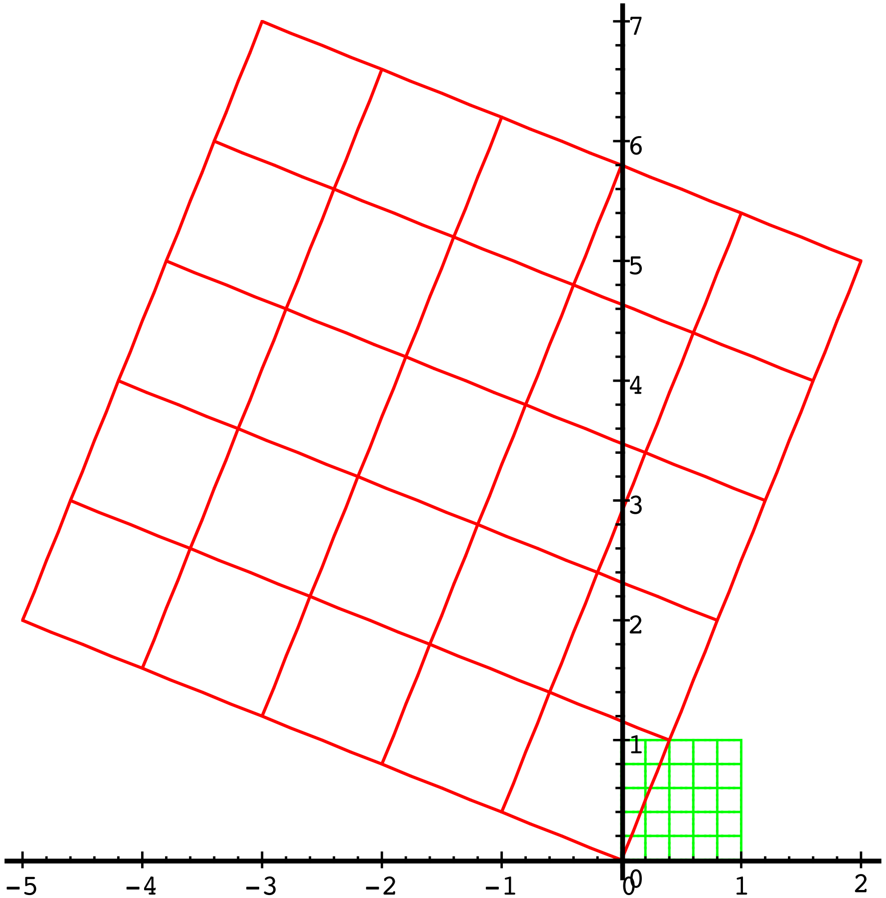
$$|w| = \sqrt{29} \approx 5.39$$

Drehwinkel:

$$\arg(w) = \arctan\left(\frac{5}{2}\right) \approx 1.190 \approx 68.20°$$

Beispiel:

Drehstreckung mit $w = 2 + 5i$



Potenzen und Wurzeln

$$\left|z^n\right| = |z|^n$$

$$\arg\left(z^n\right) = n\arg(z)$$

$$\left|\sqrt[n]{z}\right| = \sqrt[n]{|z|}$$

$$\arg\left(\sqrt[n]{z}\right) = \frac{1}{n}\arg(z)$$

Einheitswurzeln

$$z^n = 1 \quad \Leftrightarrow \quad z^n - 1 = 0$$

Lösungen?

Einheitswurzeln

$$z^n = 1 \quad \Leftrightarrow \quad z^n - 1 = 0$$

Lösungen?

Reelle Lösungen:

± 1 falls n gerade

$+1$ falls n ungerade

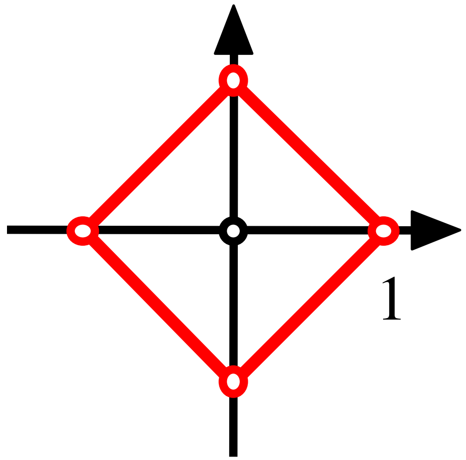
Einheitswurzeln

$$z^2 - 1 = 0 \quad \text{Lösungen} \quad \{-1, 1\}$$

Einheitswurzeln

$$z^2 - 1 = 0 \quad \text{Lösungen} \quad \{-1, 1\}$$

$$z^4 - 1 = 0 \quad \text{Lösungen} \quad \{\mathrm{i}, -1, -\mathrm{i}, 1\}$$



Einheitswurzeln

$$z^3 - 1 = 0$$

$$z^3 - 1 = (z-1)\left(z^2 + z + 1\right) = 0 \quad \Rightarrow \quad z_3 = 1$$

Nummer 3

Einheitswurzeln

$$z^3 - 1 = 0$$

$$z^3 - 1 = (z-1)\left(z^2 + z + 1\right) = 0 \quad \Rightarrow \quad z_3 = 1$$

$$z^2 + z + 1 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$$

Einheitswurzeln

$$z^3 - 1 = 0$$

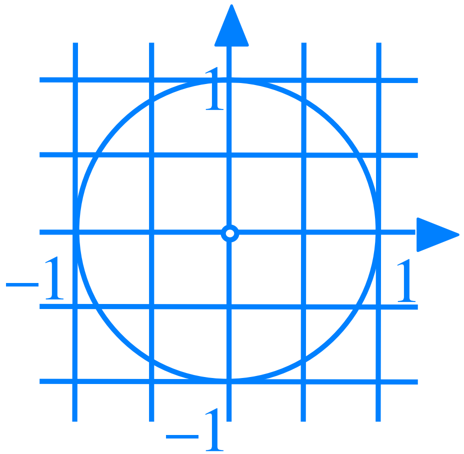
$$z^3 - 1 = (z-1)\left(z^2 + z + 1\right) = 0 \quad \Rightarrow \quad z_3 = 1$$

$$z^2 + z + 1 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \mathrm{i}\frac{\sqrt{3}}{2}$$

$$\text{Lösungsmenge} = \left\{ -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}, \quad 1 \right\}$$

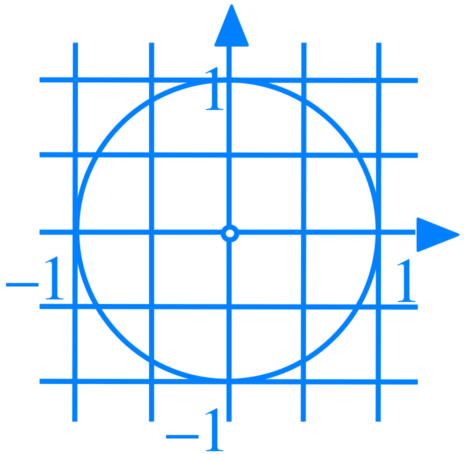
Einheitswurzeln

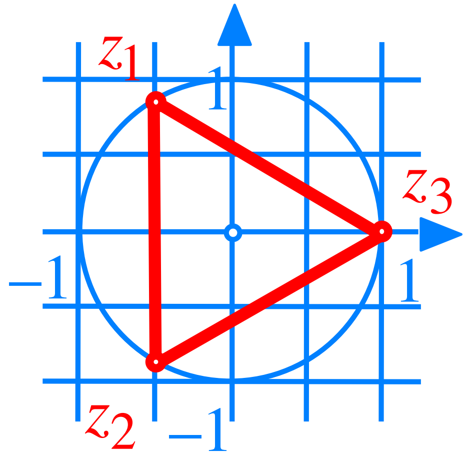
$$z^3 - 1 = 0 \qquad \text{Lösungsmenge} = \left\{ -\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}, \quad 1 \right\}$$



Einheitswurzeln

$$z^3 - 1 = 0 \qquad \text{Lösungsmenge} = \left\{ -\frac{1}{2} + \mathrm{i}\,\frac{\sqrt{3}}{2}, \quad -\frac{1}{2} - \mathrm{i}\,\frac{\sqrt{3}}{2}, \quad 1 \right\}$$





$$\text{Argumente: } \left\{ \frac{2\pi}{3}, \quad \frac{4\pi}{3}, \quad 0 \right\}$$

Einheitswurzeln

$$z^n = 1$$

Lösungen:

$$z_k = \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}} = \cos\left(k\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(k\frac{2\pi}{n}\right), \quad k \in \{1, 2, \ldots, n\}$$

Einheitswurzeln

$$z^n = 1$$

Lösungen:

$$z_k = \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}} = \cos\left(k\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(k\frac{2\pi}{n}\right), \quad k \in \{1,2,\ldots,n\}$$

Kontrolle:

$$z_k^n = \left(\mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}\right)^n = \mathrm{e}^{\mathrm{i}2k\pi} = \left(\mathrm{e}^{\mathrm{i}2\pi}\right)^k = 1^k = 1$$

Einheitswurzeln

$$z^n = 1$$

Lösungen:

$$z_k = e^{ik\frac{2\pi}{n}} = \cos\left(k\frac{2\pi}{n}\right) + i\sin\left(k\frac{2\pi}{n}\right), \quad k \in \{1, 2, \ldots, n\}$$

Regelmäßiges *n*-Eck
Eine Ecke bei 1

Beispiel

$z^5 = 32$

Beispiel

$$z^5 = 32$$

Lösungen:

$$z_k = 2\mathrm{e}^{\mathrm{i}k\frac{2\pi}{5}} = 2\left(\cos\left(k\,\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(k\,\frac{2\pi}{5}\right)\right), \quad k \in \{1,2,\ldots,5\}$$

Beispiel

$$z^5 = 32$$

Lösungen:

$$z_k = 2\mathrm{e}^{\mathrm{i}k\frac{2\pi}{5}} = 2\left(\cos\left(k\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(k\frac{2\pi}{5}\right)\right), \quad k \in \{1,2,\ldots,5\}$$

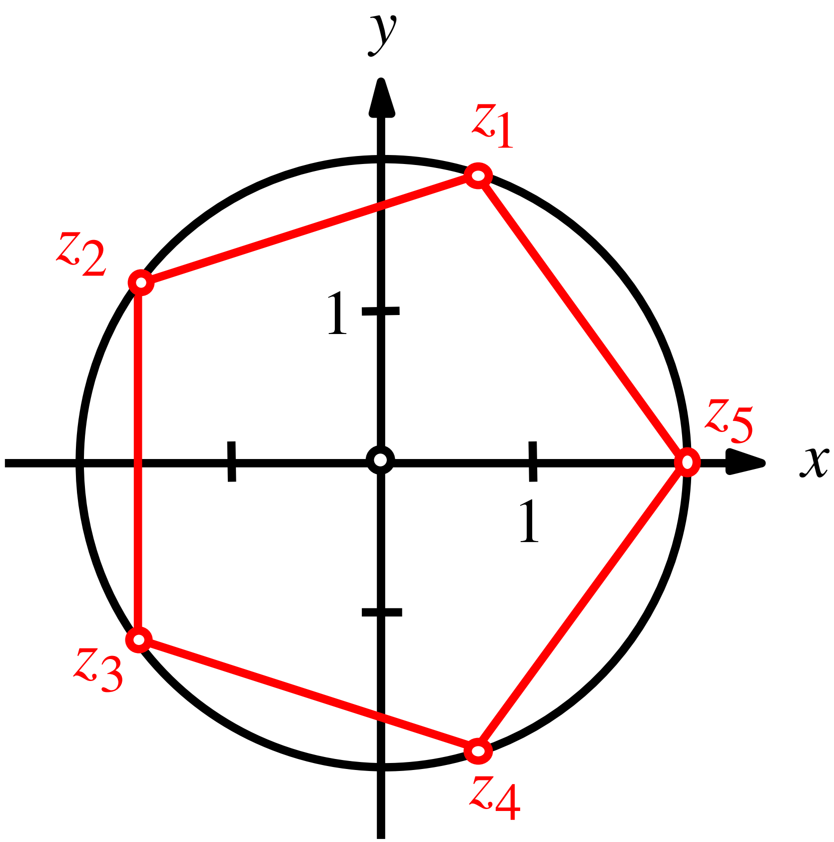
Lösungsmenge =

$$= \left\{z_1 = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{5}}, z_2 = 2\mathrm{e}^{\mathrm{i}2\frac{2\pi}{5}}, z_3 = 2\mathrm{e}^{\mathrm{i}3\frac{2\pi}{5}}, z_4 = 2\mathrm{e}^{\mathrm{i}4\frac{2\pi}{5}}, z_5 = 2\mathrm{e}^{\mathrm{i}5\frac{2\pi}{5}}\right\}$$

Regelmäßiges 5-Eck
Eine Ecke bei 2

Beispiel

$$z^5 = 32$$



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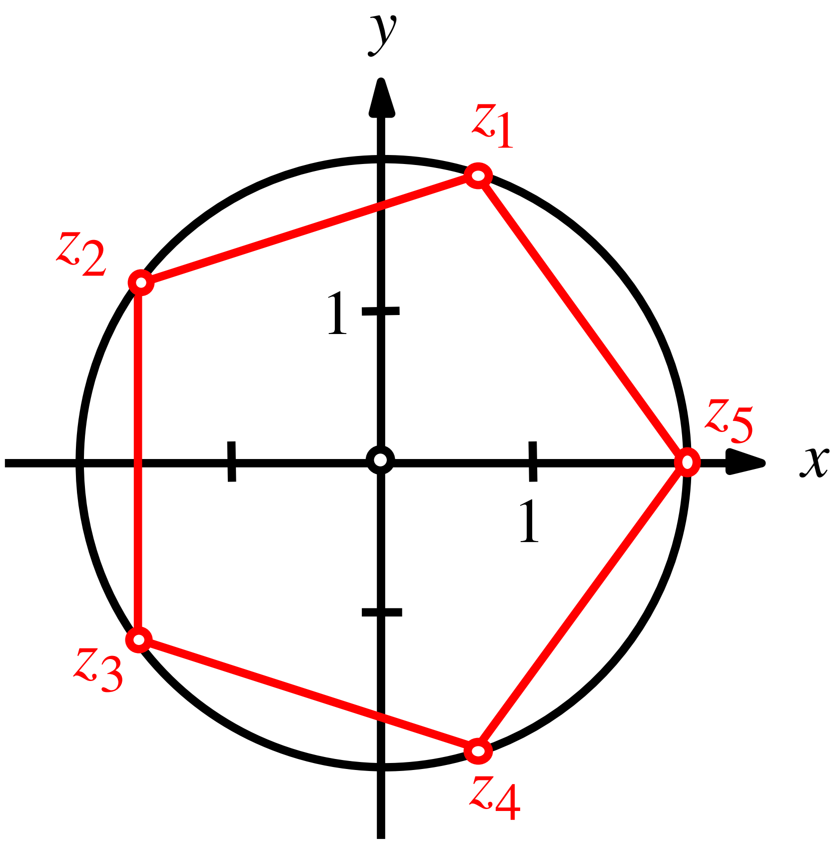

Eine reelle Lösung,
zwei Paare konjugiert komplexer Lösungen