

Modul 107 Fixpunkte

Fixpunkt:  
da läuft nichts mehr

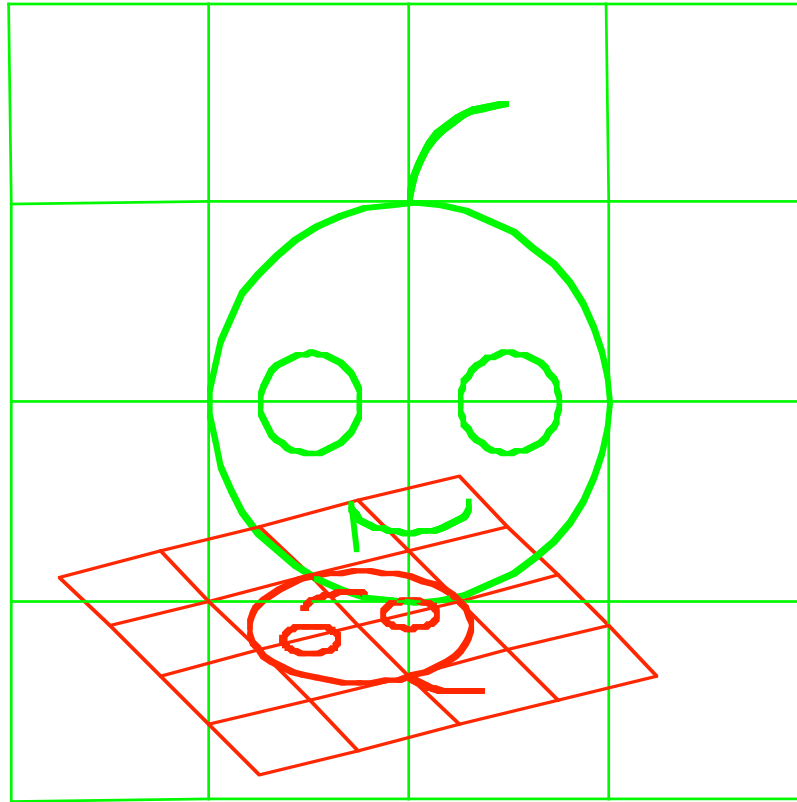
Fixpunkt:  
da läuft nichts mehr

$$f \left( \underset{\substack{\uparrow \\ \text{input}}}{x} \right) = \underset{\substack{\uparrow \\ \text{output}}}{x}$$

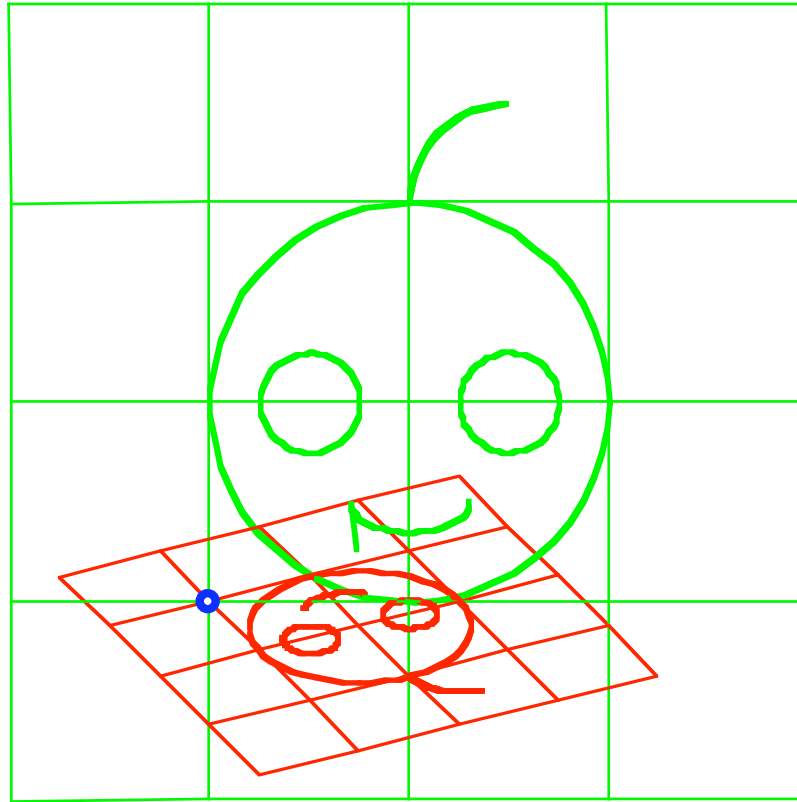
Fixpunkt:  
da läuft nichts mehr

$$\begin{array}{ccc} f & (x) & = & x \\ & \uparrow & & \uparrow \\ & \text{input} & & \text{output} \end{array}$$

gigo: garbage in, garbage out



Fixpunkt?

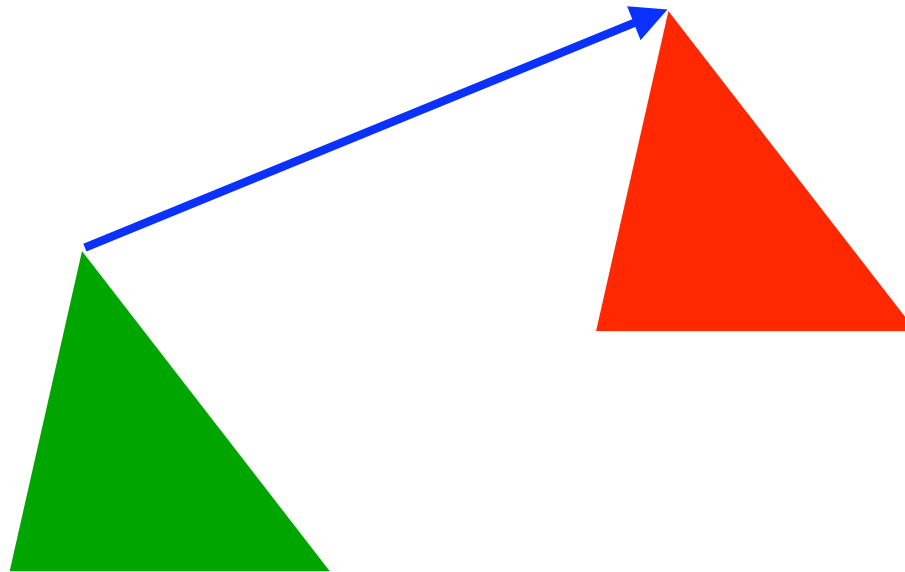


Fixpunkt?

# Erinnerungen an die Schulgeometrie

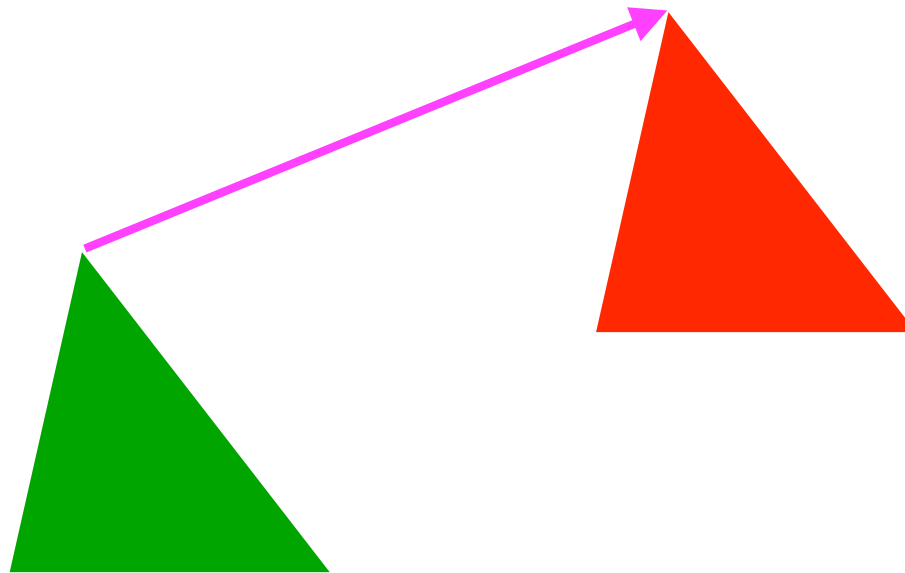


# Erinnerungen an die Schulgeometrie



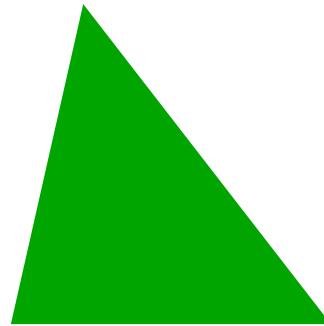


## Erinnerungen an die Schulgeometrie

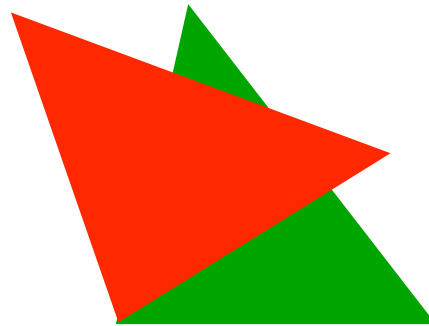


Translation: *kein* Fixpunkt

# Erinnerungen an die Schulgeometrie

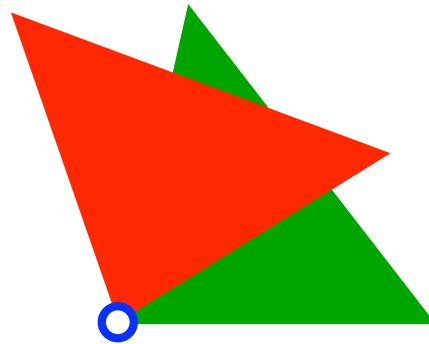


# Erinnerungen an die Schulgeometrie



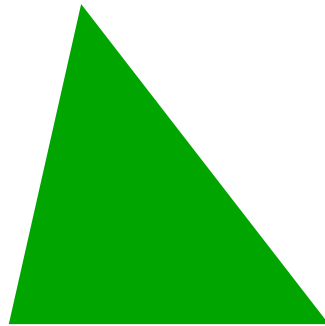
Drehung:

## Erinnerungen an die Schulgeometrie

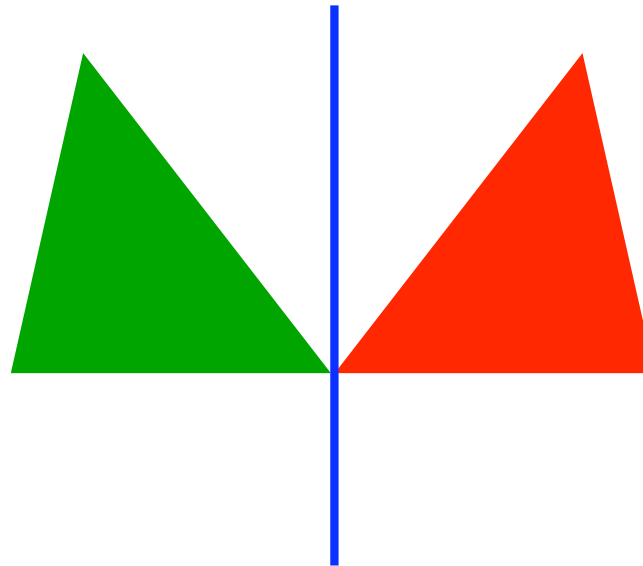


Drehung: Drehzentrum ist Fixpunkt

# Erinnerungen an die Schulgeometrie

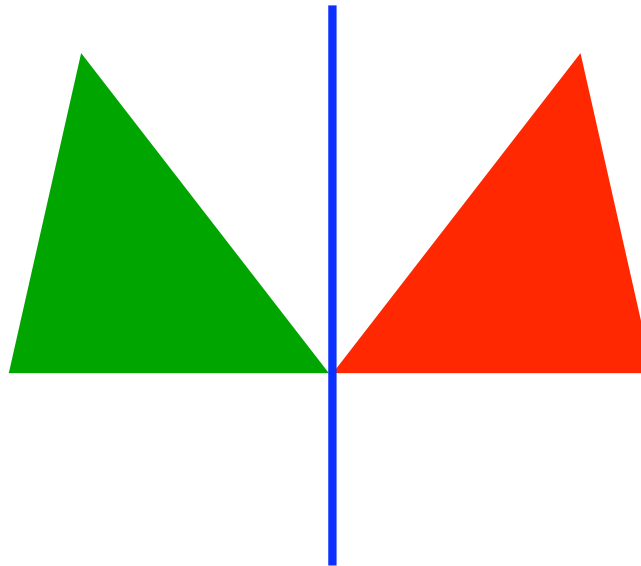


# Erinnerungen an die Schulgeometrie



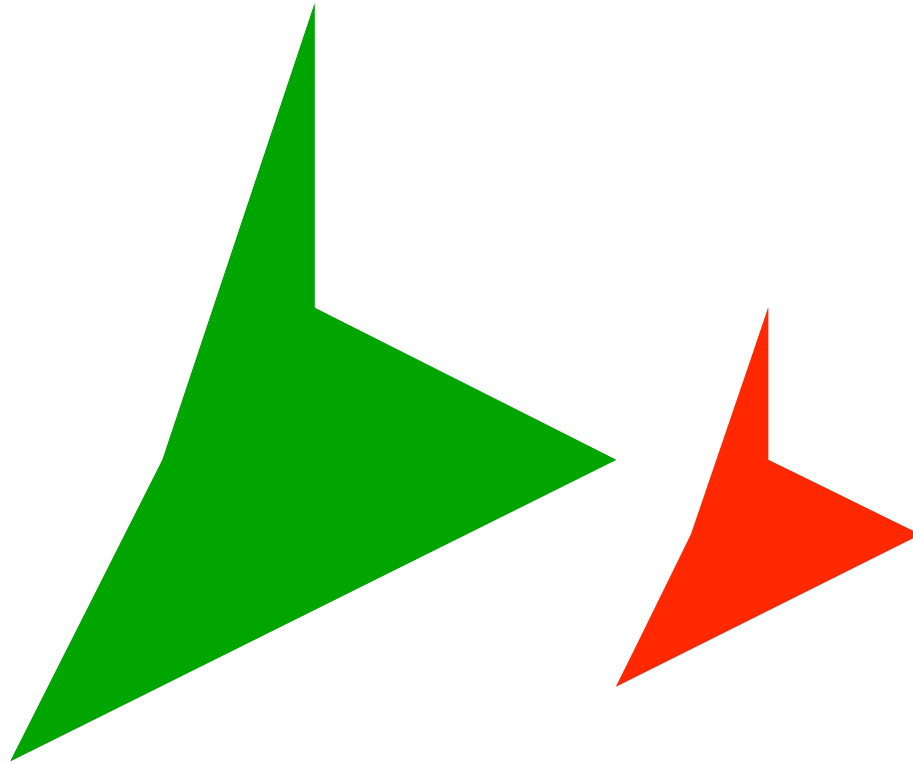
Geradenspiegelung:

## Erinnerungen an die Schulgeometrie



Geradenspiegelung: **Spiegelachse ist Fixpunktgerade**

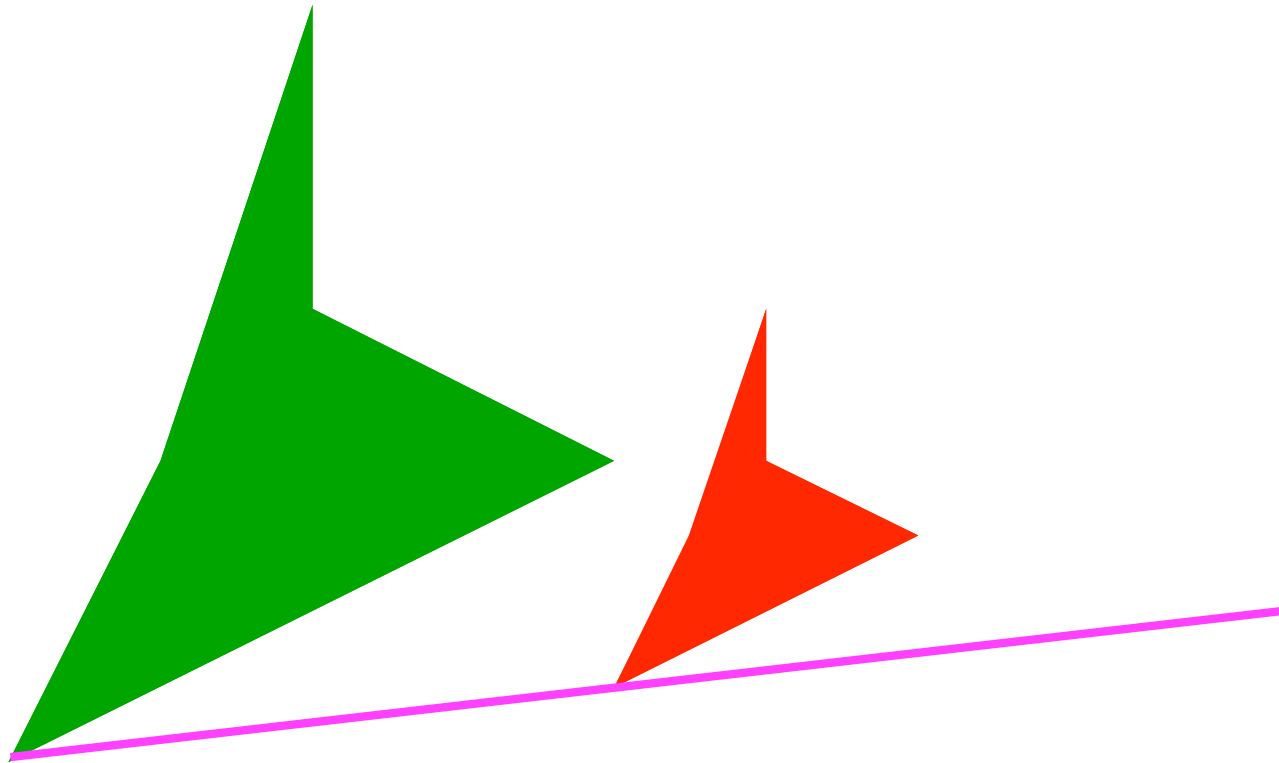
# Erinnerungen an die Schulgeometrie



Zoom:

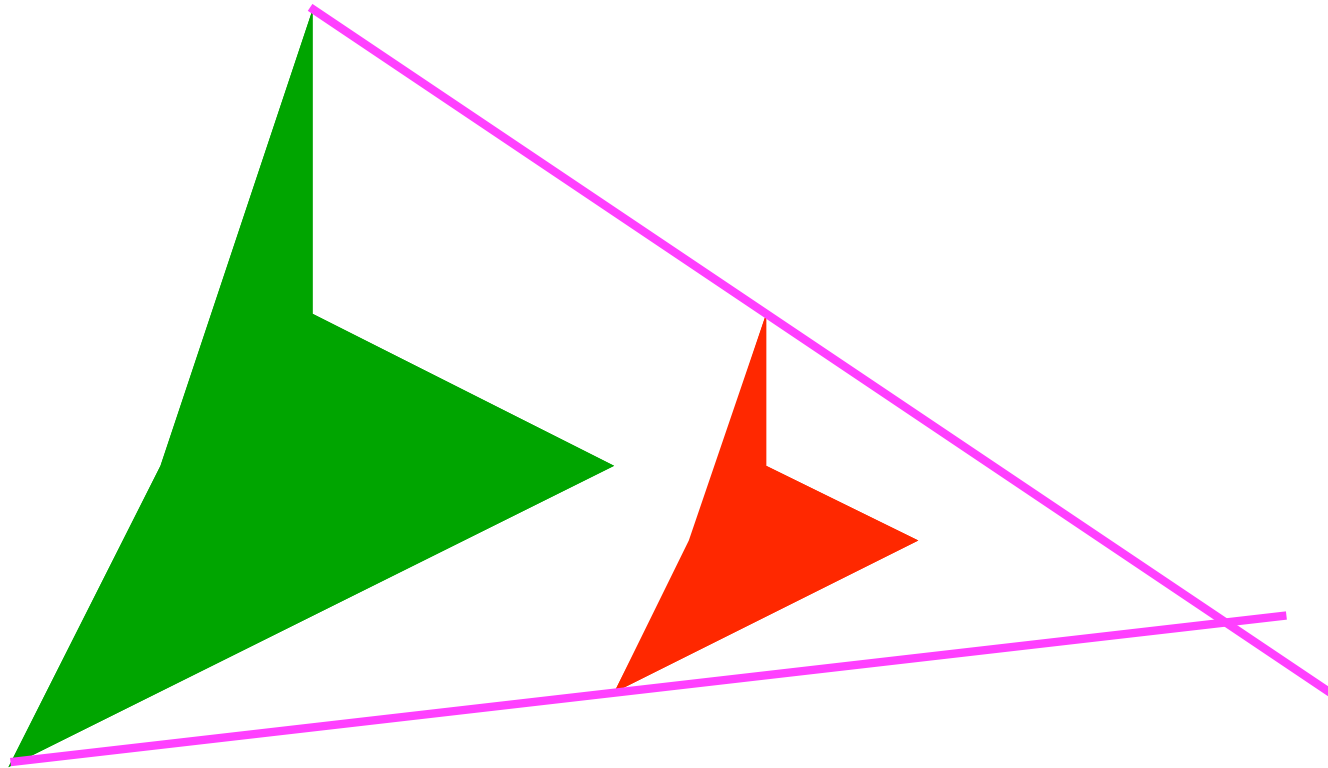


# Erinnerungen an die Schulgeometrie



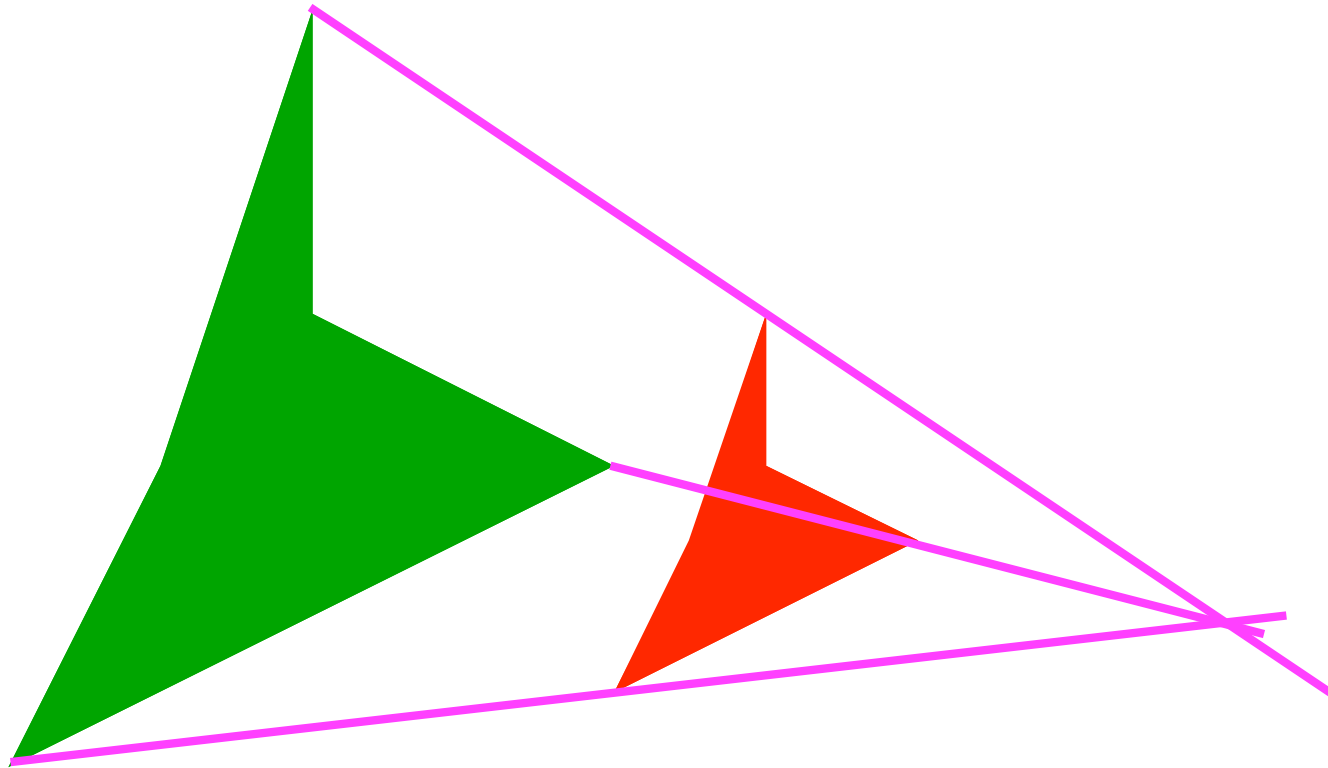
Zoom:

# Erinnerungen an die Schulgeometrie



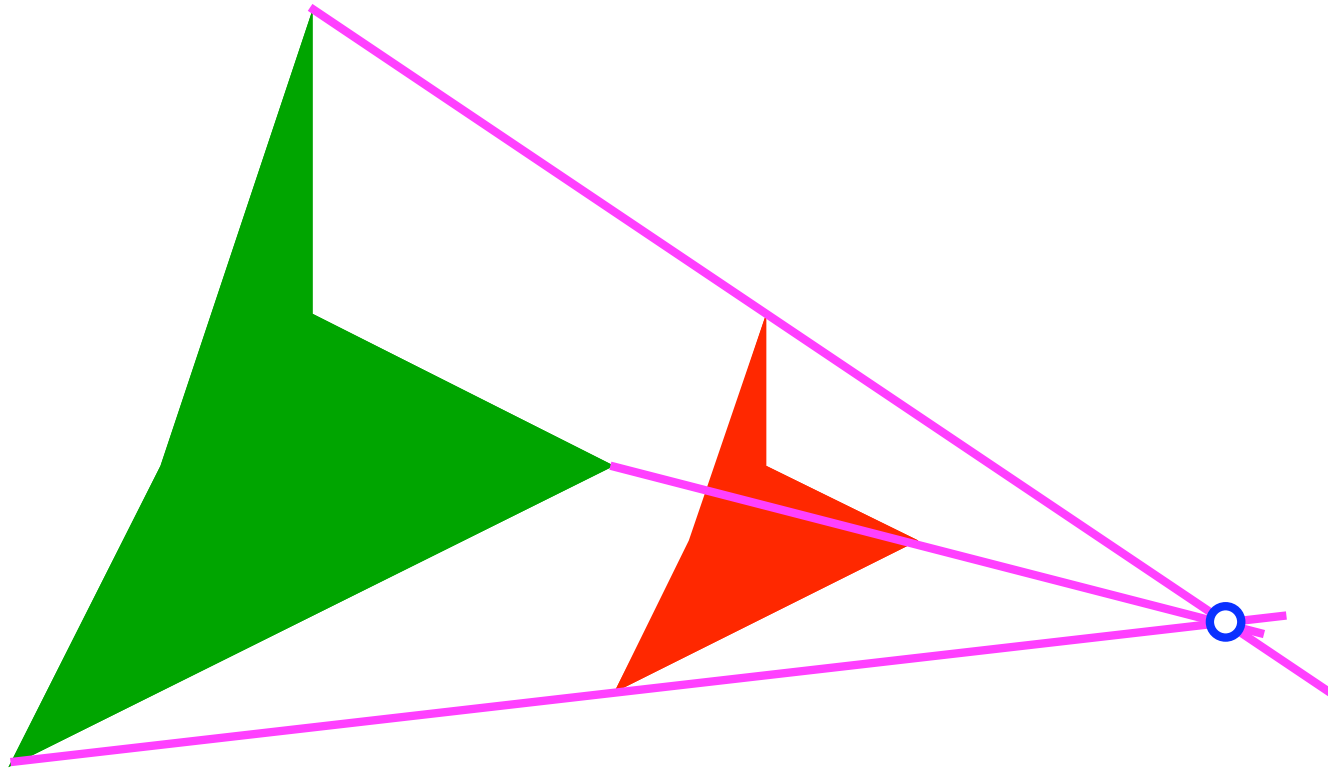
Zoom:

# Erinnerungen an die Schulgeometrie



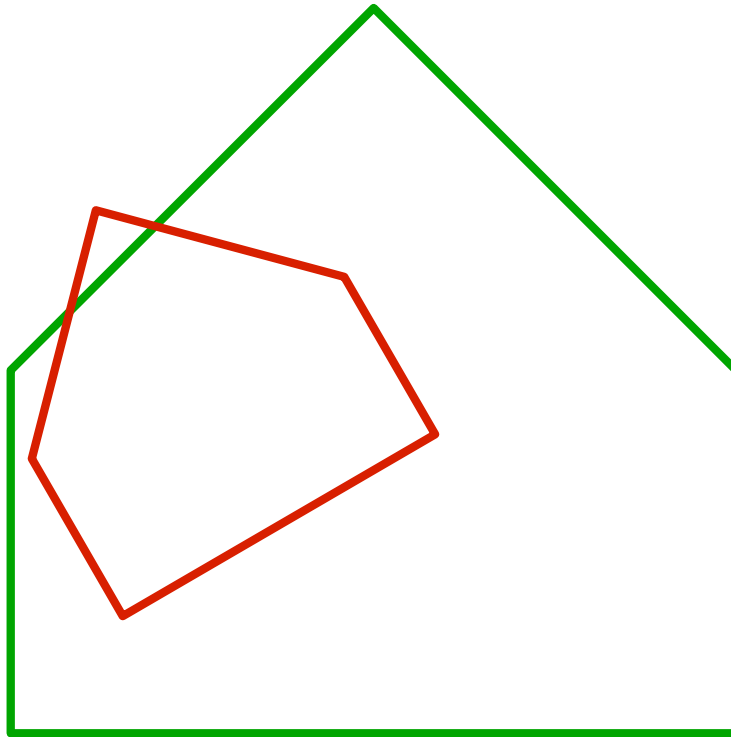
Zoom:

# Erinnerungen an die Schulgeometrie



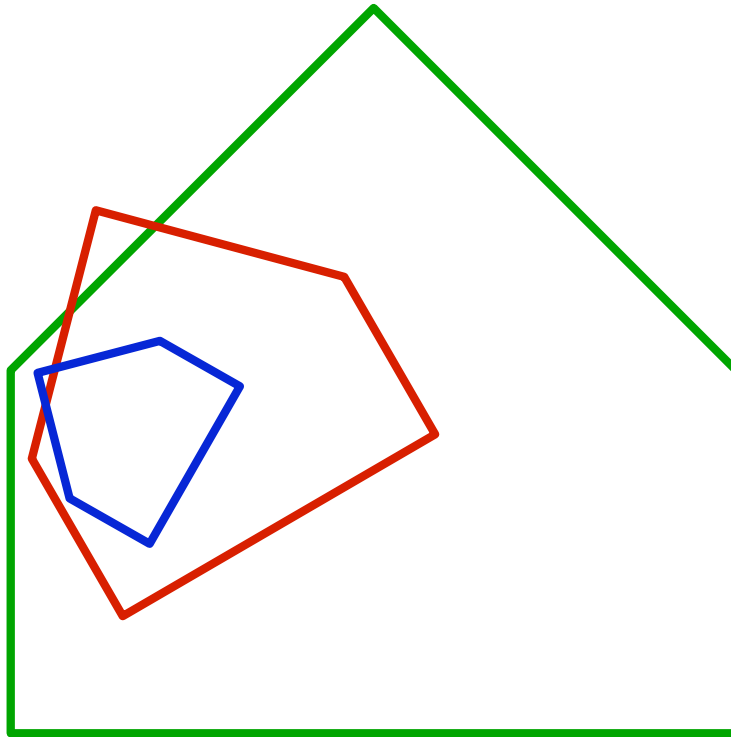
Zoom: Streckzentrum ist Fixpunkt

# Erinnerungen an die Schulgeometrie



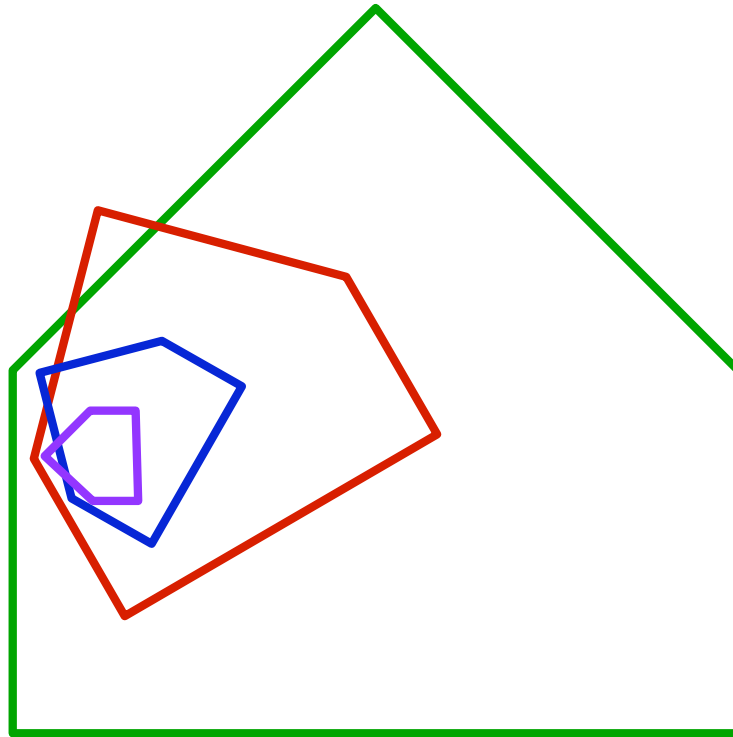
Drehstreckung

# Erinnerungen an die Schulgeometrie



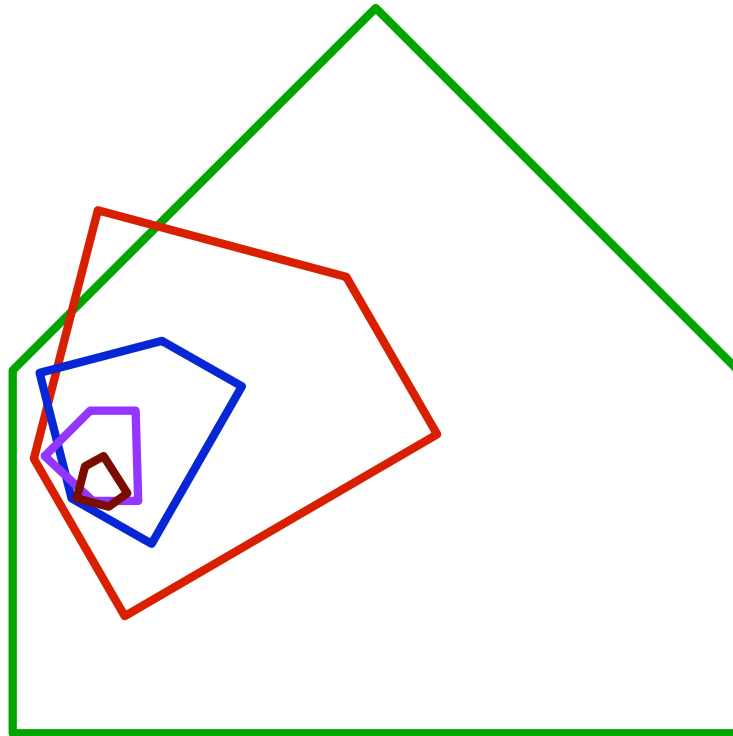
Drehstreckung

# Erinnerungen an die Schulgeometrie



Drehstreckung

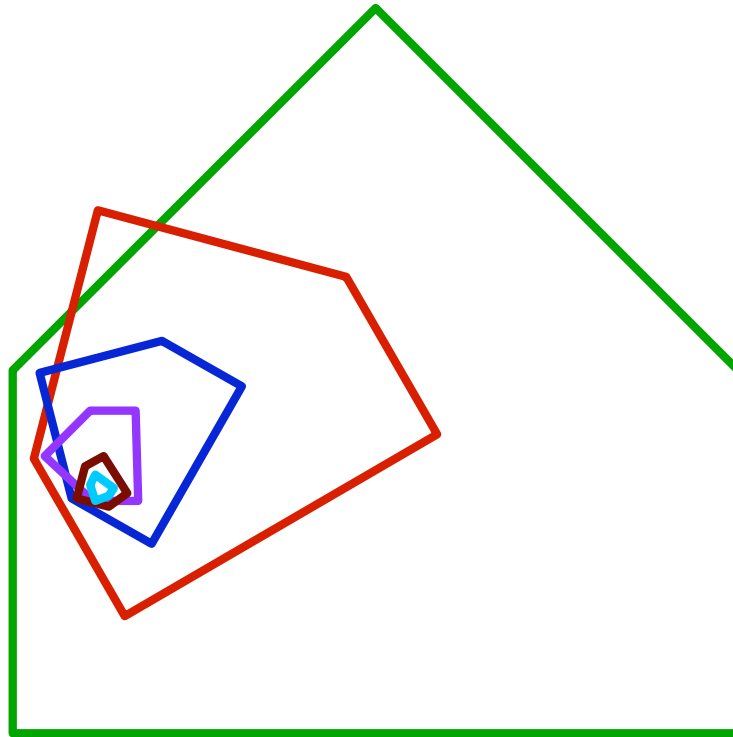
# Erinnerungen an die Schulgeometrie



Drehstreckung

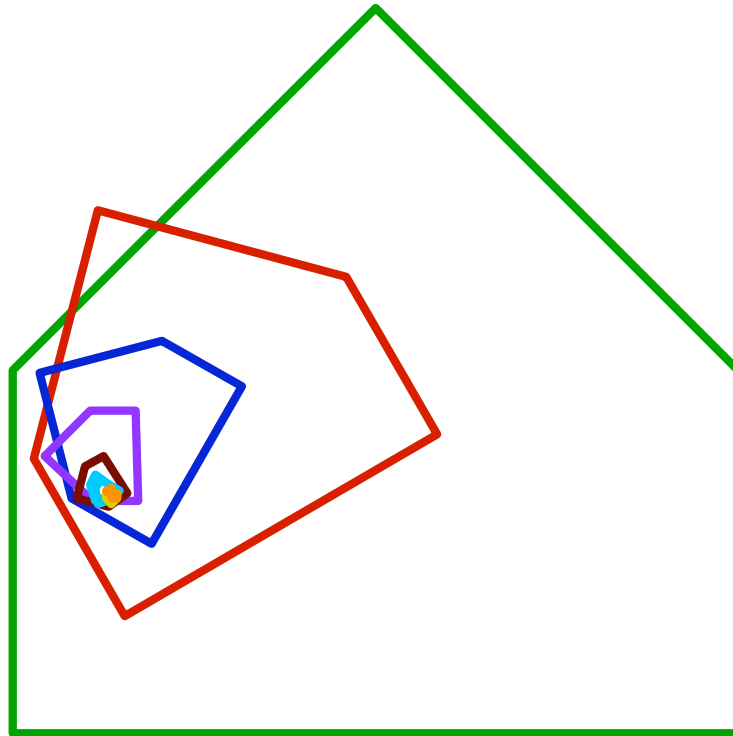


# Erinnerungen an die Schulgeometrie



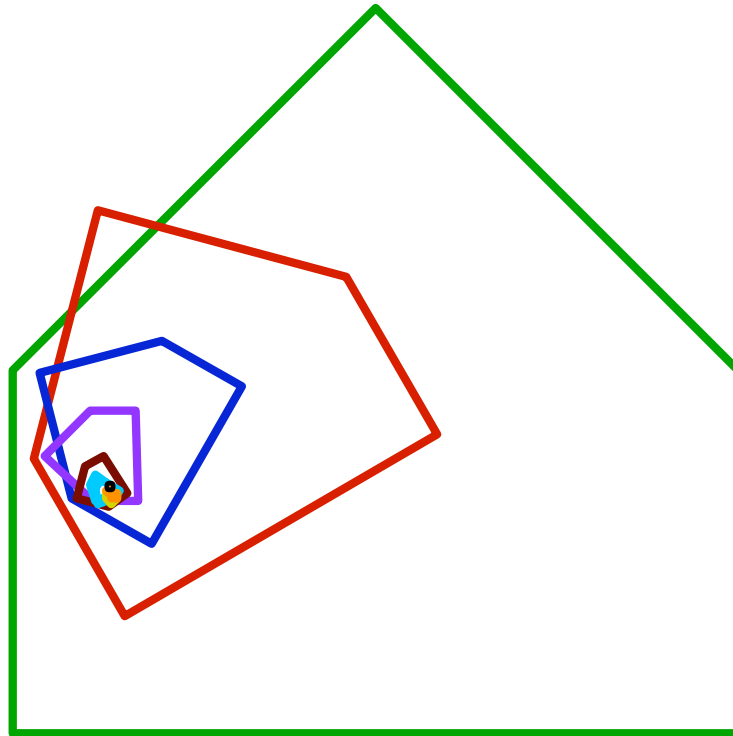
Drehstreckung

# Erinnerungen an die Schulgeometrie



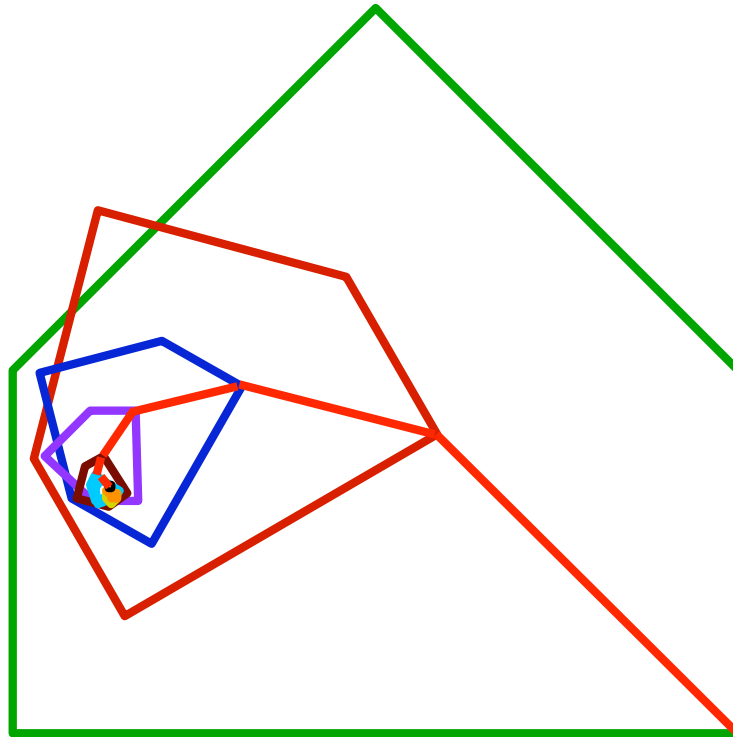
Drehstreckung

# Erinnerungen an die Schulgeometrie



Drehstreckung

# Erinnerungen an die Schulgeometrie



Drehstreckung

Fixpunktsatz:

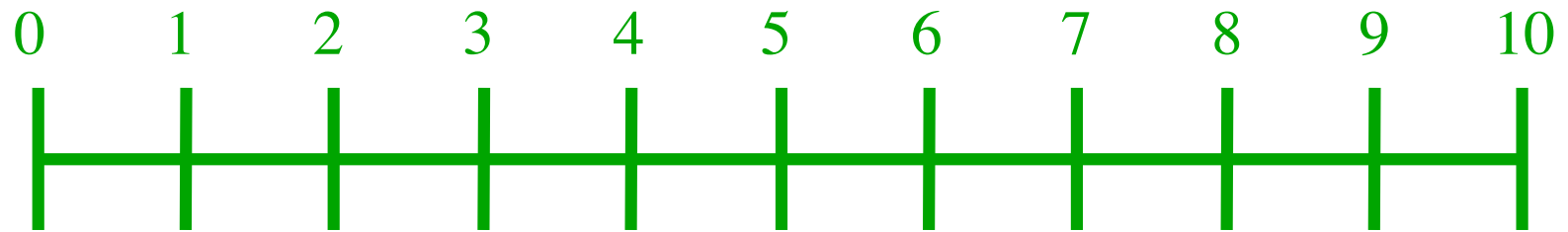
Jede kontrahierende Abbildung  $f$   
mit

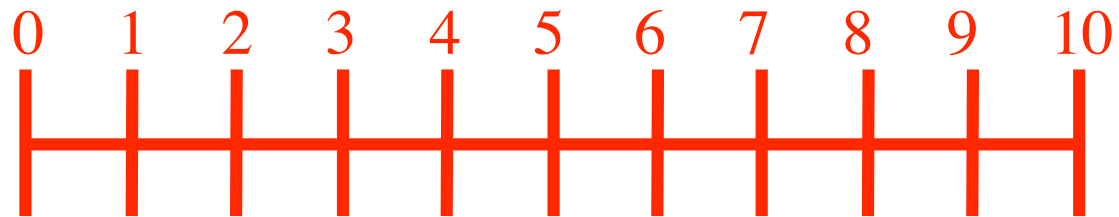
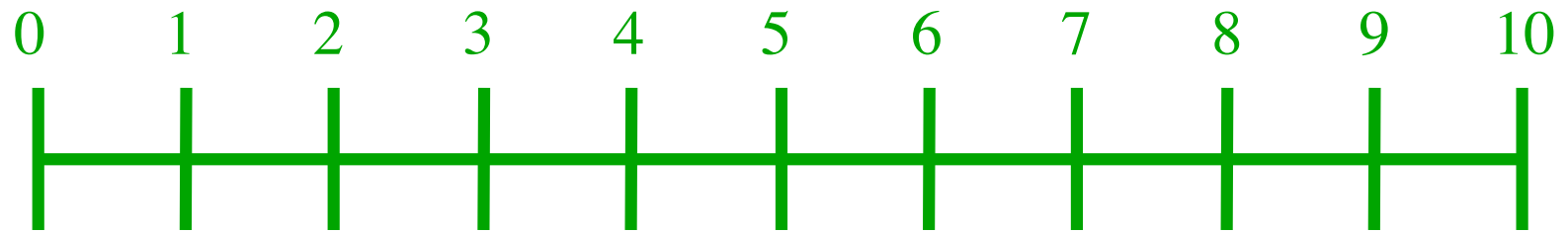
$$f(A) \subset A$$

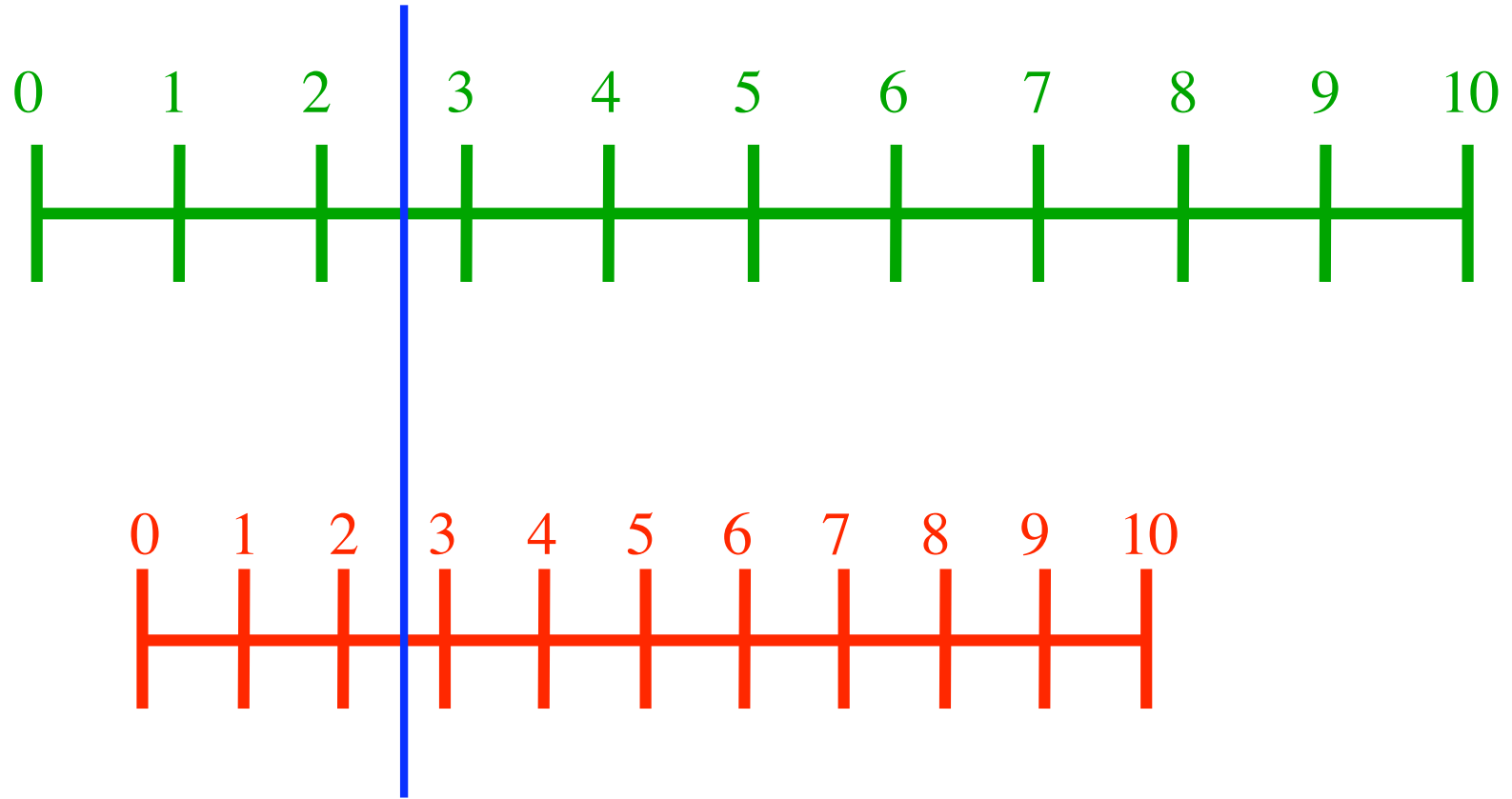
hat **genau einen** Fixpunkt.

$A$  ist der Definitionsbereich von  $f$ .

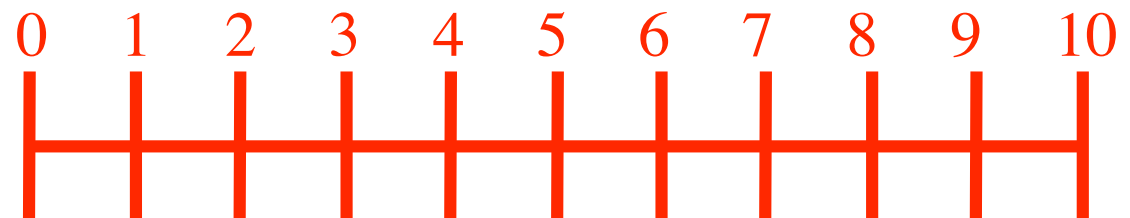
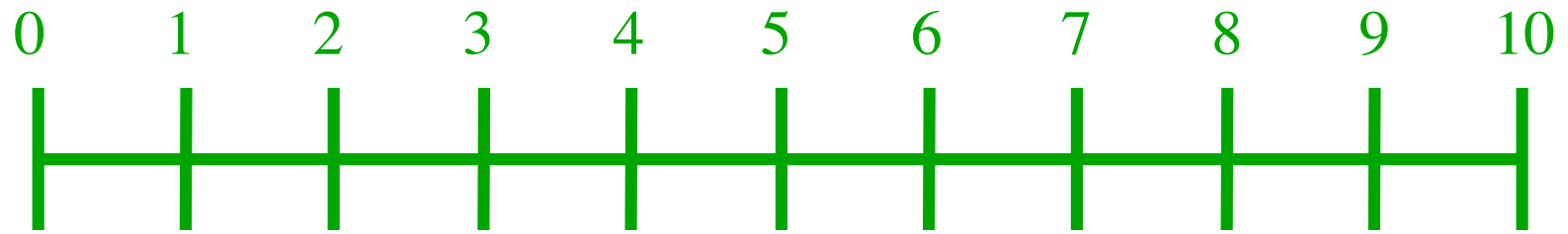
Mit Folien zeigen

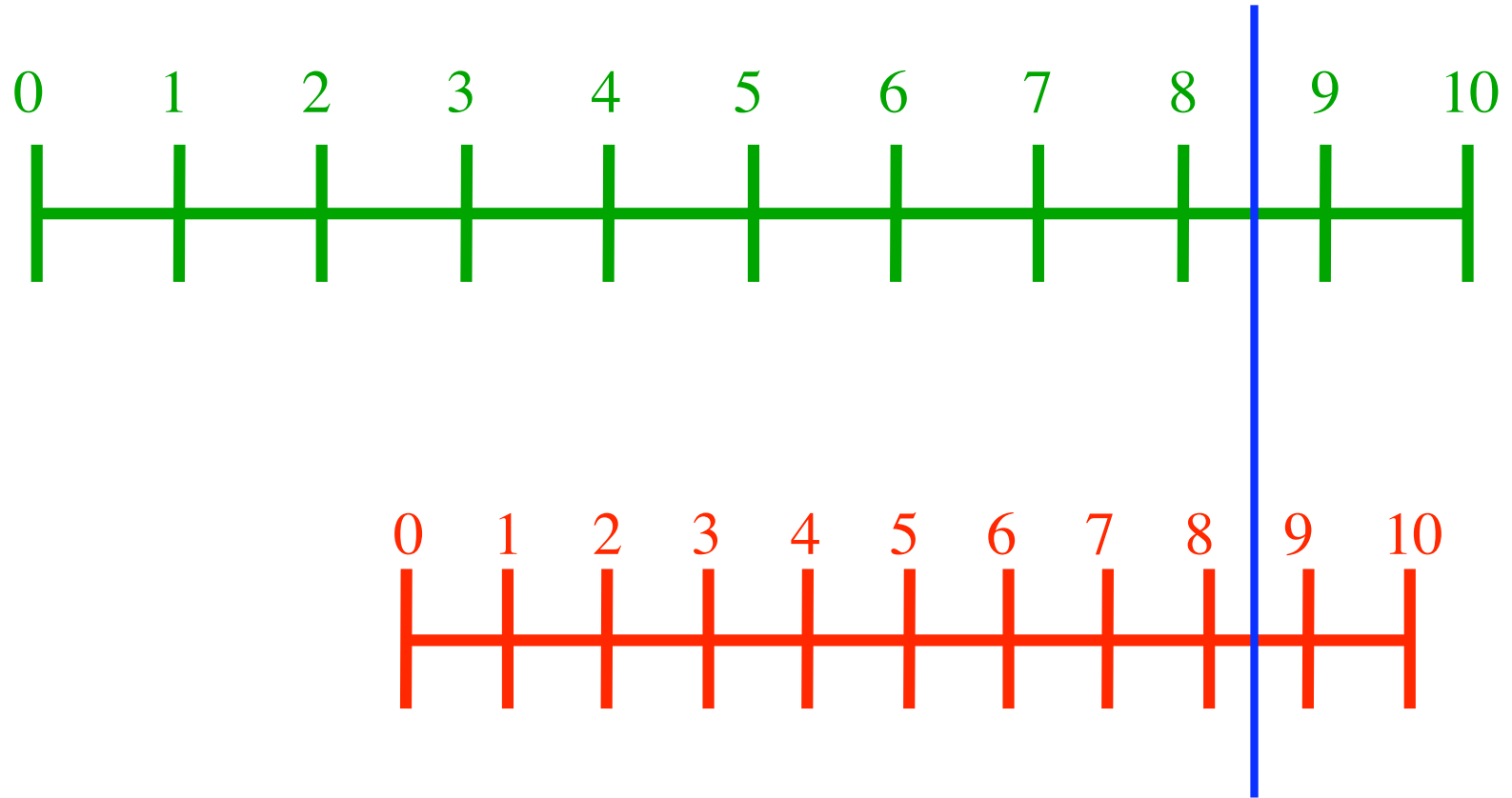


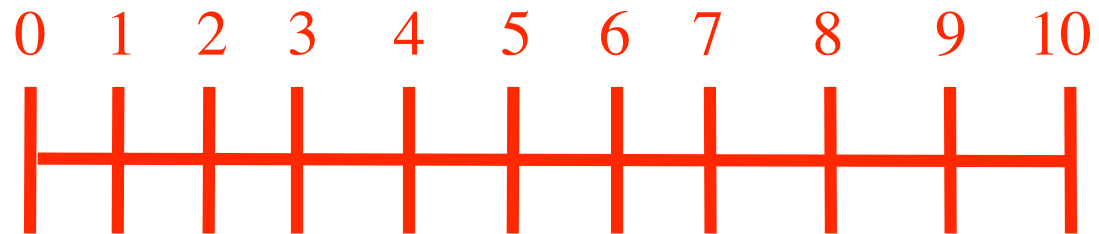
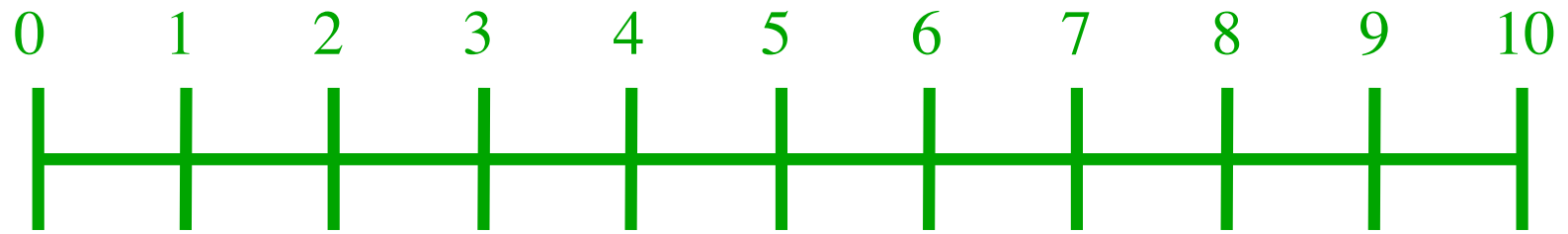




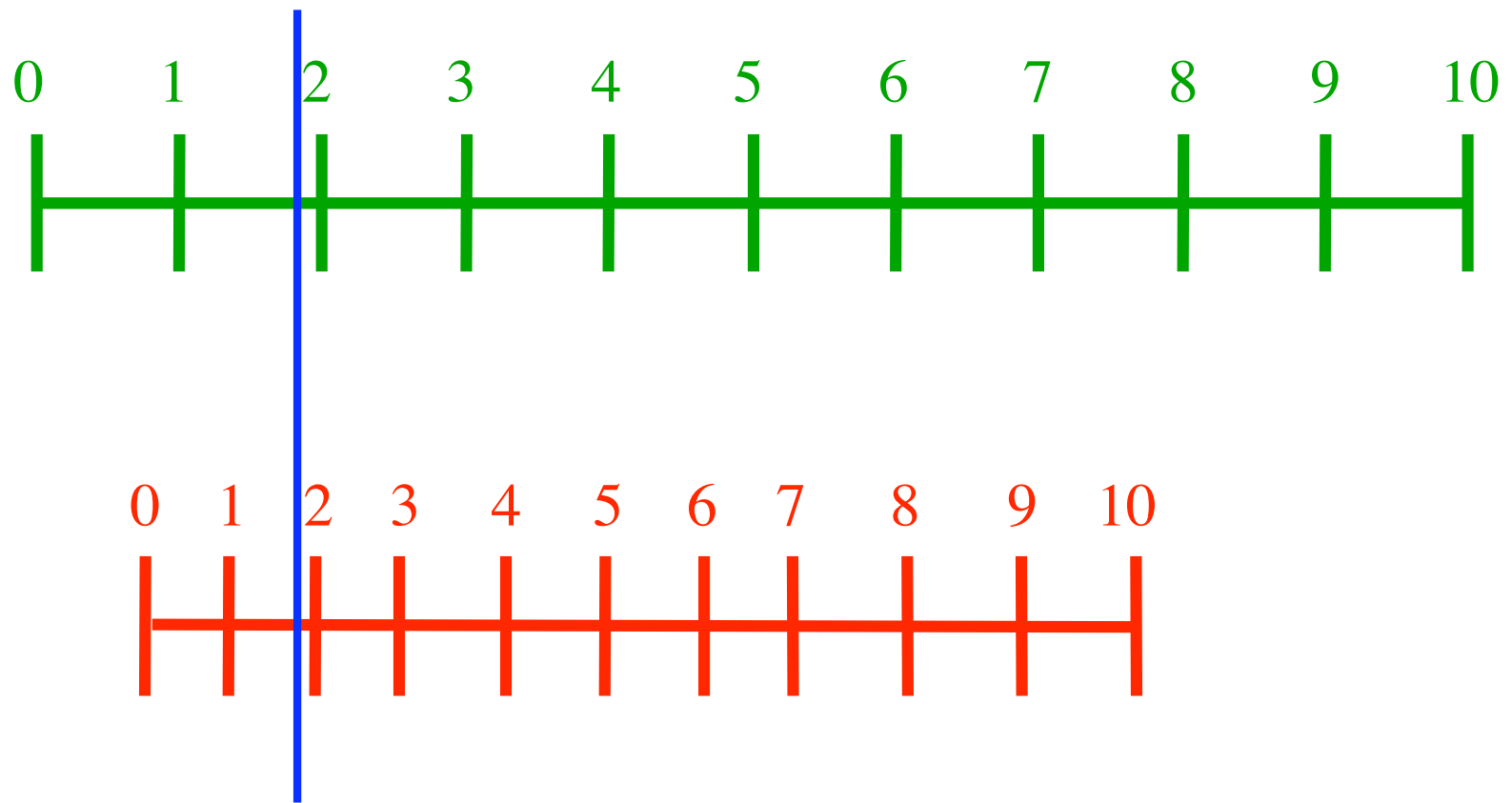




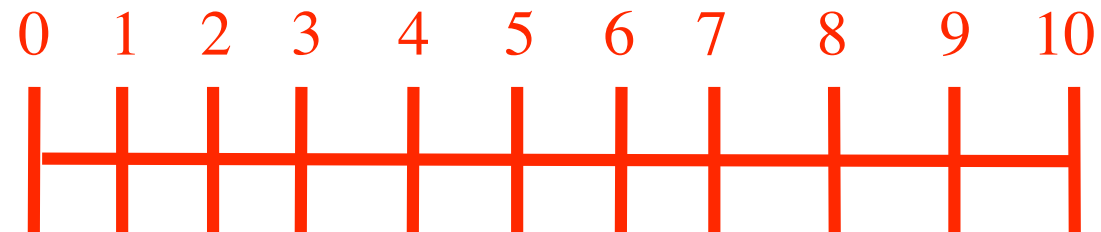
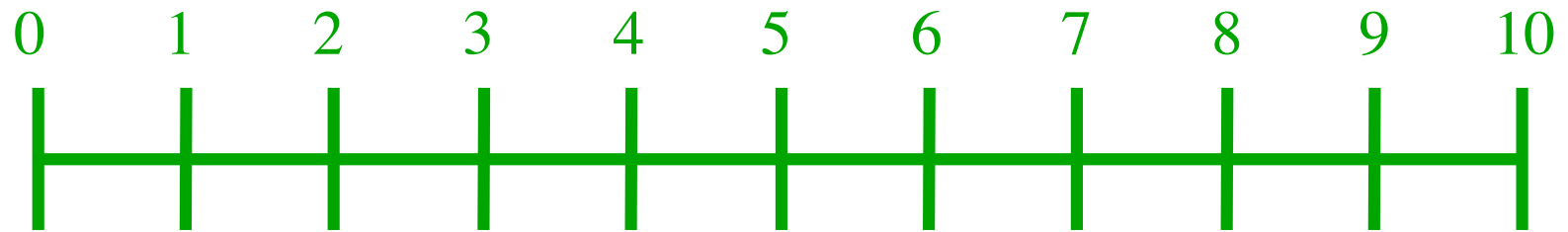




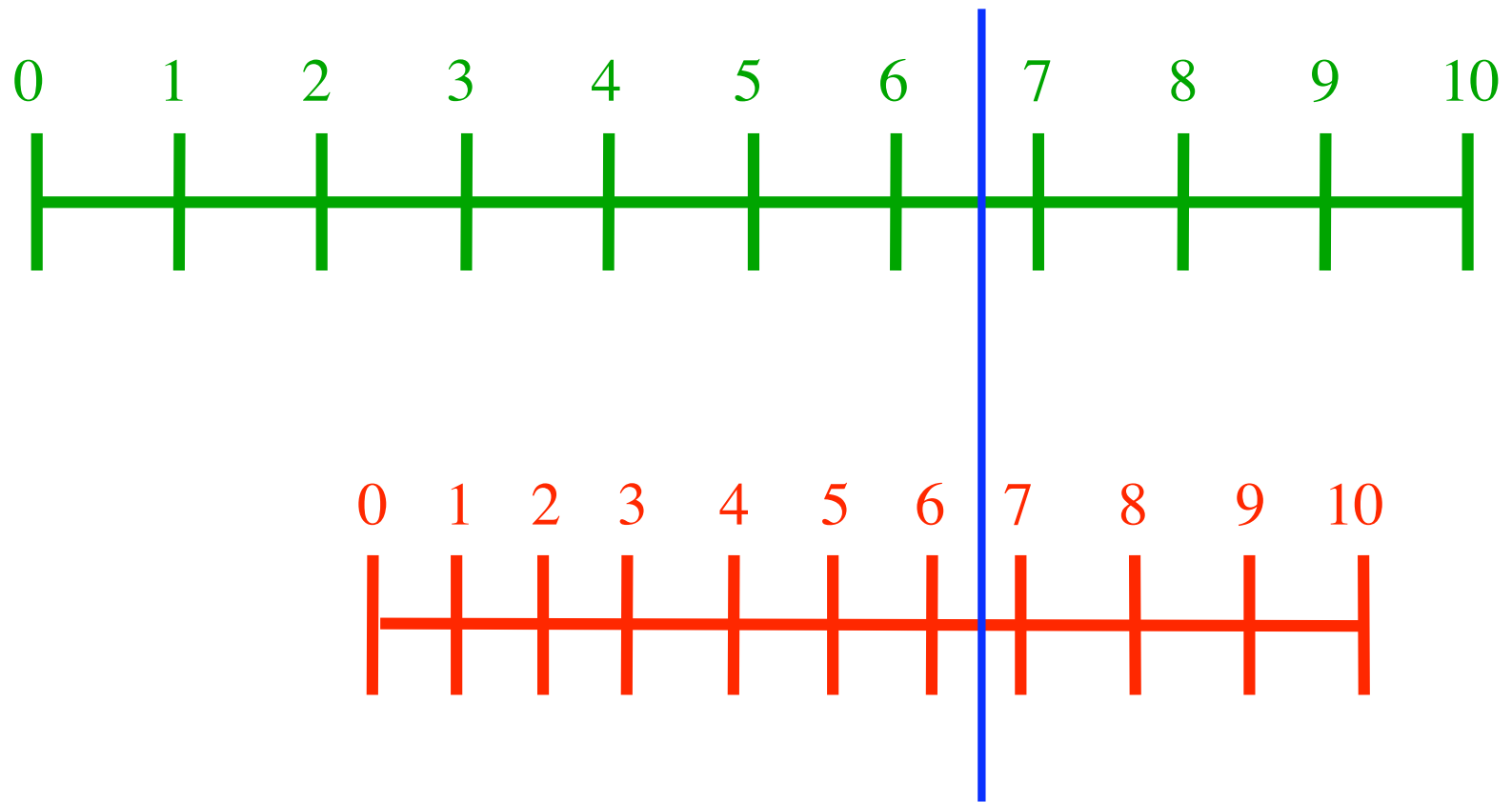
Unregelmäßige Kontraktion



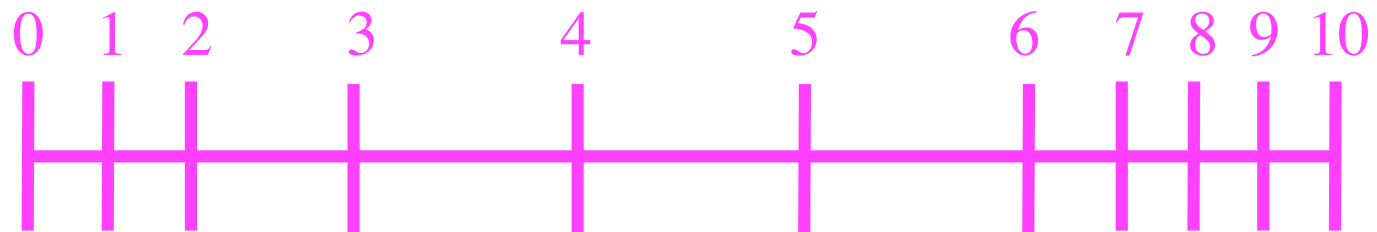
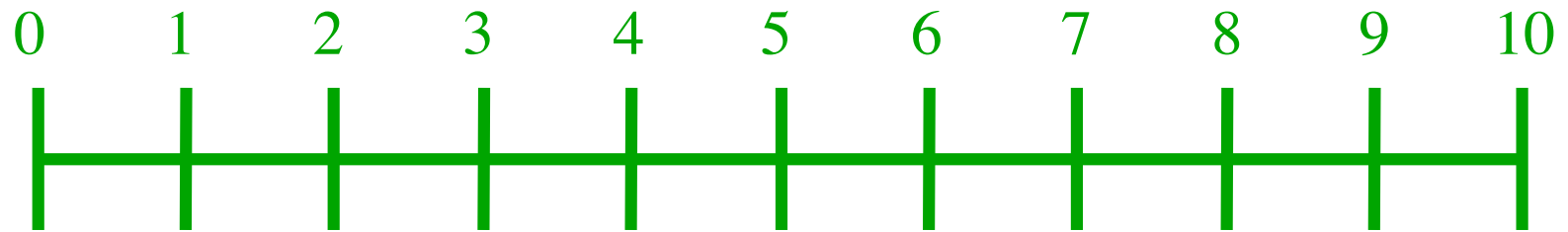
Unregelmäßige Kontraktion



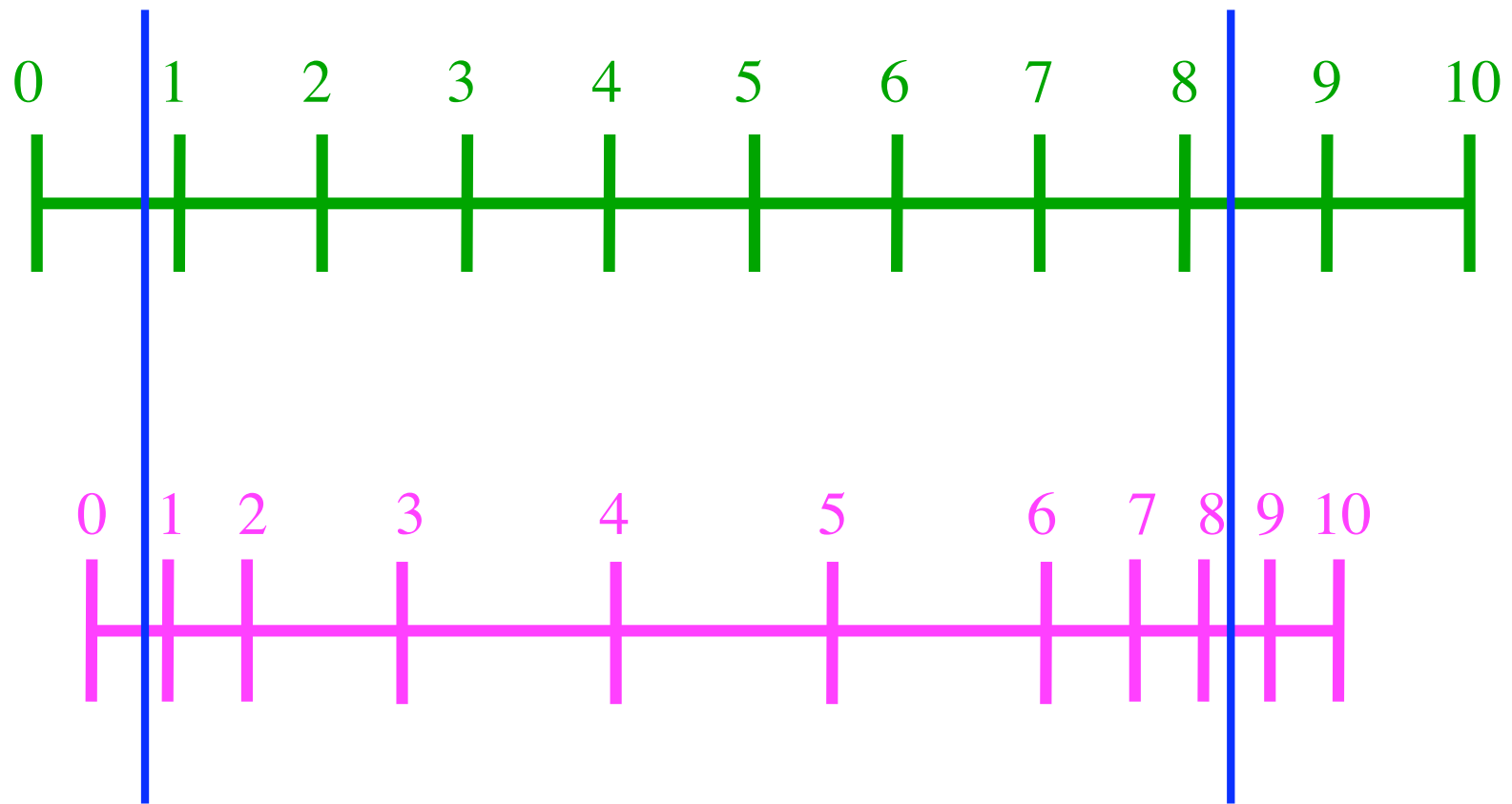
Unregelmäßige Kontraktion



Unregelmäßige Kontraktion



Nicht überall kontrahierend



Nicht überall kontrahierend

Zwei Fixpunkte



## Demo mit Taschenrechner

$$\cos(x) = x$$

Taschenrechner mit UPN (umgekehrte polnische Notation),  
bei welchem man zuerst die Zahl  
und dann die Funktion (zum Beispiel cos) eintippen muss.

Bogenmaß (Radian) einstellen.

Irgend eine Zahl eingeben.

Fortlaufend auf cos-Taste drücken.

$$\cos(x) = x$$

0		1.0
1		.5403023059
2		.8575532158
3		.6542897905
4		.7934803587
5		.7013687737
6		.7639596829
7		.7221024250
8		.7504177618
9		.7314040424

$$\cos(x) = x$$

0		1.0	10		.7442373549
1		.5403023059	11		.7356047404
2		.8575532158	12		.7414250866
3		.6542897905	13		.7375068905
4		.7934803587	14		.7401473356
5		.7013687737	15		.7383692041
6		.7639596829	16		.7395672022
7		.7221024250	17		.7387603199
8		.7504177618	18		.7393038924
9		.7314040424	19		.7389377567

$$\cos(x) = x$$

20		.7391843998
21		.7390182624
22		.7391301765
23		.7390547908
24		.7391055719
25		.7390713653
26		.7390944074
27		.7390788860
28		.7390893414
29		.7390822985

$$\cos(x) = x$$

20		.7391843998	30		.7390870427
21		.7390182624	31		.7390838470
22		.7391301765	32		.7390859996
23		.7390547908	33		.7390845496
24		.7391055719	34		.7390855263
25		.7390713653	35		.7390848684
26		.7390944074	36		.7390853116
27		.7390788860	37		.7390850131
28		.7390893414	38		.7390852141
29		.7390822985	39		.7390850787

$$\cos(x) = x$$

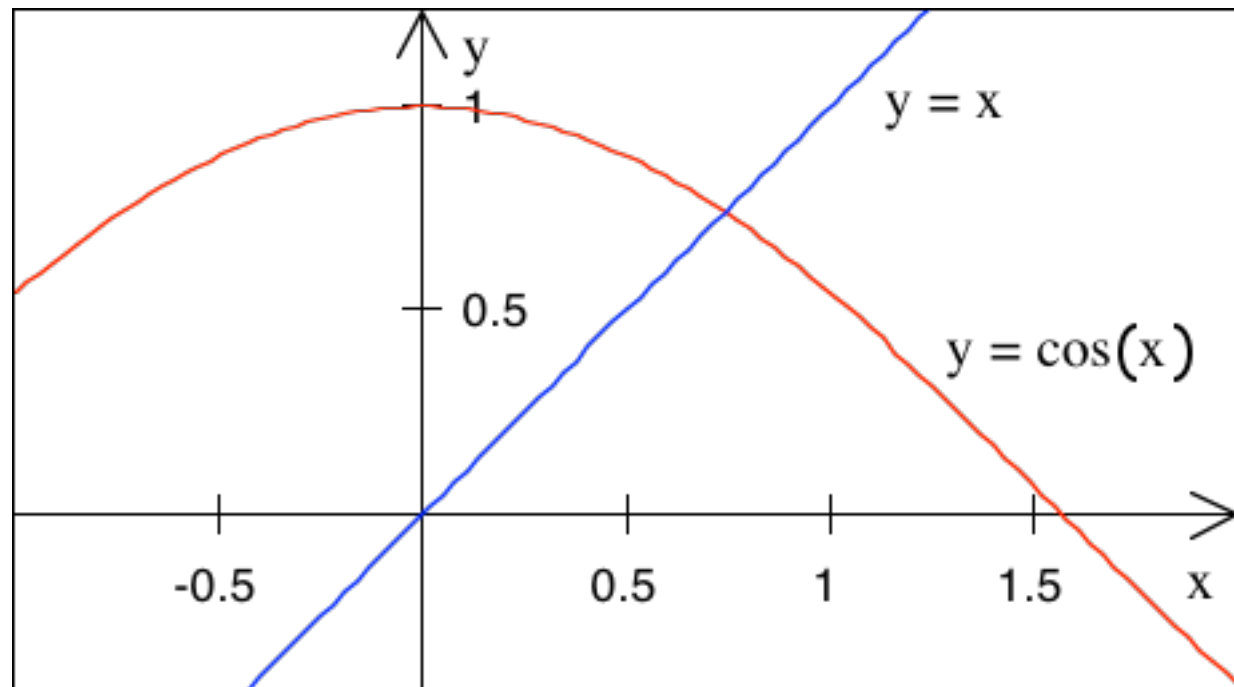
40		.7390851699
41		.7390851085
42		.7390851499
43		.7390851220
44		.7390851408
45		.7390851281
46		.7390851367
47		.7390851309
48		.7390851348
49		.7390851321

$$\cos(x) = x$$

40		.7390851699	50		.7390851340
41		.7390851085	51		.7390851327
42		.7390851499	52		.7390851336
43		.7390851220	53		.7390851330
44		.7390851408	54		.7390851334
45		.7390851281	55		.7390851331
46		.7390851367	56		.7390851333
47		.7390851309	57		.7390851332
48		.7390851348	58		.7390851332
49		.7390851321	59		.7390851332

Fixzahl

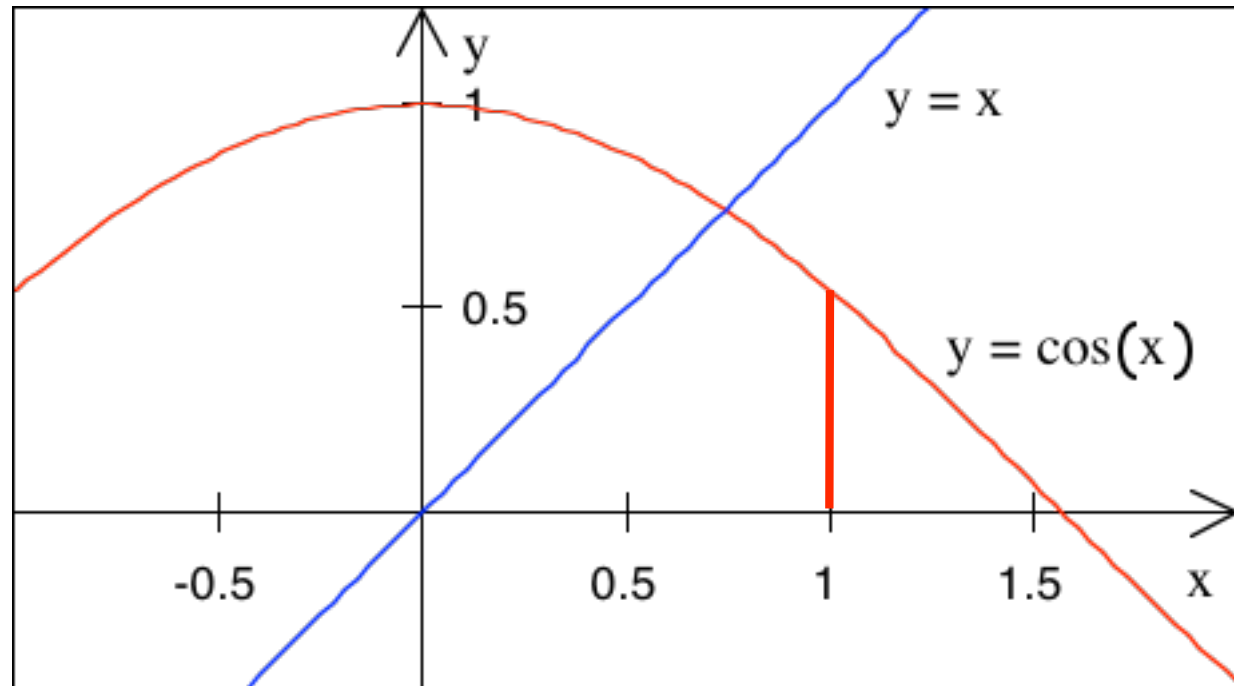
$$\cos(x) = x$$





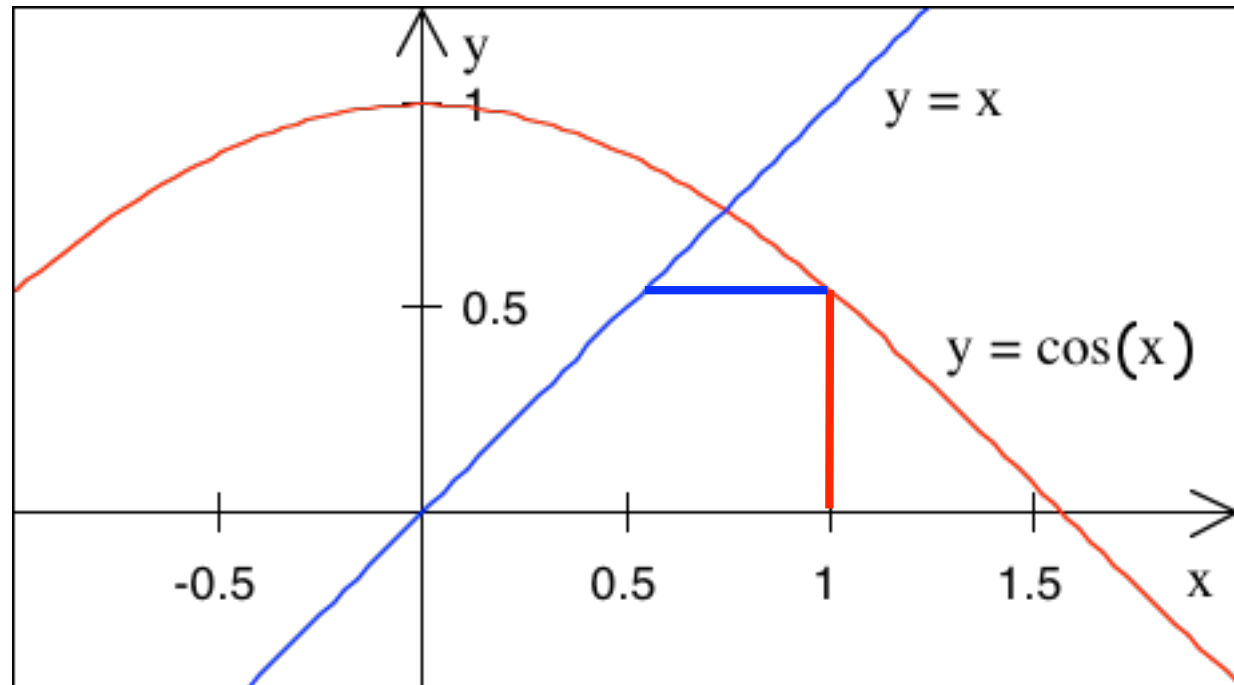
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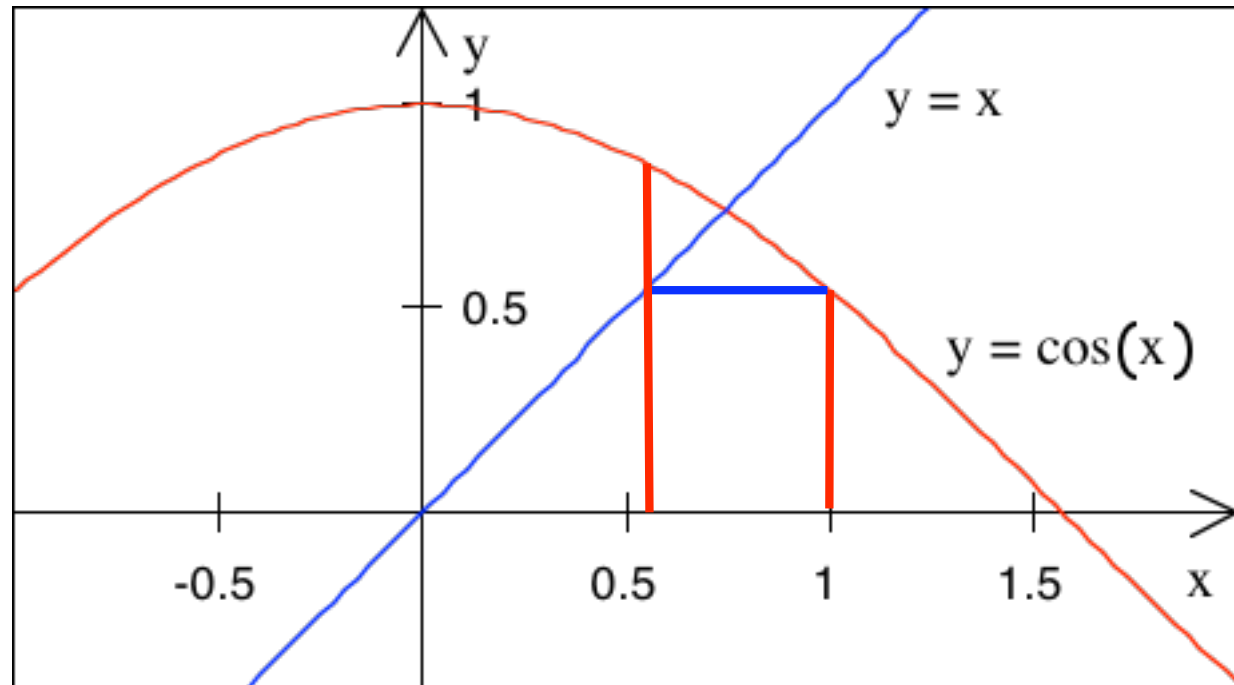
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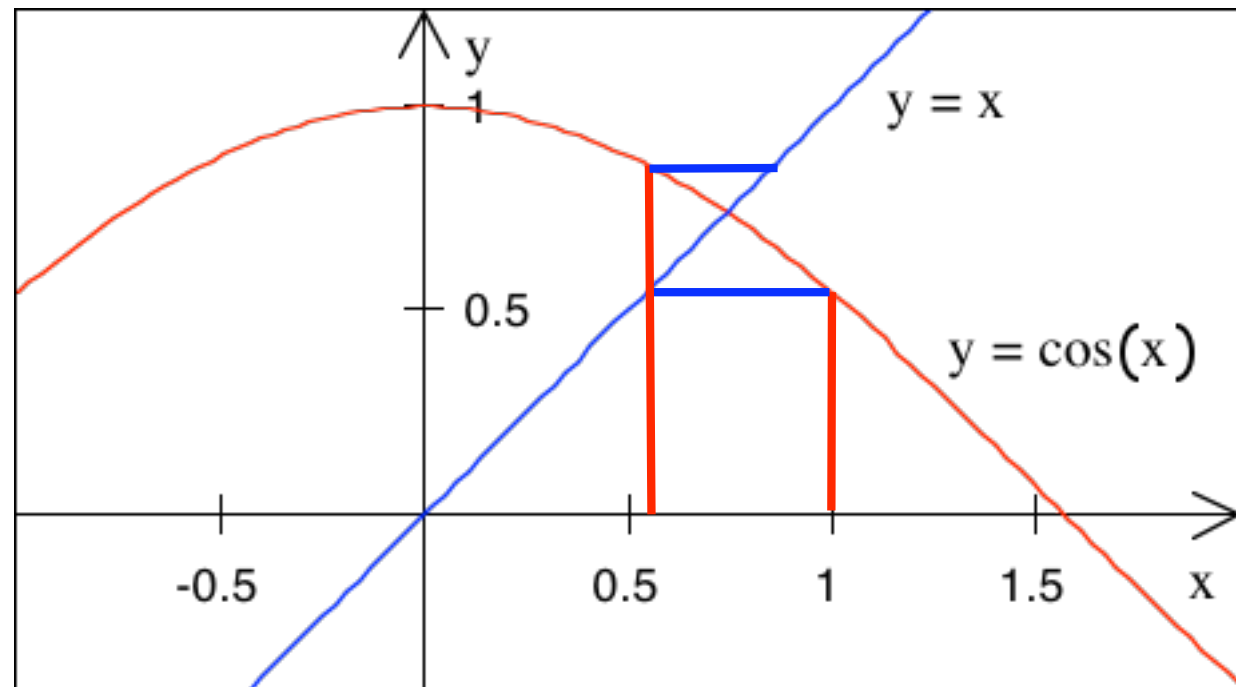
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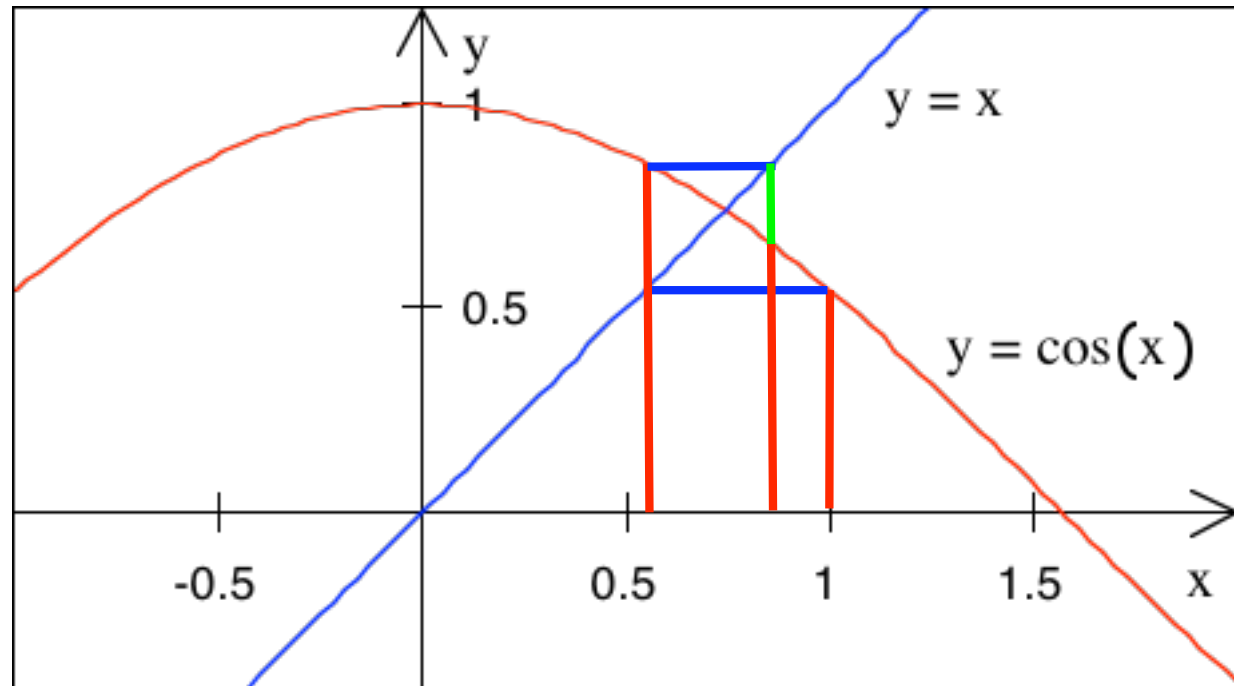
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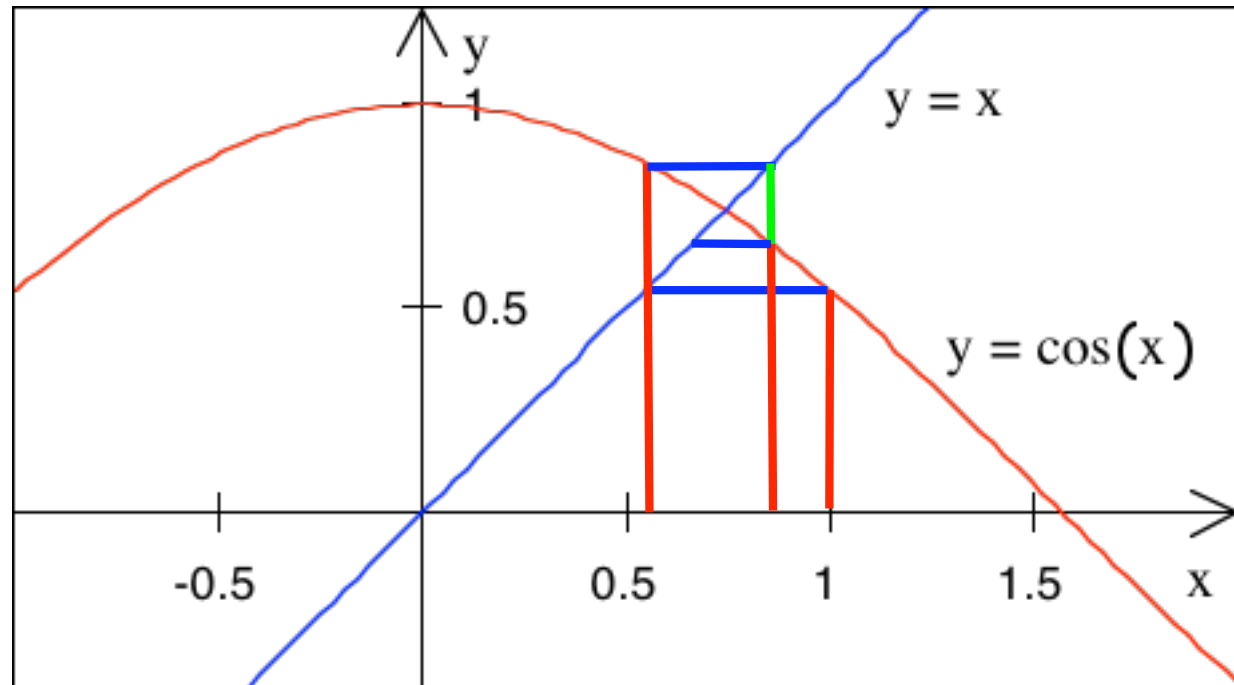
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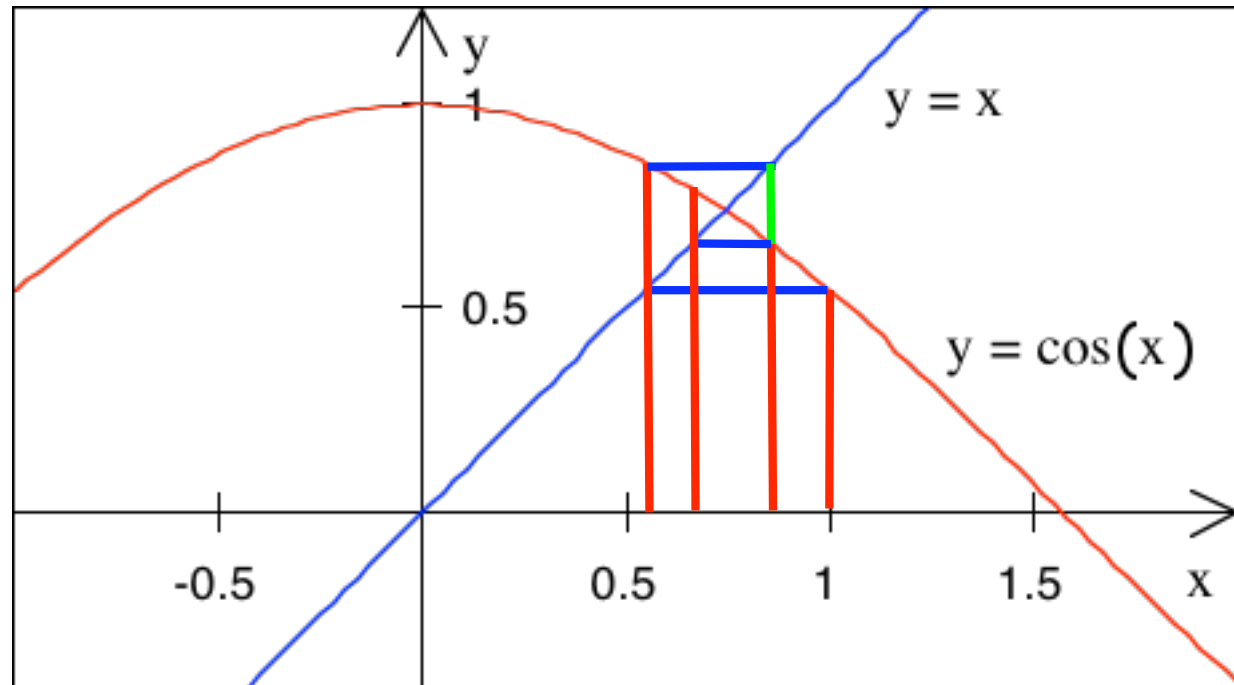
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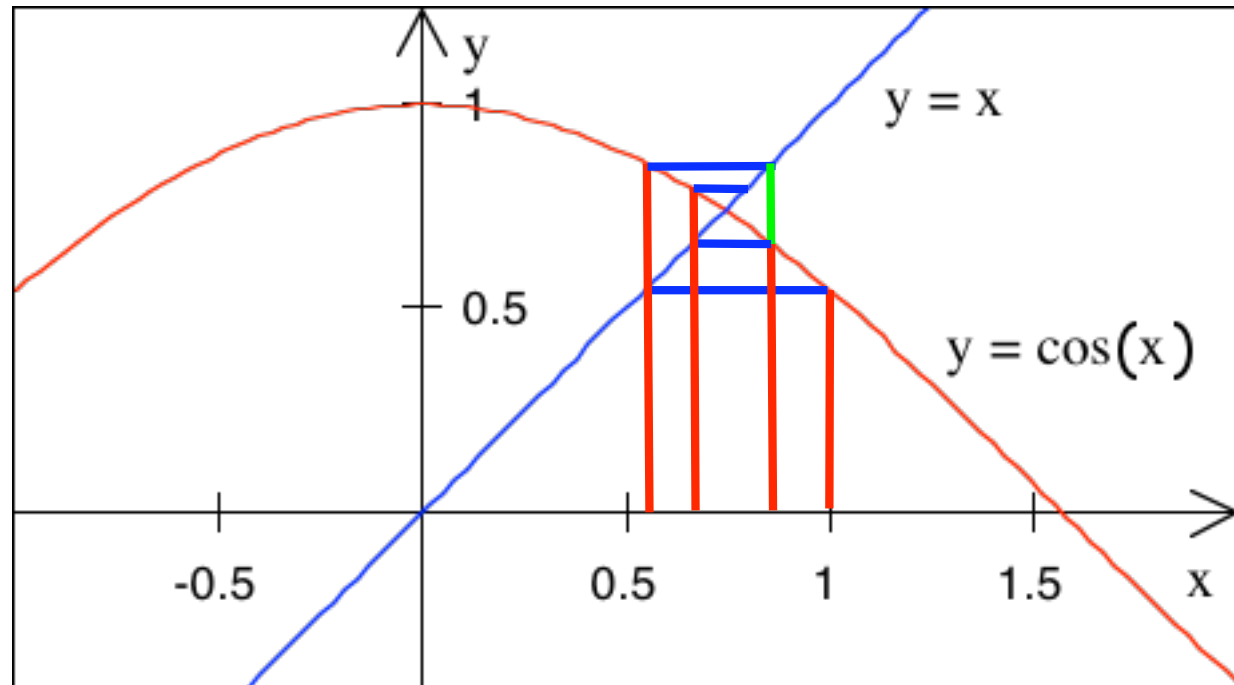
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$$\cos(x) = x$$

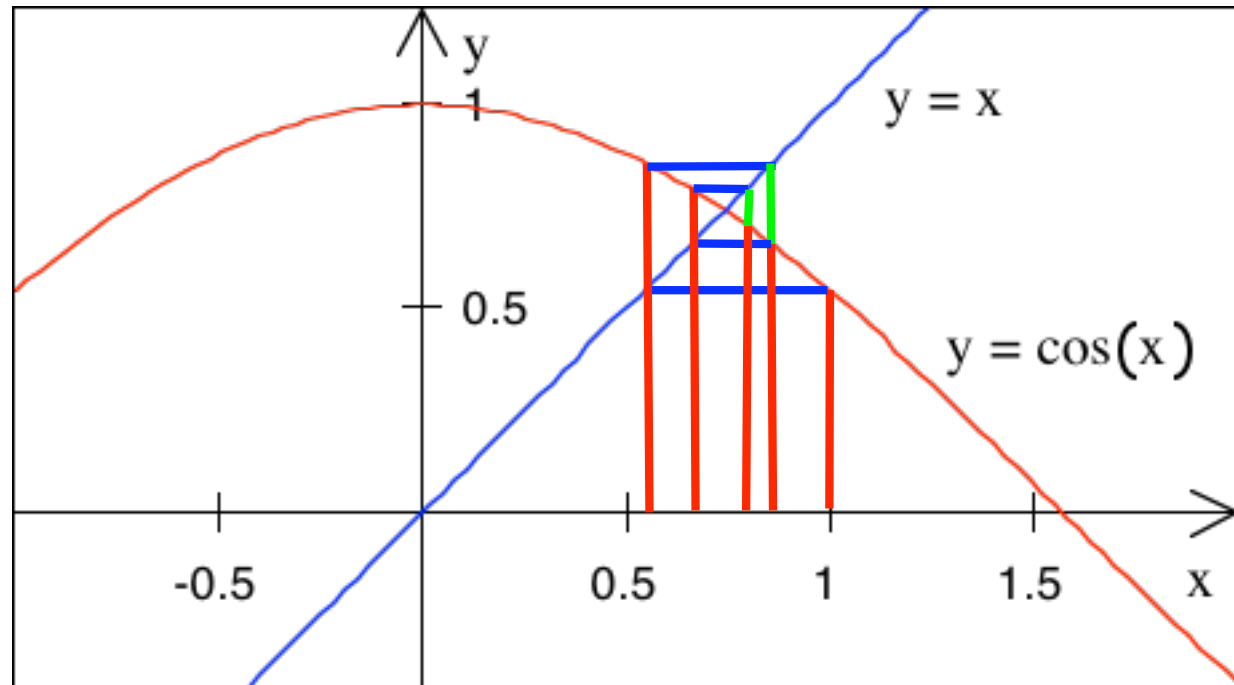
0		1.0
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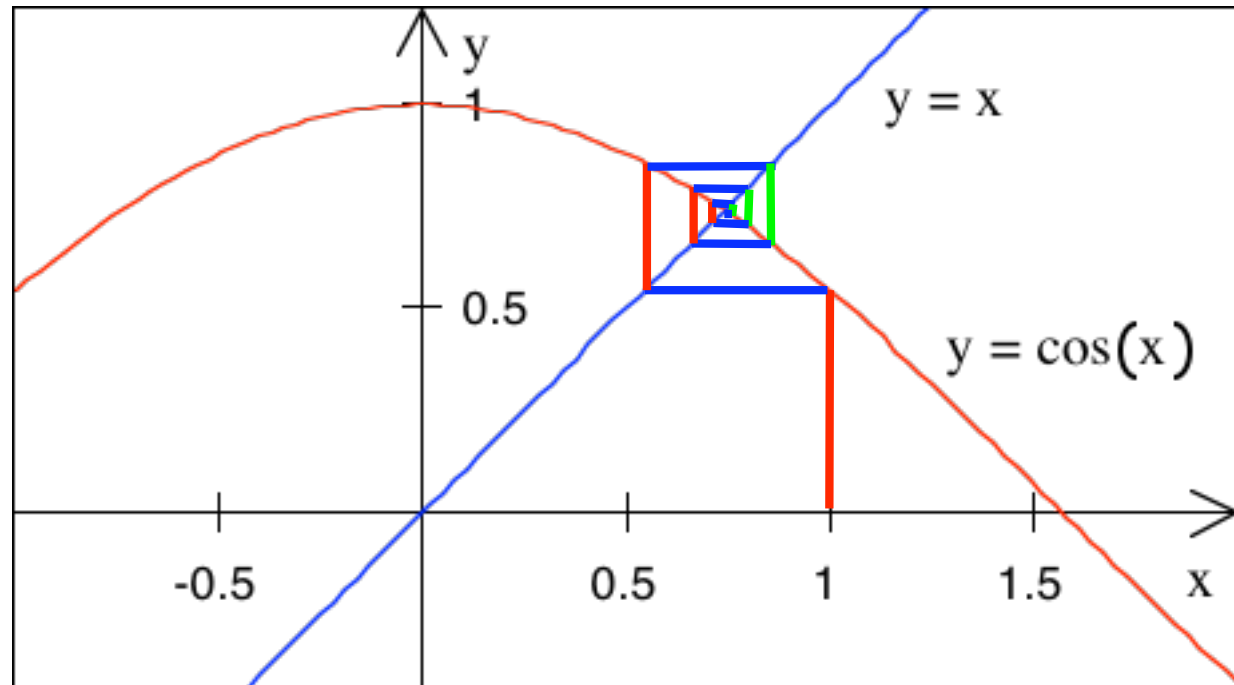
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2		.8575532158
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$$\cos(x) = x$$

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## Erinnerung: Newton

$$\cos(x) = x \iff \cos(x) - x = 0$$

$$f(x) = \cos(x) - x$$

$$f'(x) = -\sin(x) - 1$$

$$x_{n+1} = x_n - \frac{\cos(x_n) - x_n}{-\sin(x_n) - 1}$$

## Erinnerung: Newton

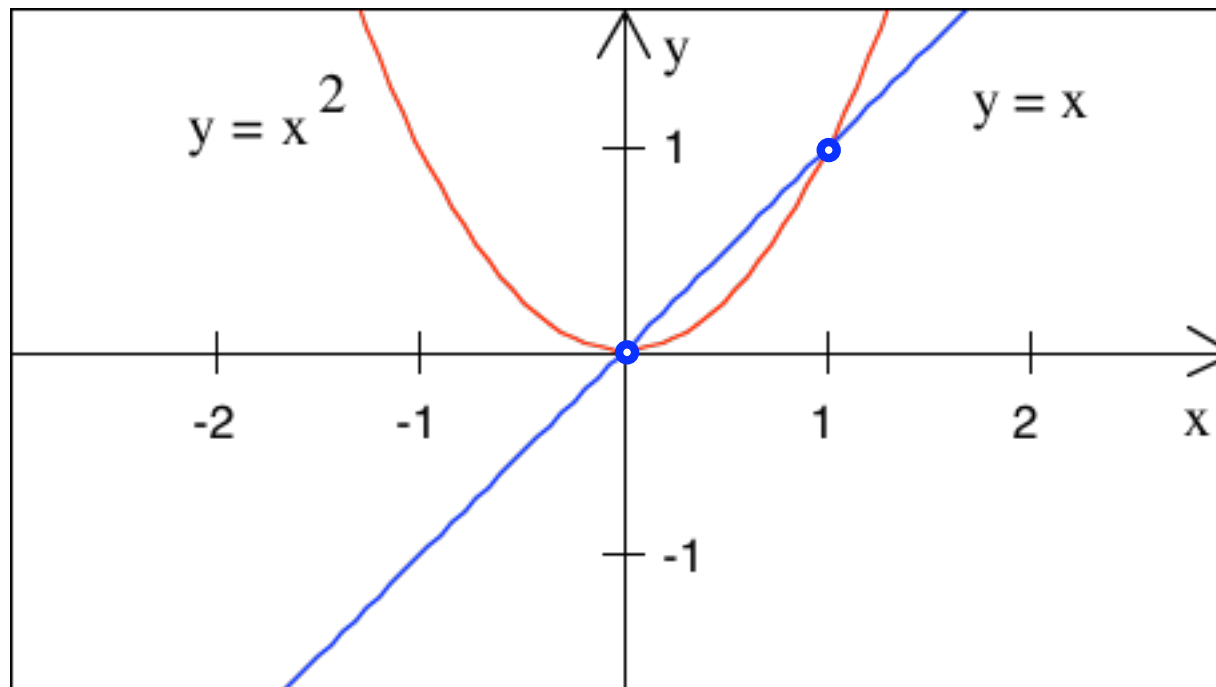
Schritt	x[n]
0	1.0
1	.7503638679
2	.7391128909
3	.7390851334
4	.7390851332
5	.7390851332

Newton ist viel schneller als die Fixpunktmethode.

Beispiel:  $x^2 = x$

Beispiel:  $x^2 = x \implies x = 0$  oder  $x = 1$

Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



Zu welchem Schnittpunkt geht die Fixpunktmethode?

## Demo mit Taschenrechner

Beispiel:  $x^2 = x \implies x = 0$  oder  $x = 1$

Taschenrechner mit UPN

Irgend eine Zahl als Startwert eingeben.

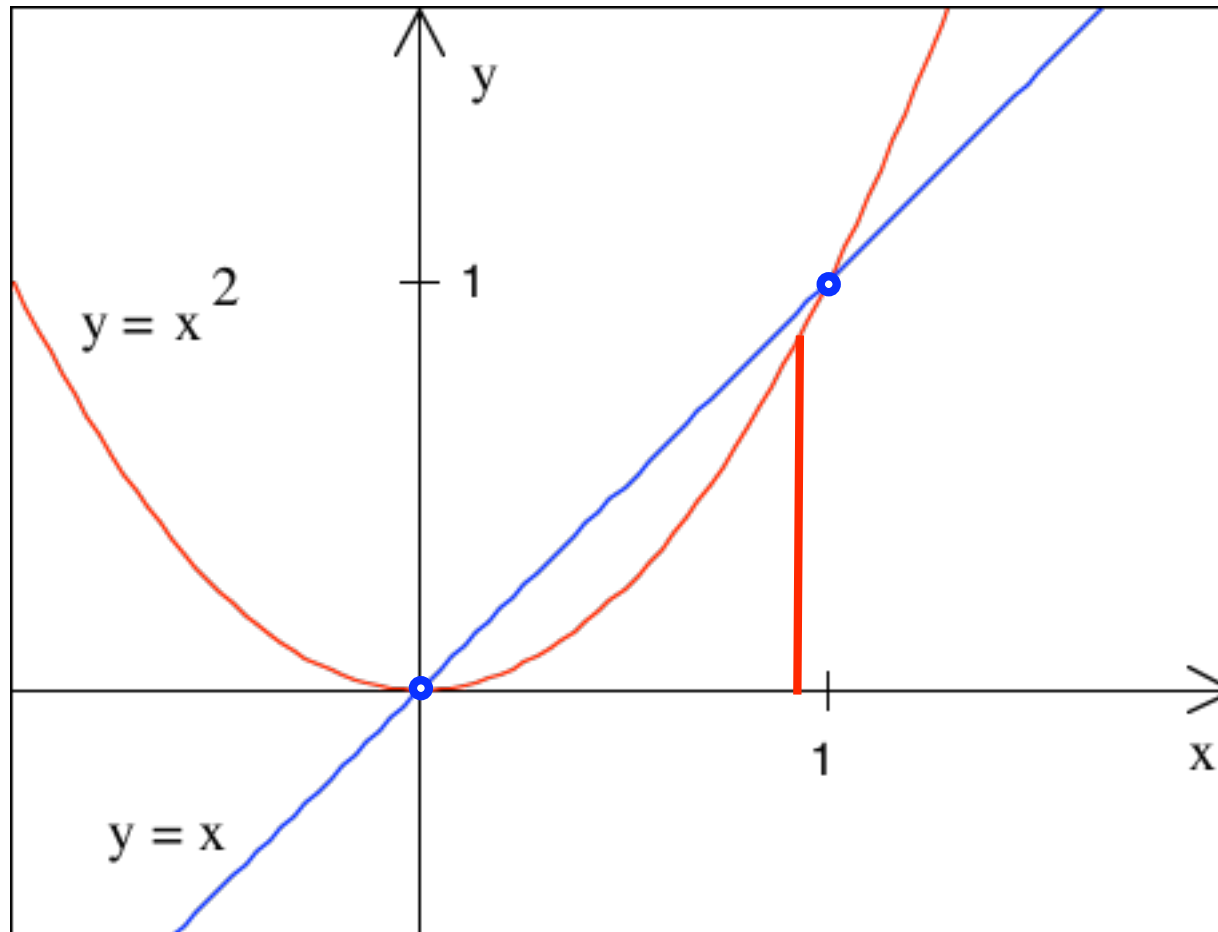
Fortlaufend auf Quadrat-Taste drücken.



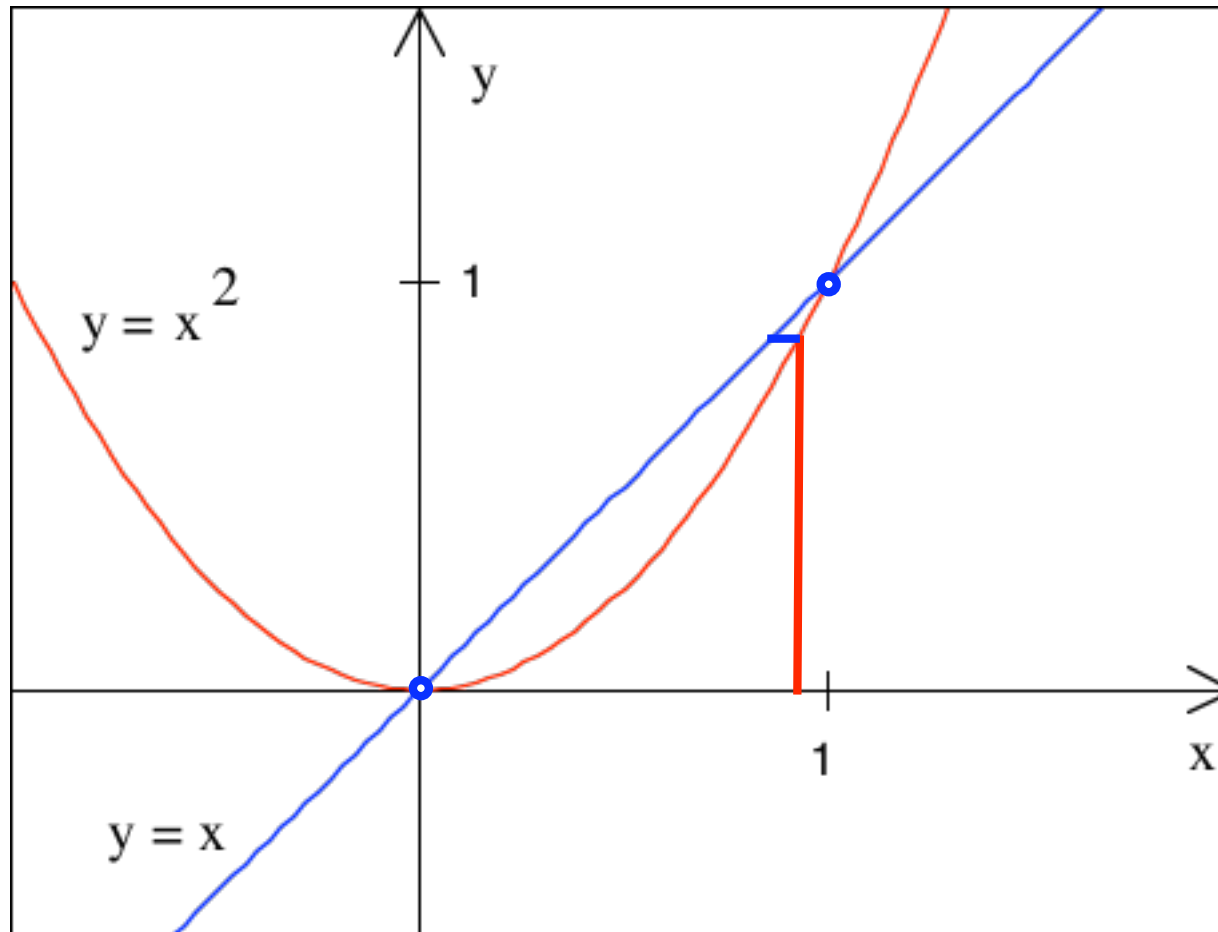
Beispiel:  $x^2 = x \implies x = 0$  oder  $x = 1$

Schritt		x[n]
0		.9 Startwert
1		.81
2		.6561
3		.43046721
4		.1853020189
5		.03433683821
6		.001179018458
7		.1390084524e-5
8		.1932334984e-11
9		.3733918490e-23

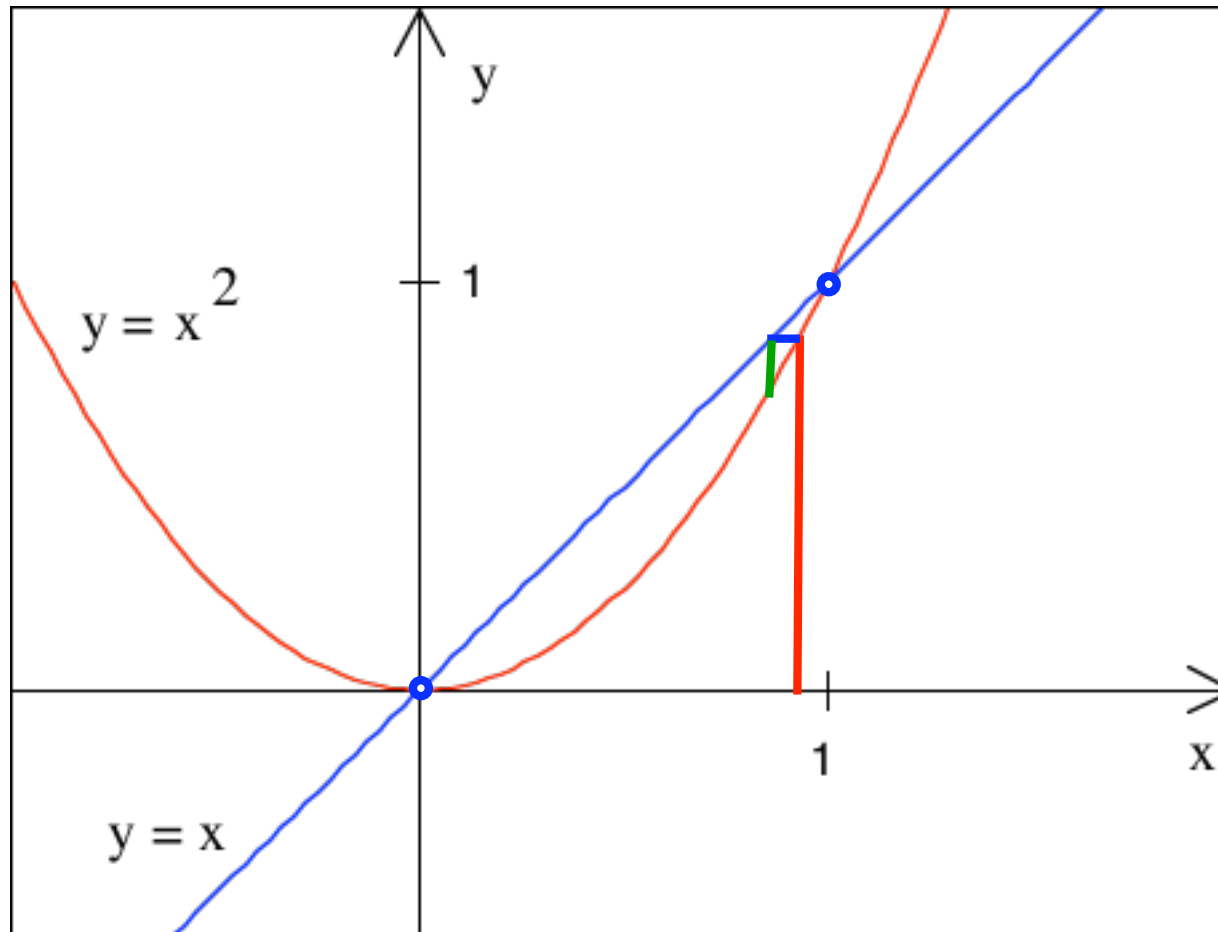
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



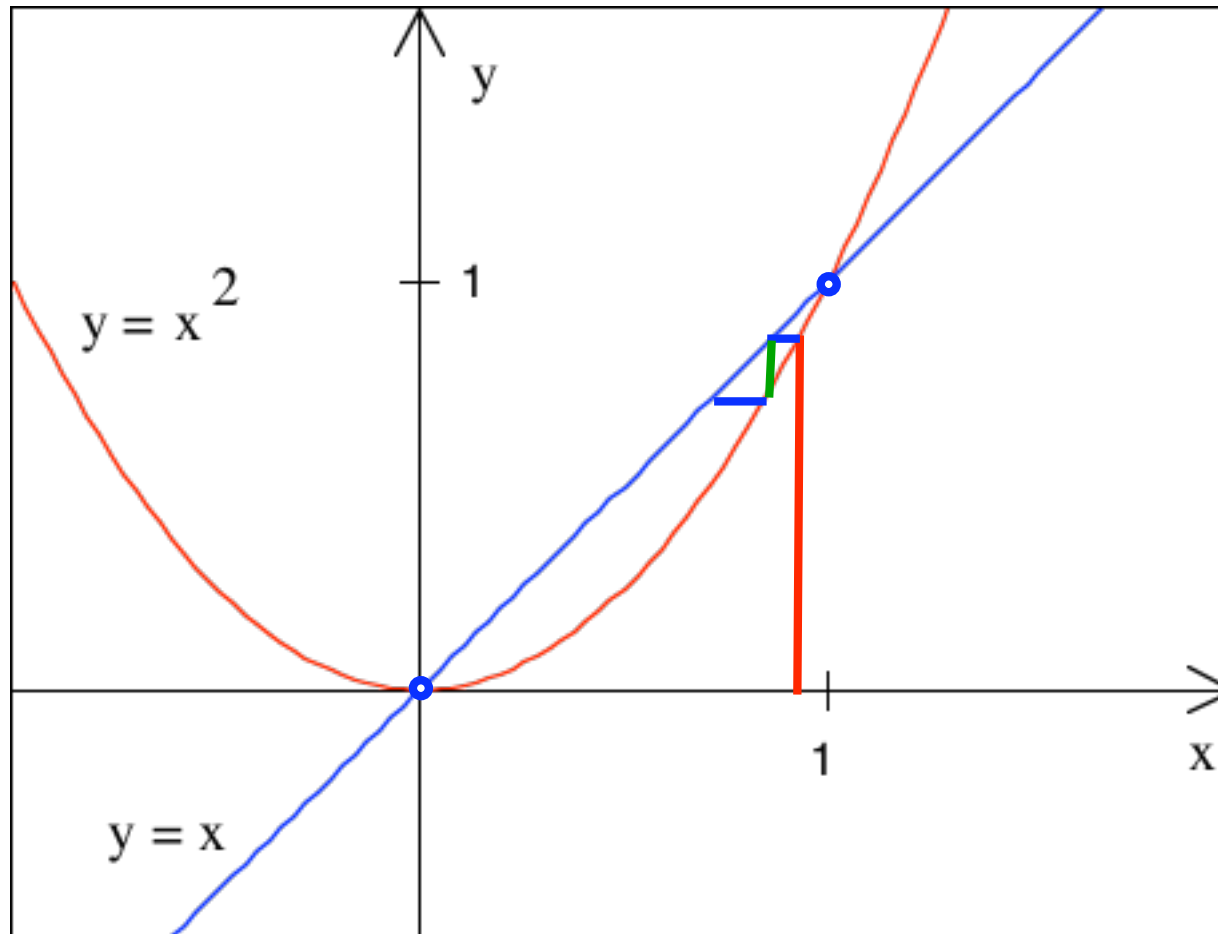
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



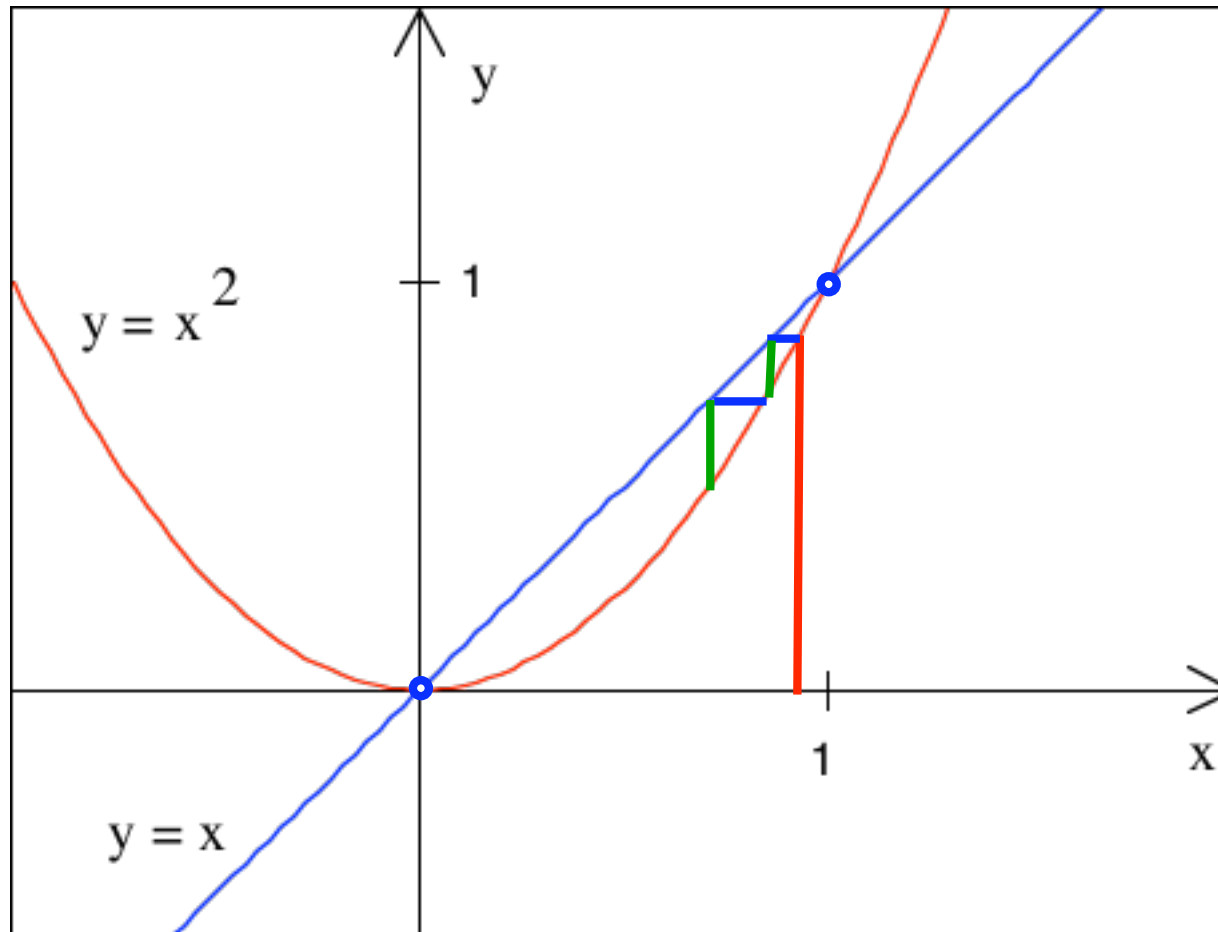
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



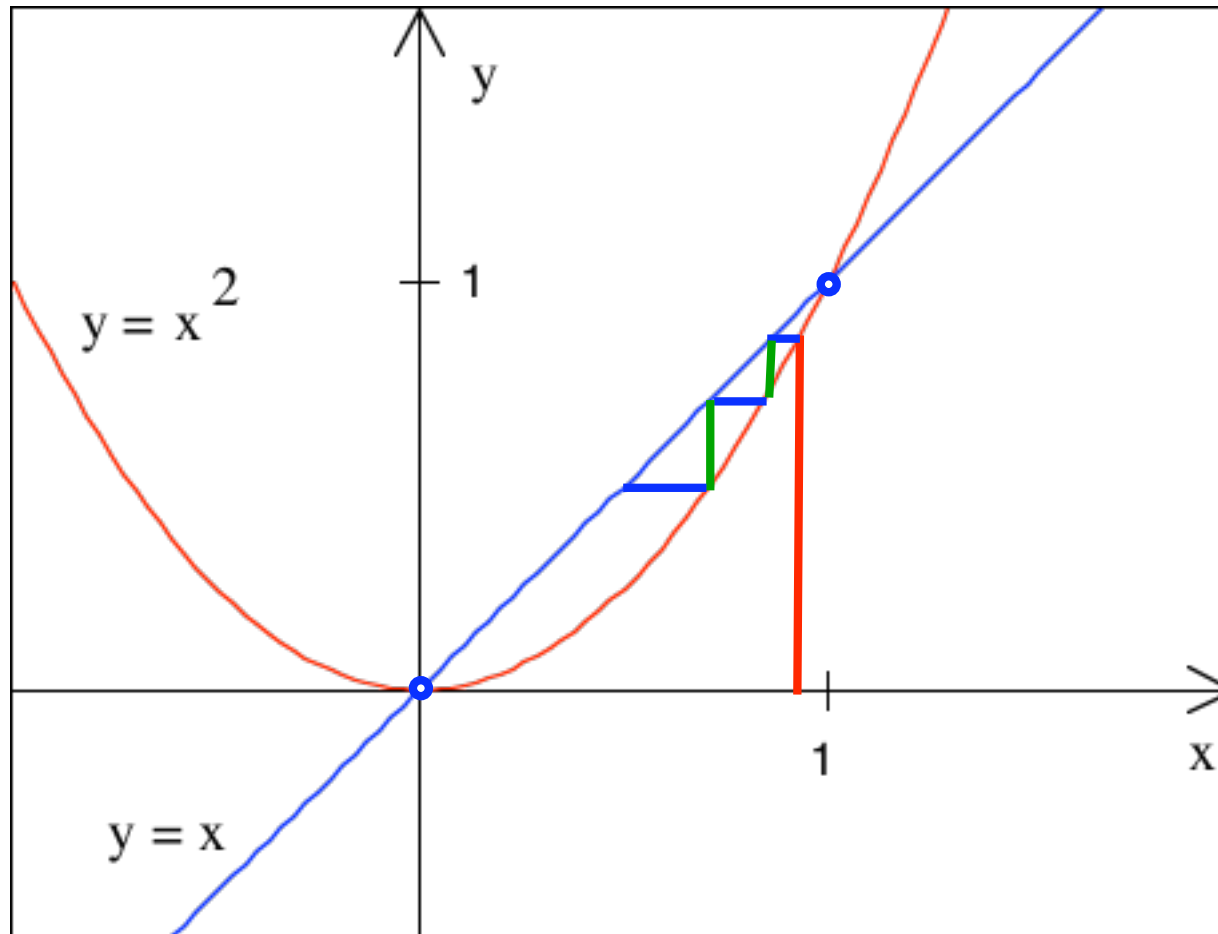
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



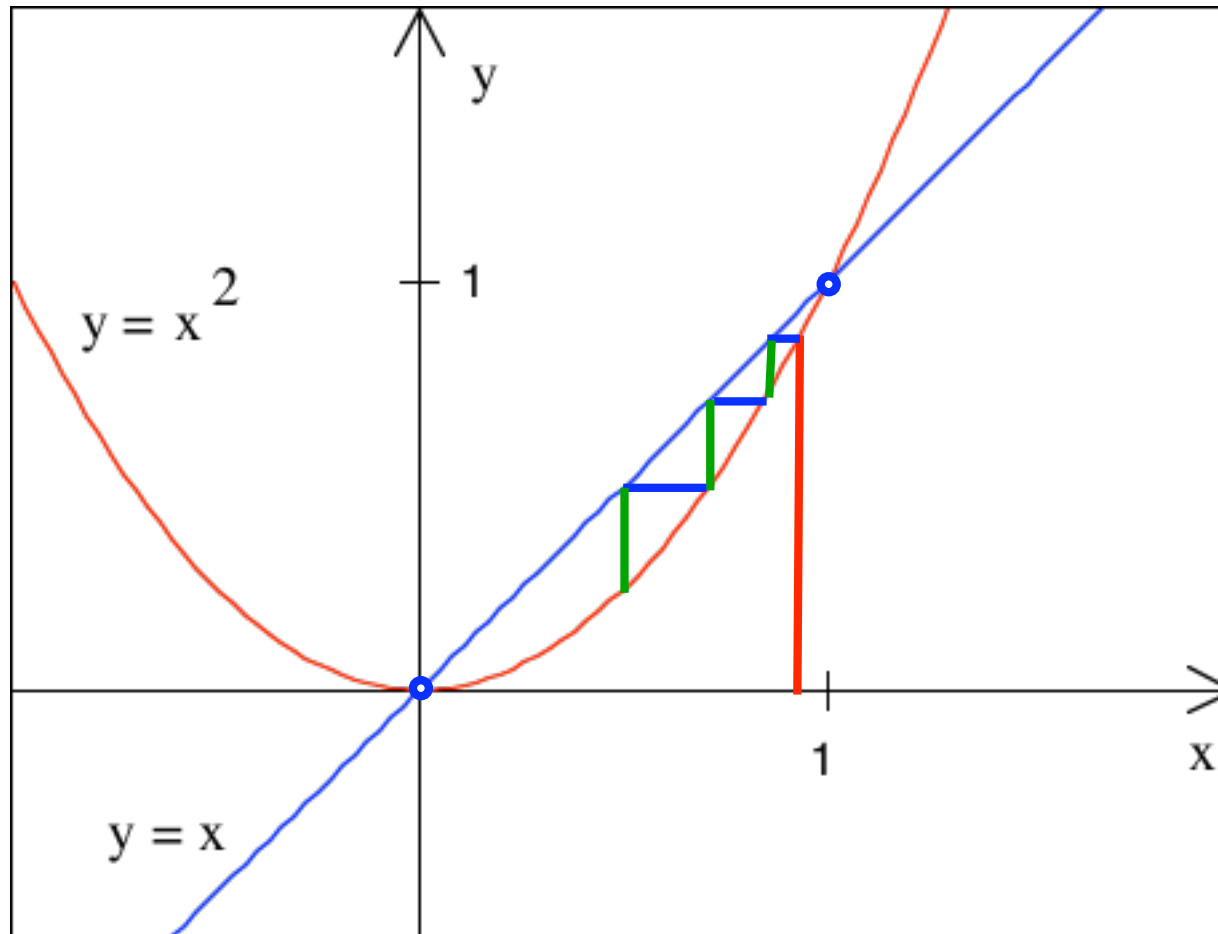
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



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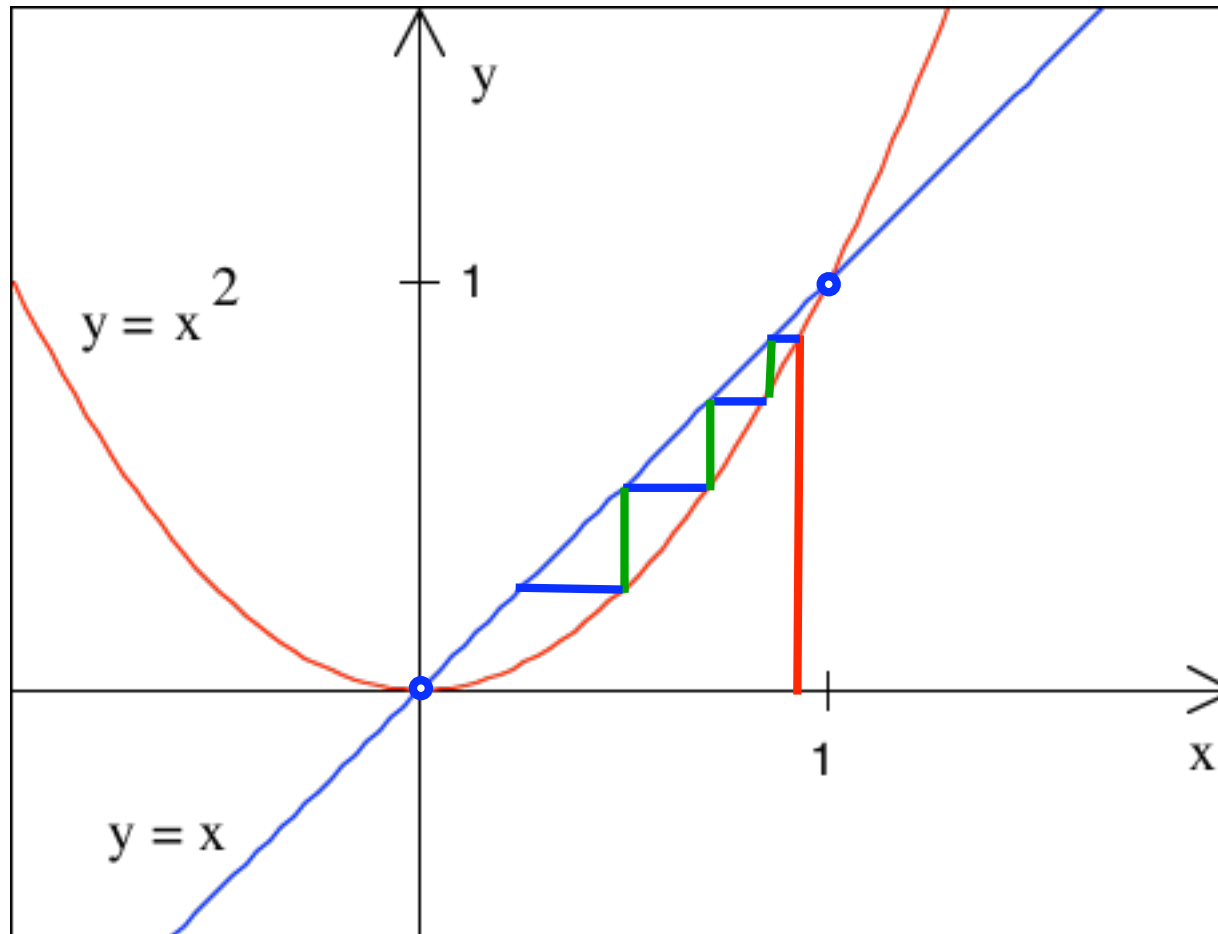


Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$

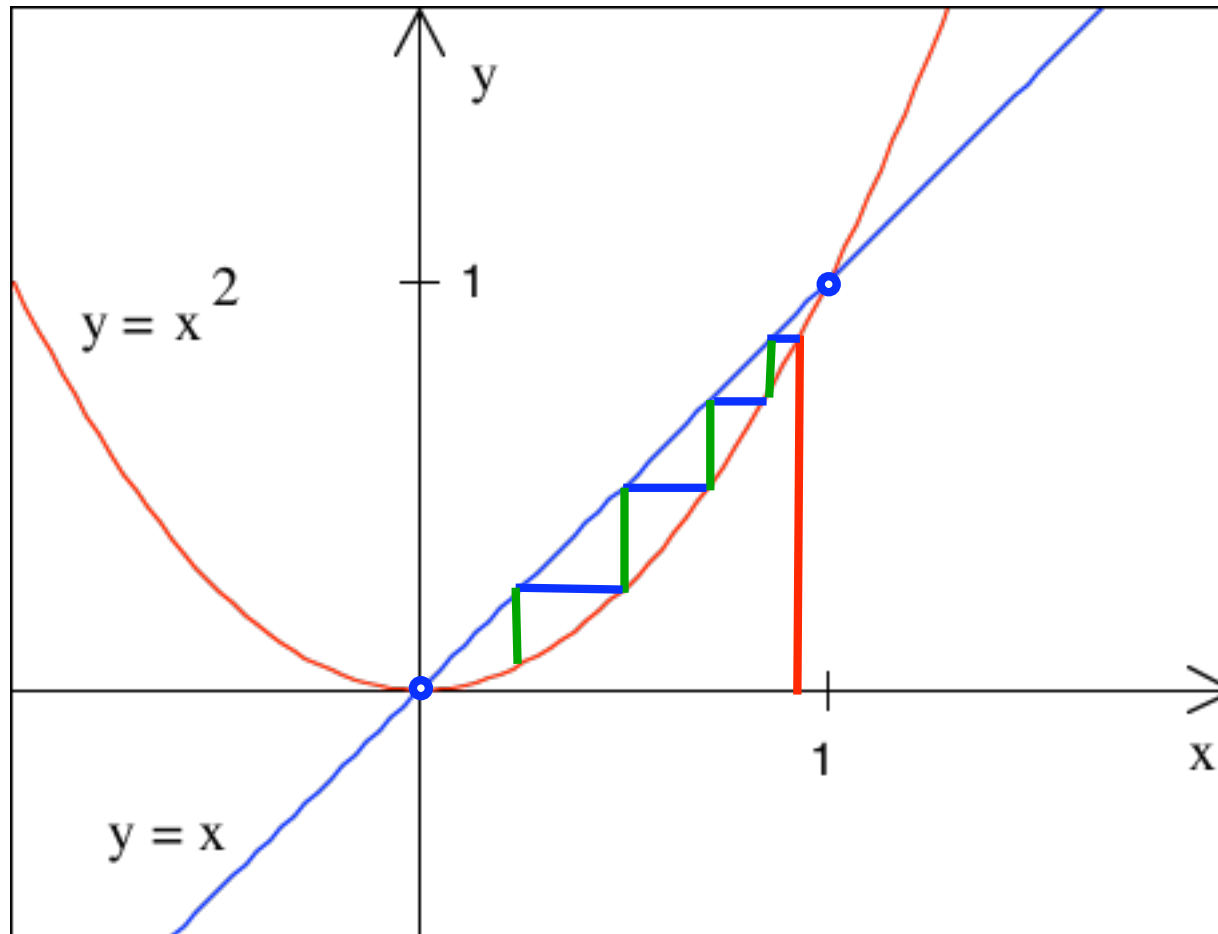




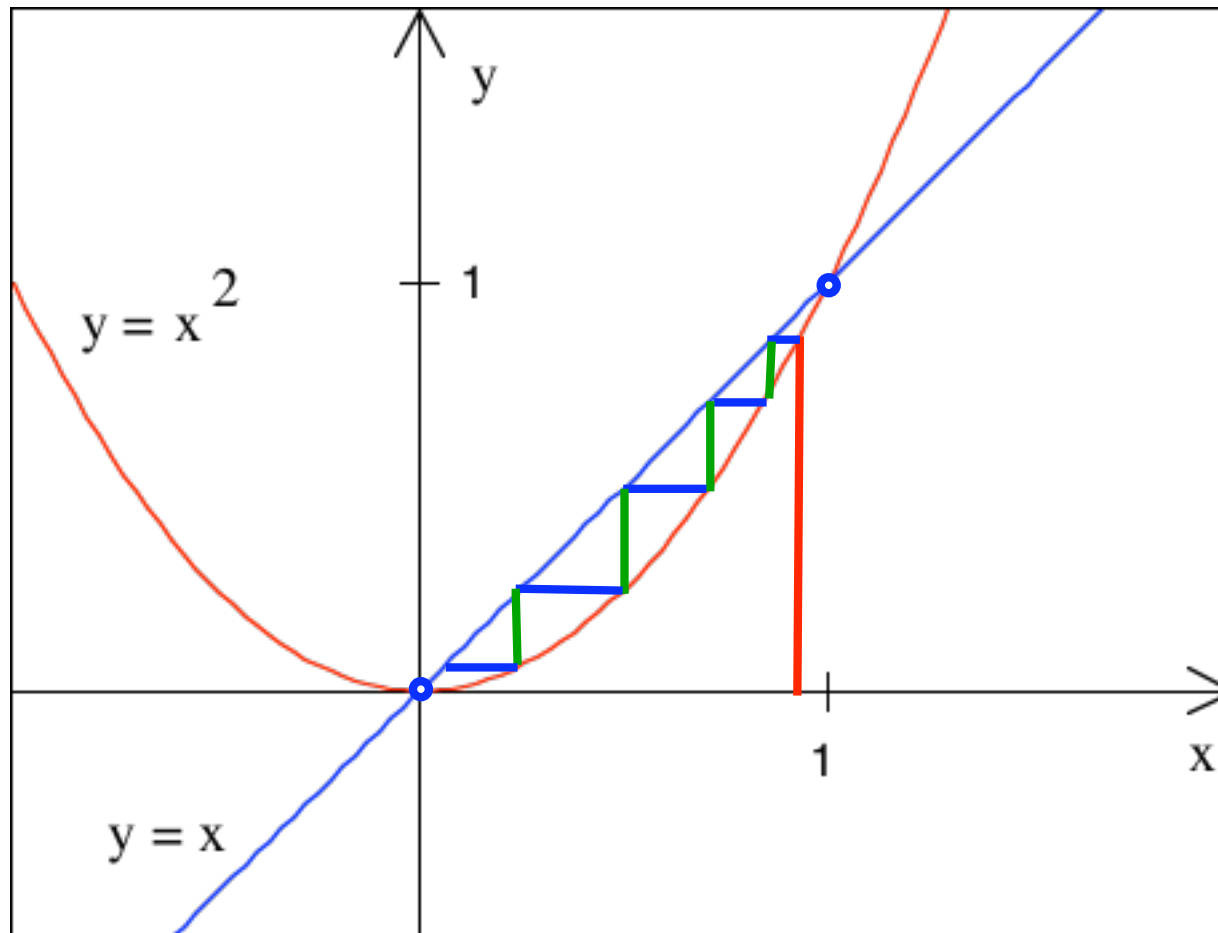
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



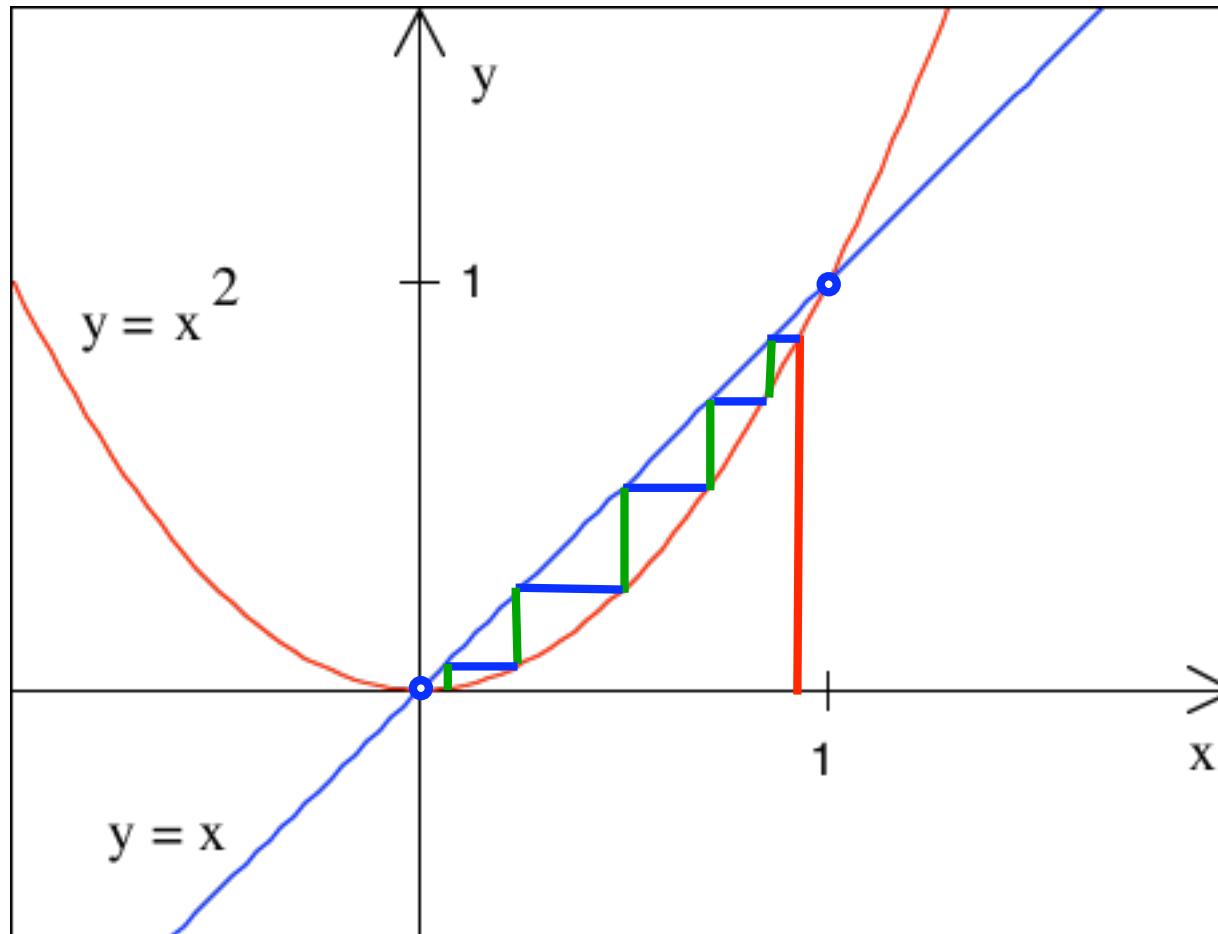
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



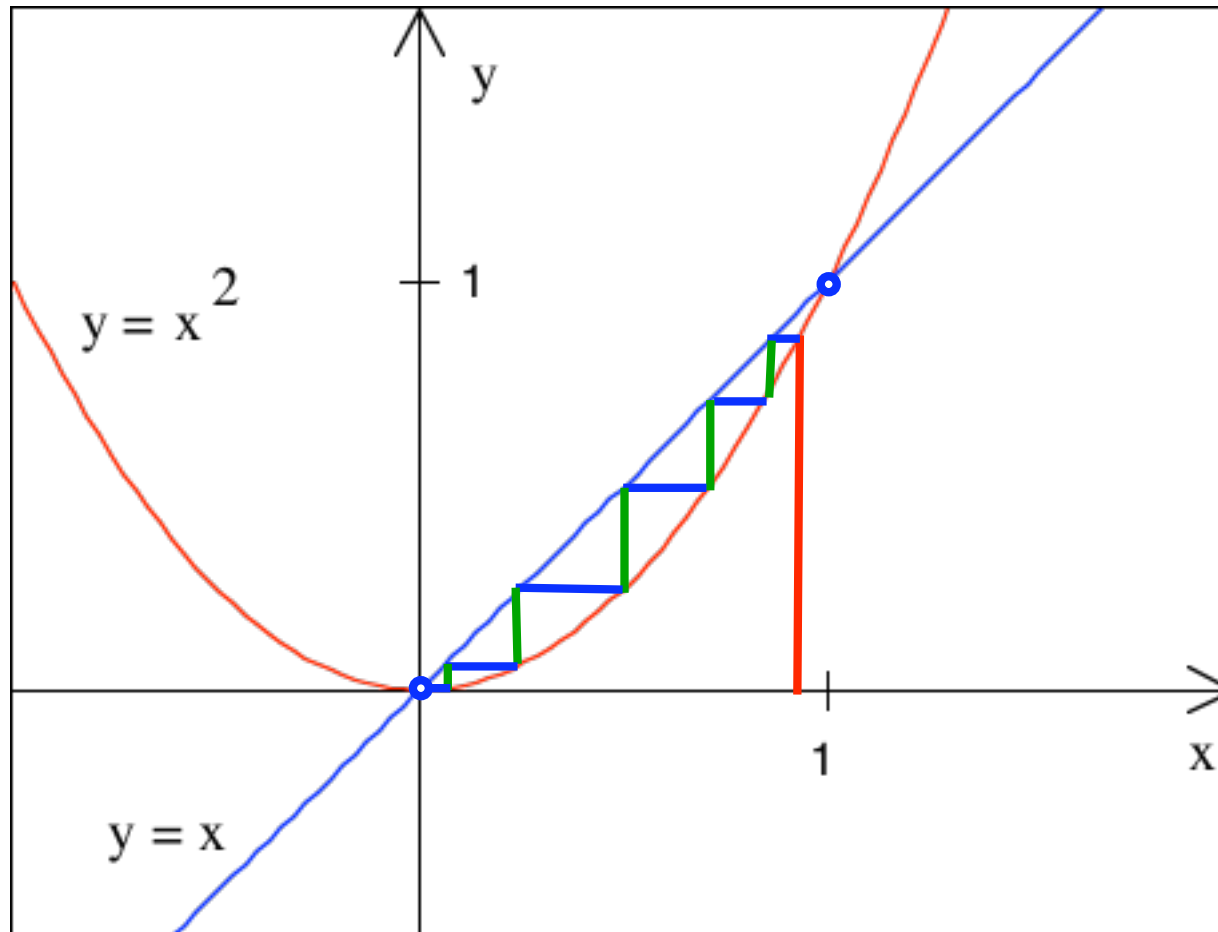
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



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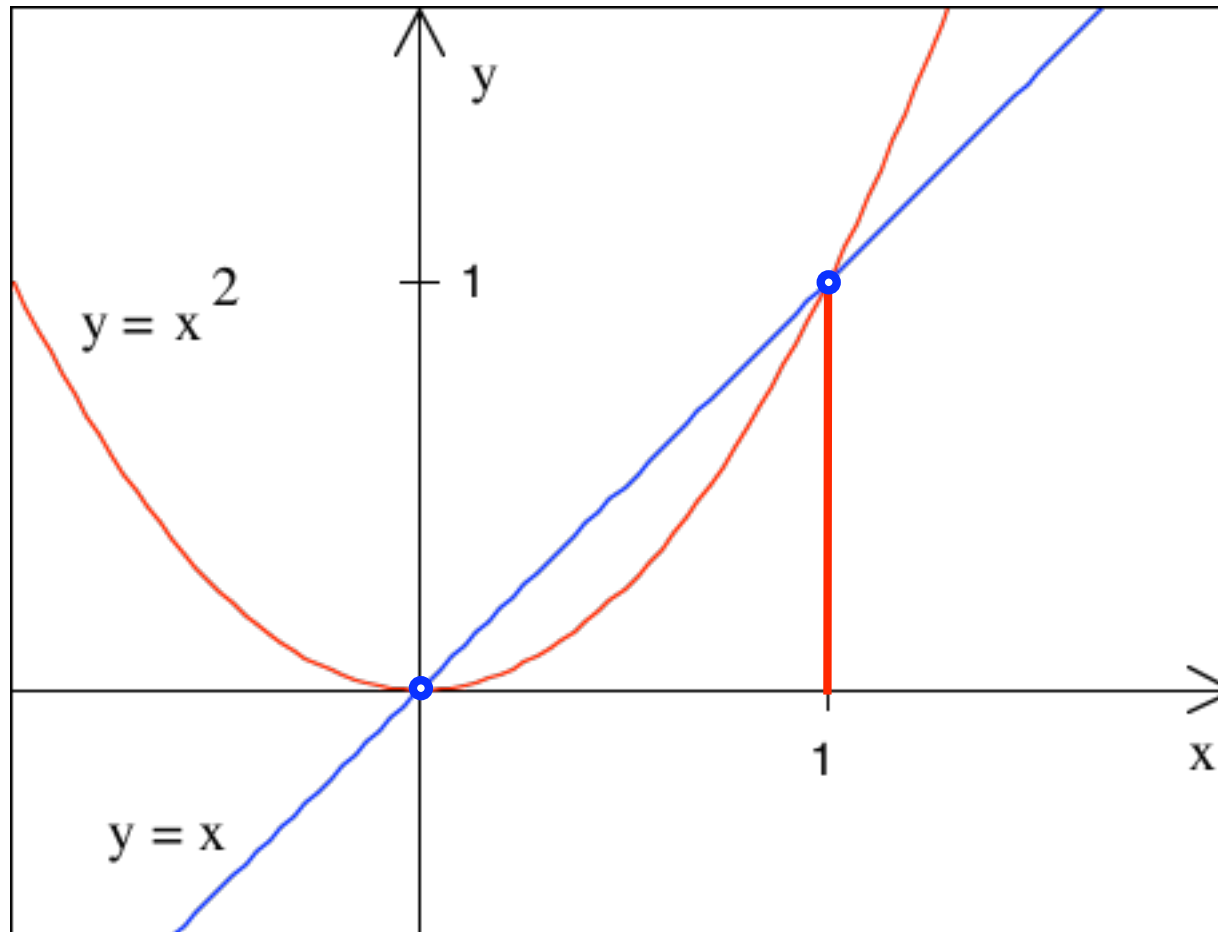
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



Beispiel:  $x^2 = x \implies x = 0$  oder  $x = 1$

Schritt		x[n]
0		1 Startwert
1		1
2		1
3		1
4		1
5		1
6		1
7		1
8		1
9		1

Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$

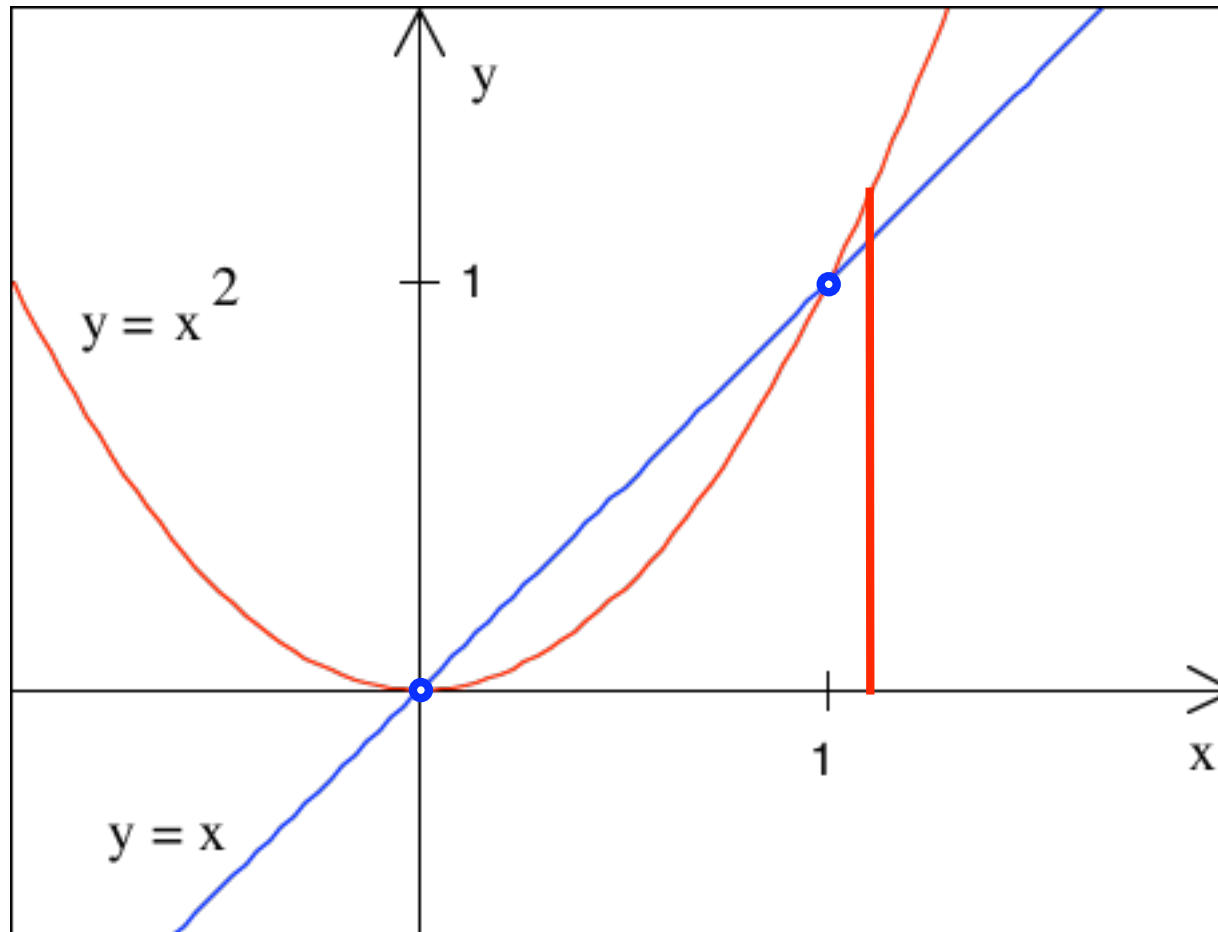


Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$

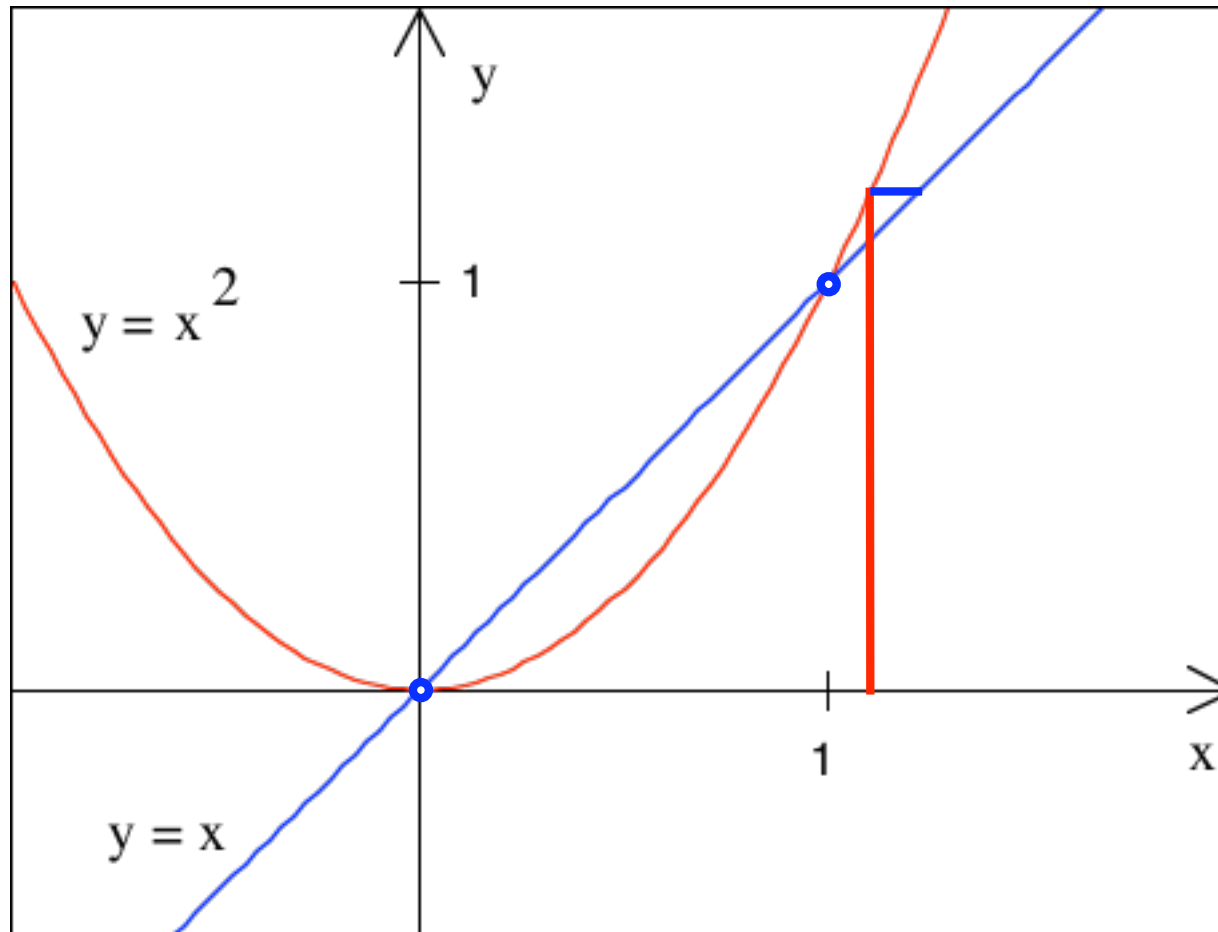
Schritt		x[n]
0		1.1 Startwert
1		1.21
2		1.4641
3		2.14358881
4		4.594972986
5		21.11377674
6		445.7915682
7		198730.1223
8		.3949366151e11
9		.1559749299e22



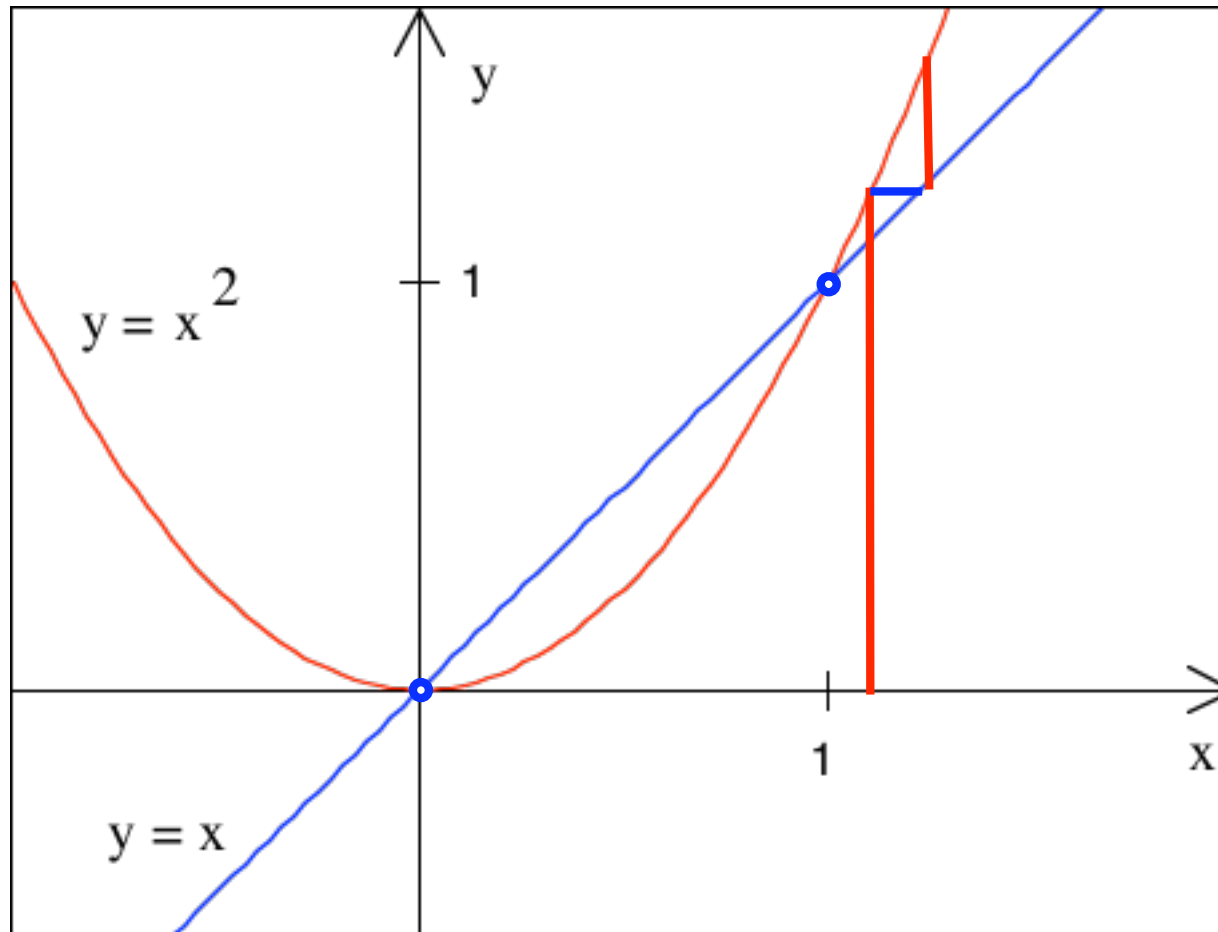
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



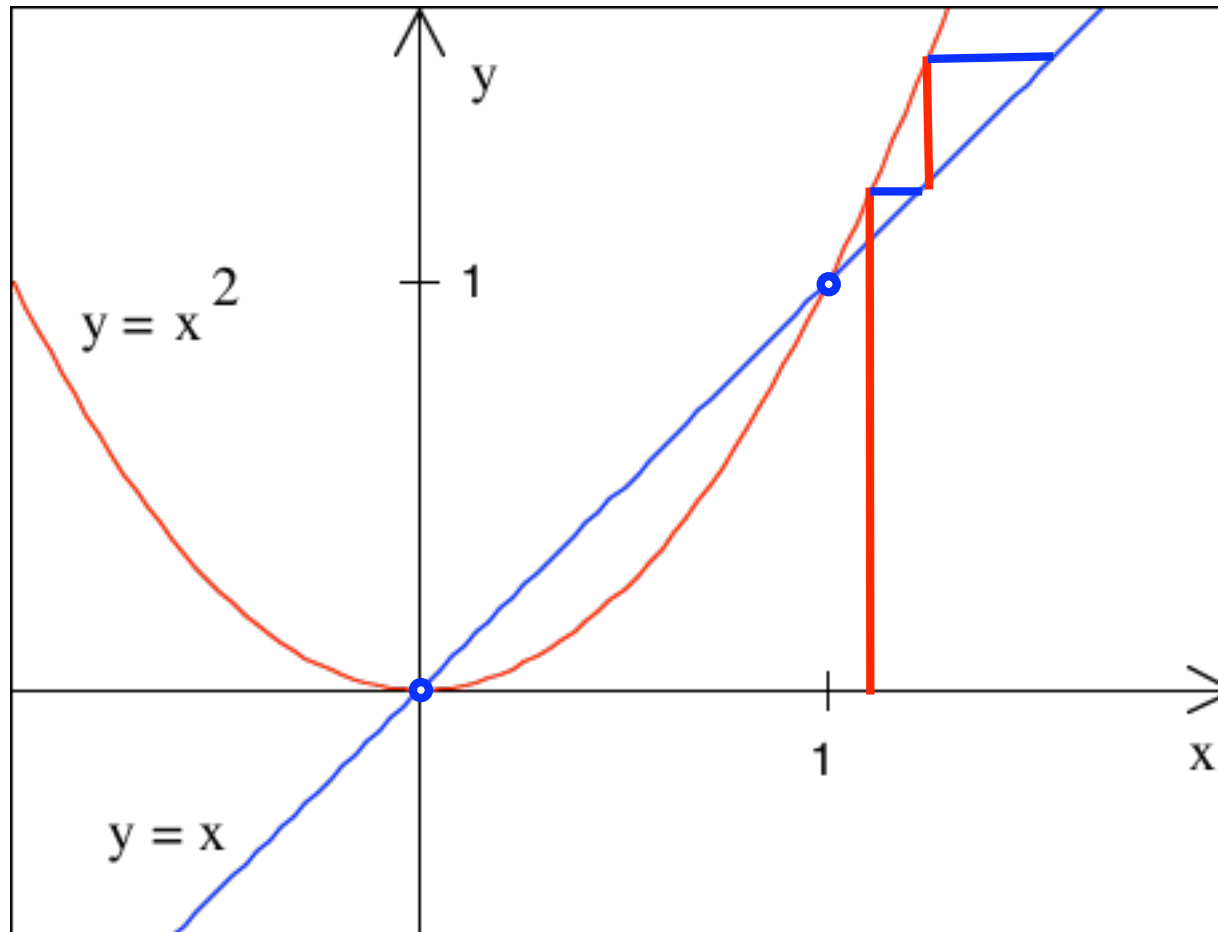
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



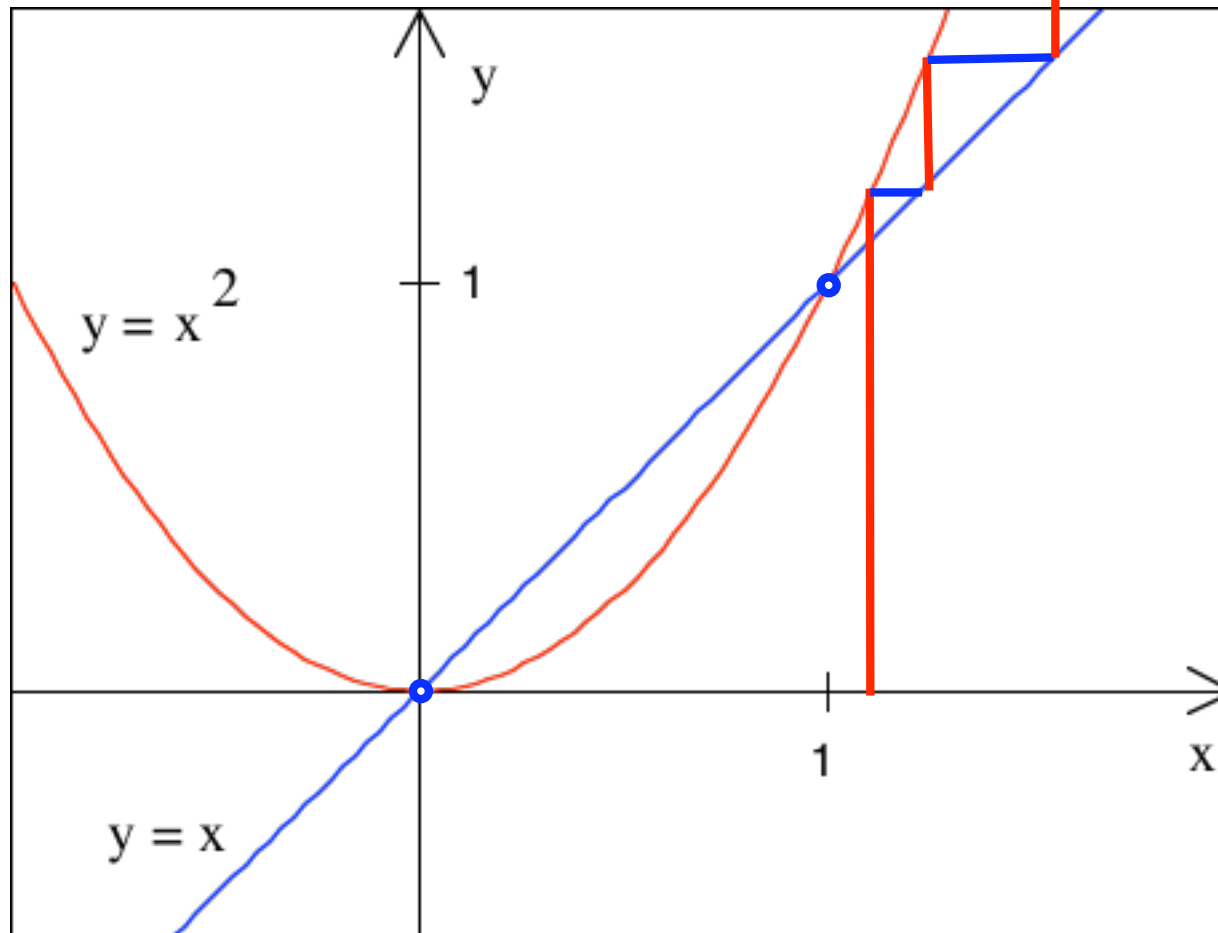
Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



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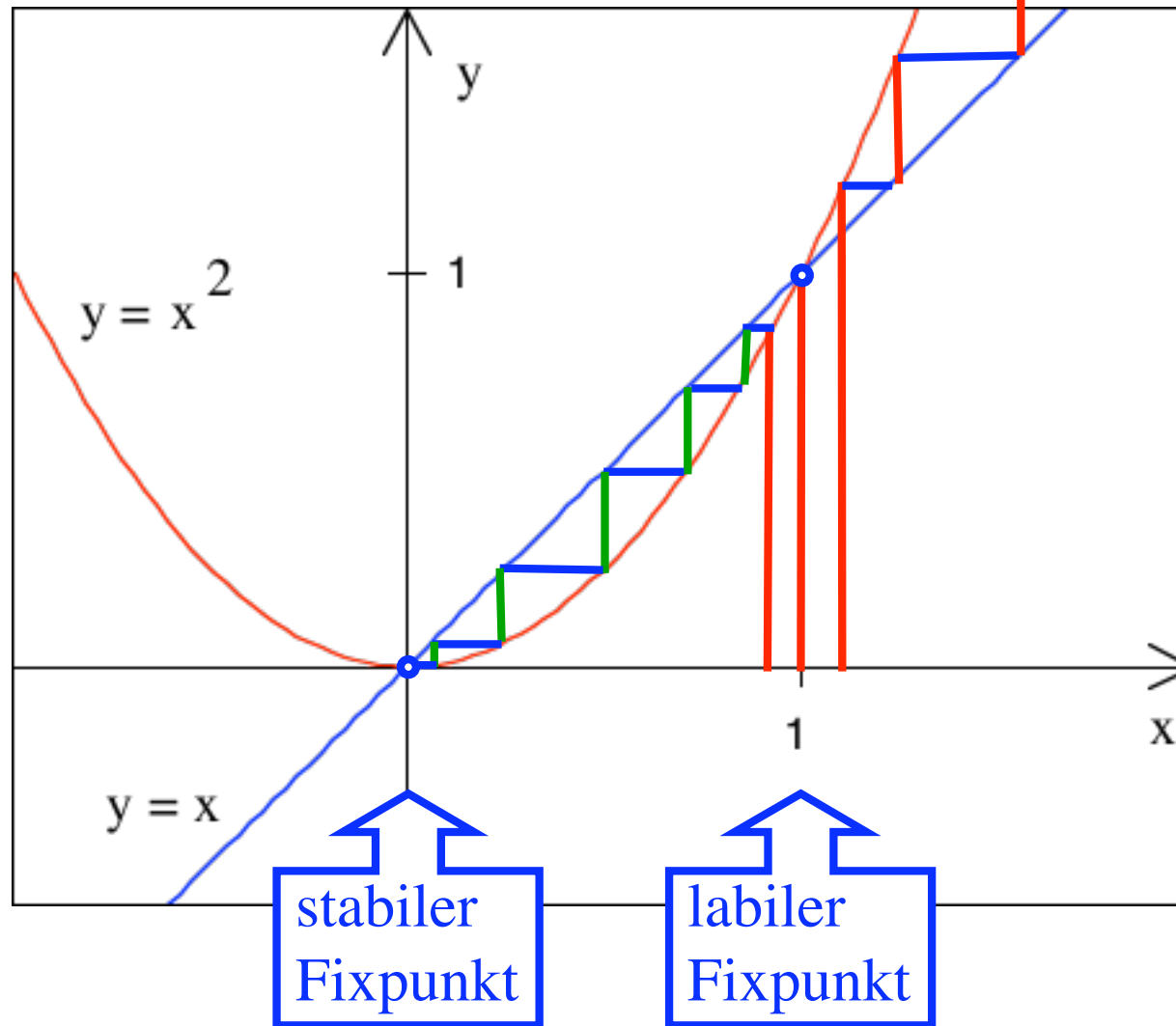


Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



Ab in die Wolken

Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



## Vorgehen beim Fixpunktverfahren


1. Gleichung umformen in die Form:  $dadamdadam = x$
2. Linken Teil als Funktion auffassen:  $f(x) = x$
3. Startwert  $x_0$  wählen
4. Rekursion:  $x_{n+1} = f(x_n)$
5. Vorsicht: Labile Fixpunkte gehen verloren


Neues Beispiel:

$$\frac{1}{10}x^3 - \frac{1}{5}x^2 - \frac{11}{10}x + \frac{6}{5} = 0$$

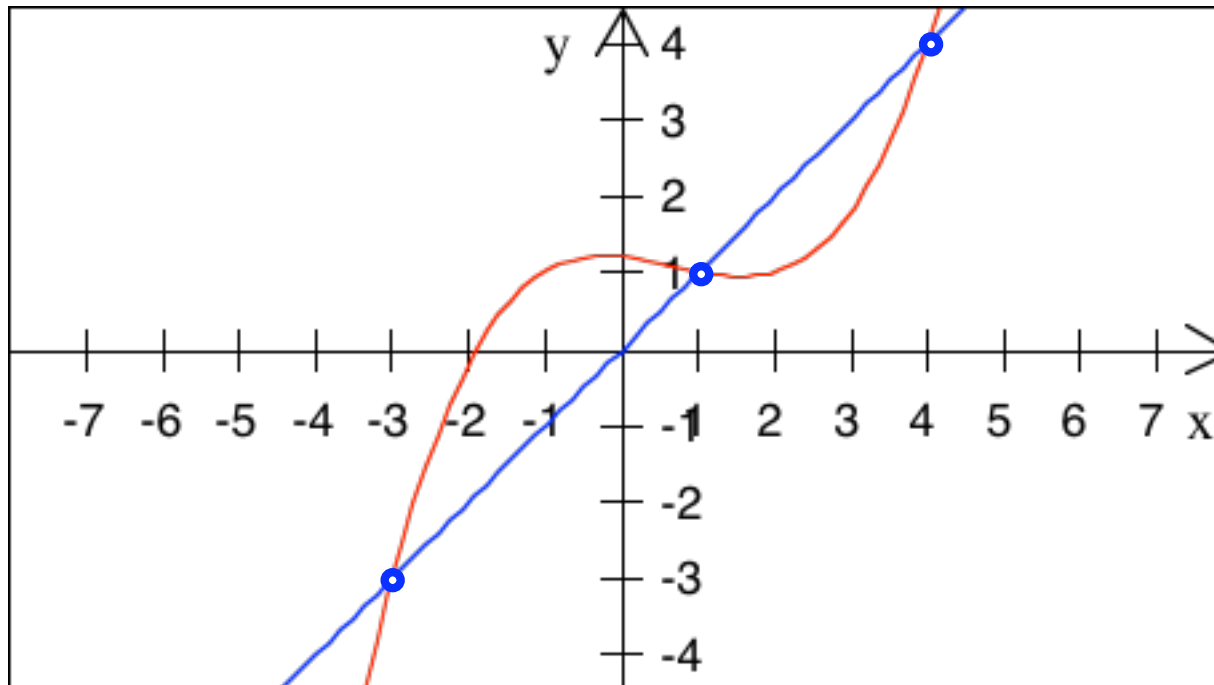


Neues Beispiel:

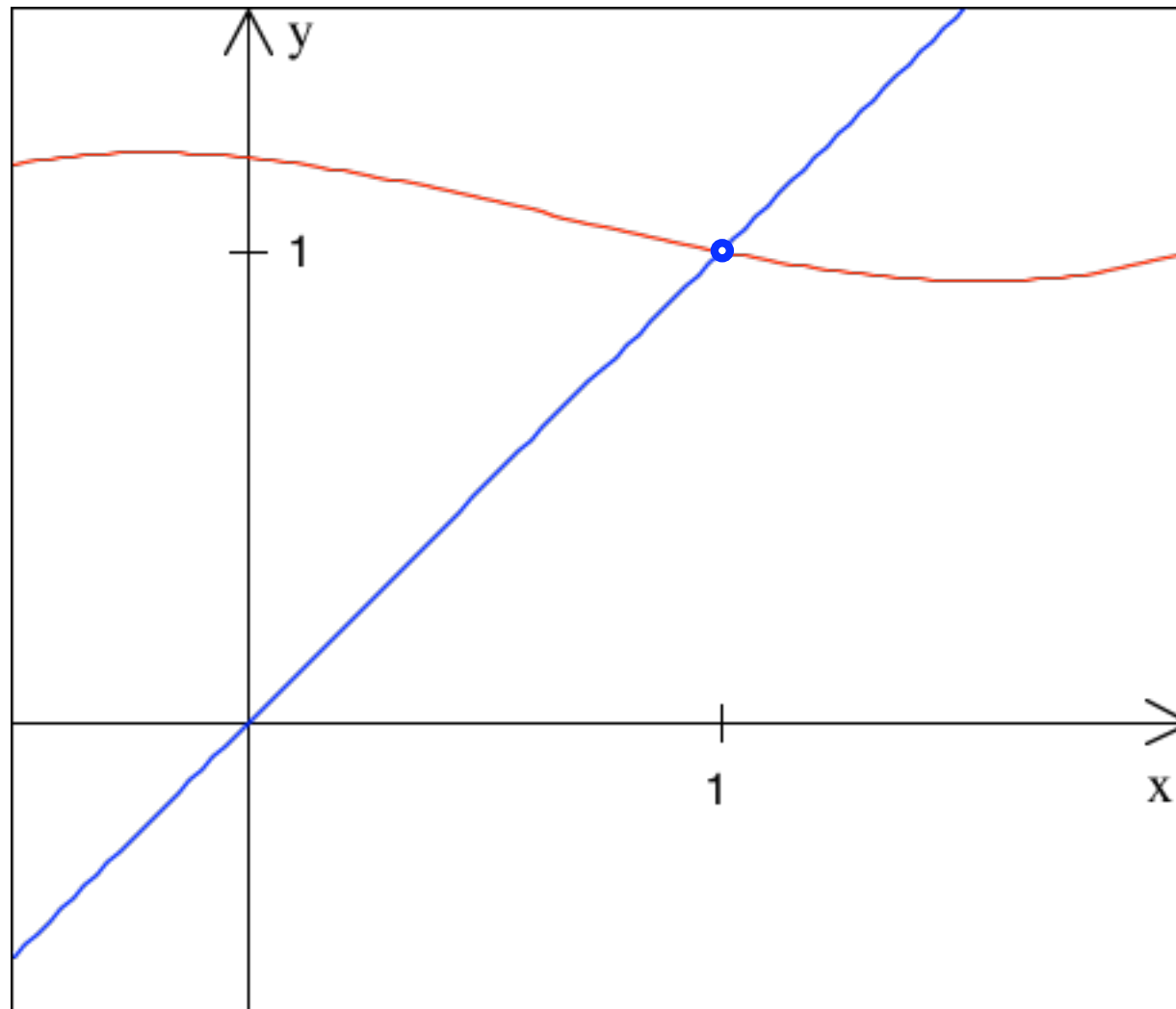
$$\frac{1}{10}x^3 - \frac{1}{5}x^2 - \frac{11}{10}x + \frac{6}{5} = 0$$


$$\underbrace{\frac{1}{10}x^3 - \frac{1}{5}x^2 - \frac{1}{10}x + \frac{6}{5}}_{f(x)} = x$$


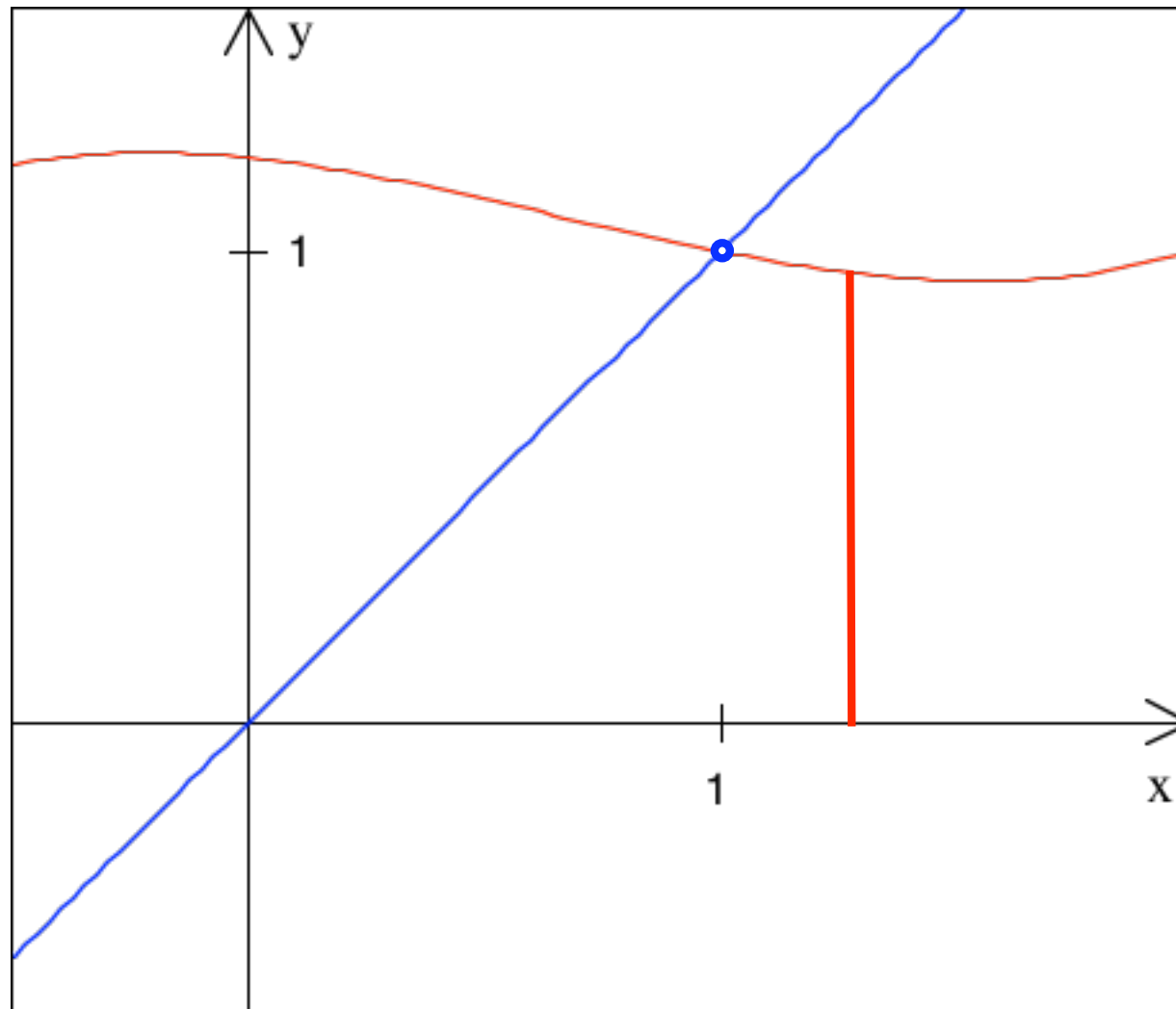
$$\underbrace{\frac{1}{10}x^3 - \frac{1}{5}x^2 - \frac{1}{10}x + \frac{6}{5}}_{f(x)} = x$$

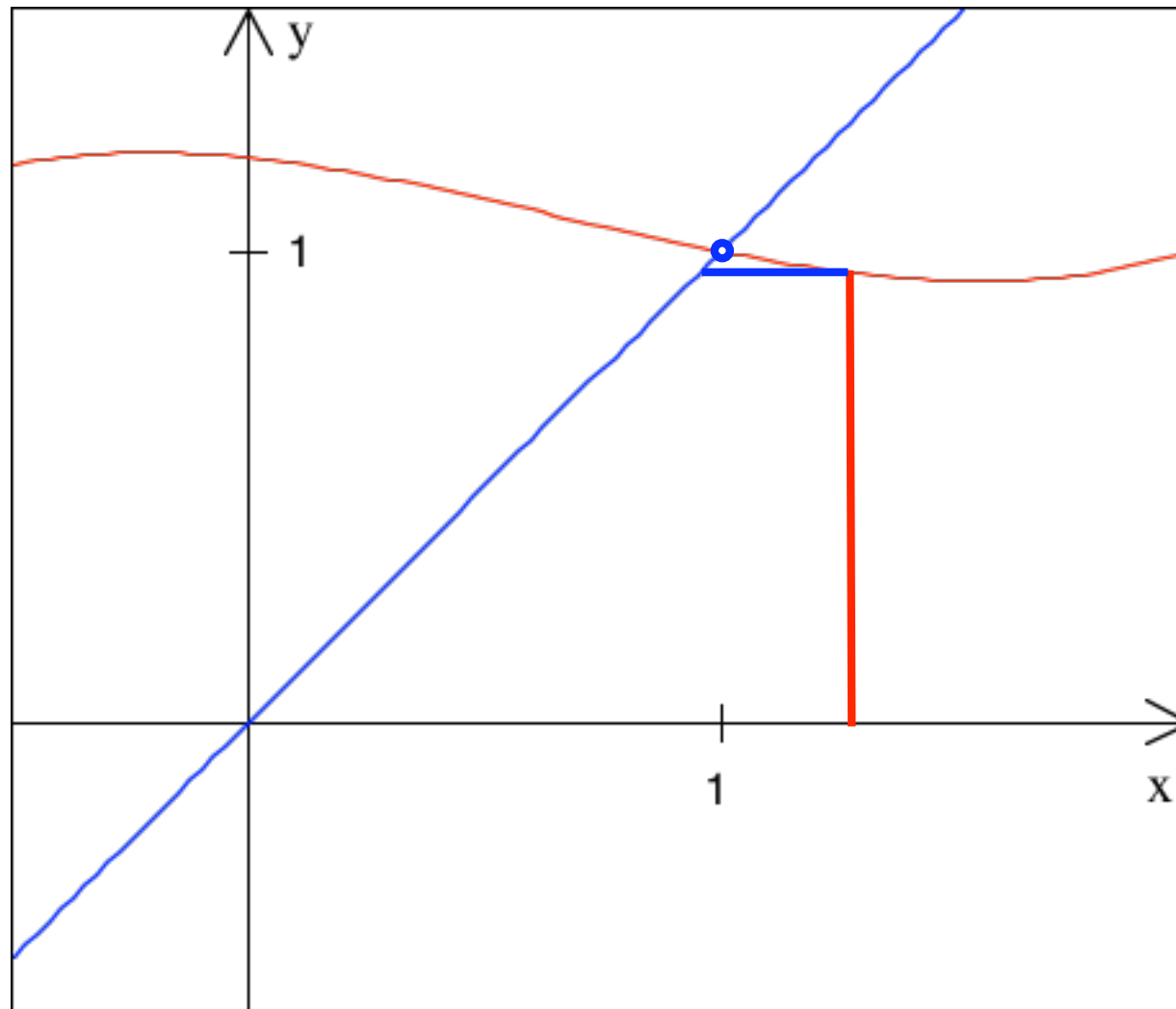


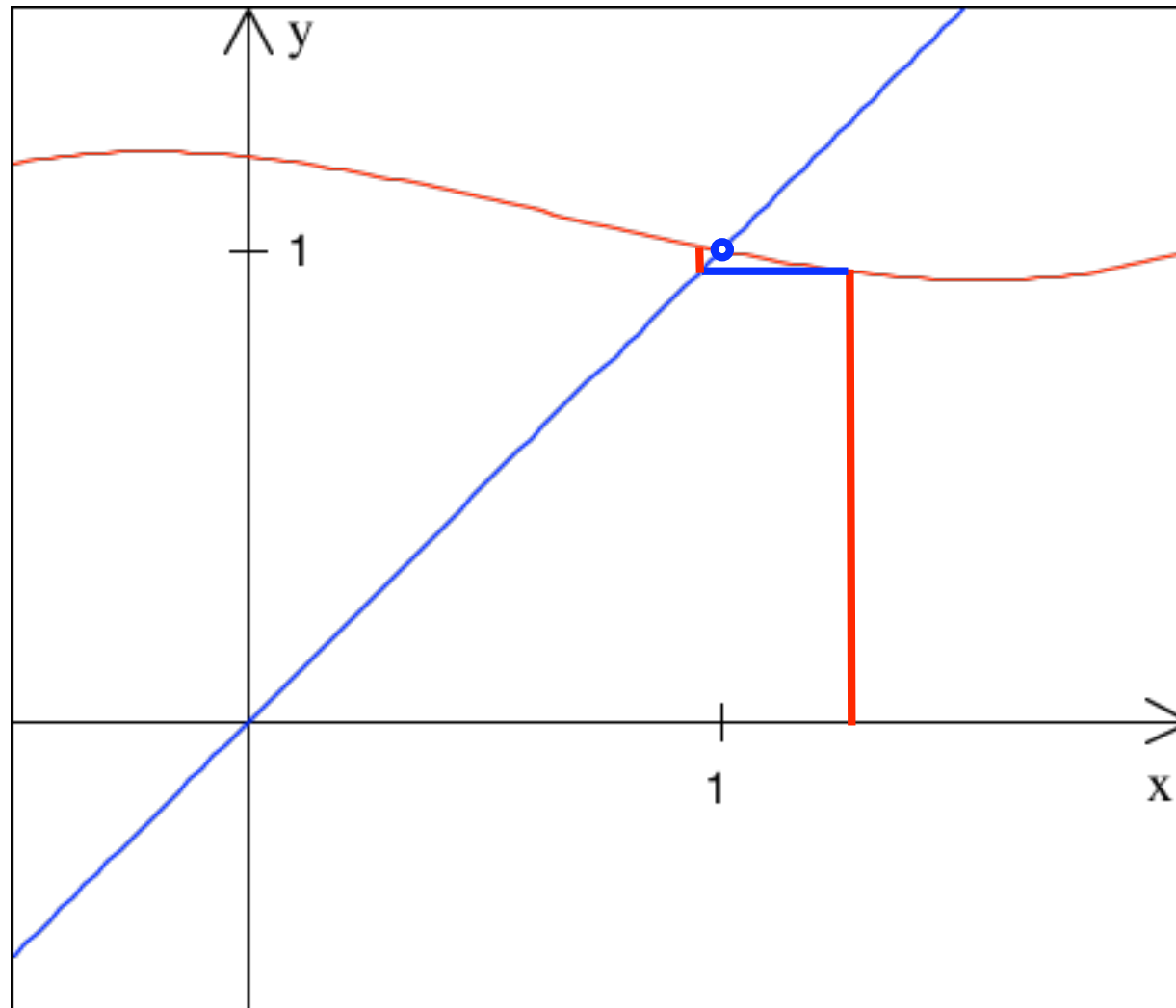
Labile / stabile Fixpunkte?

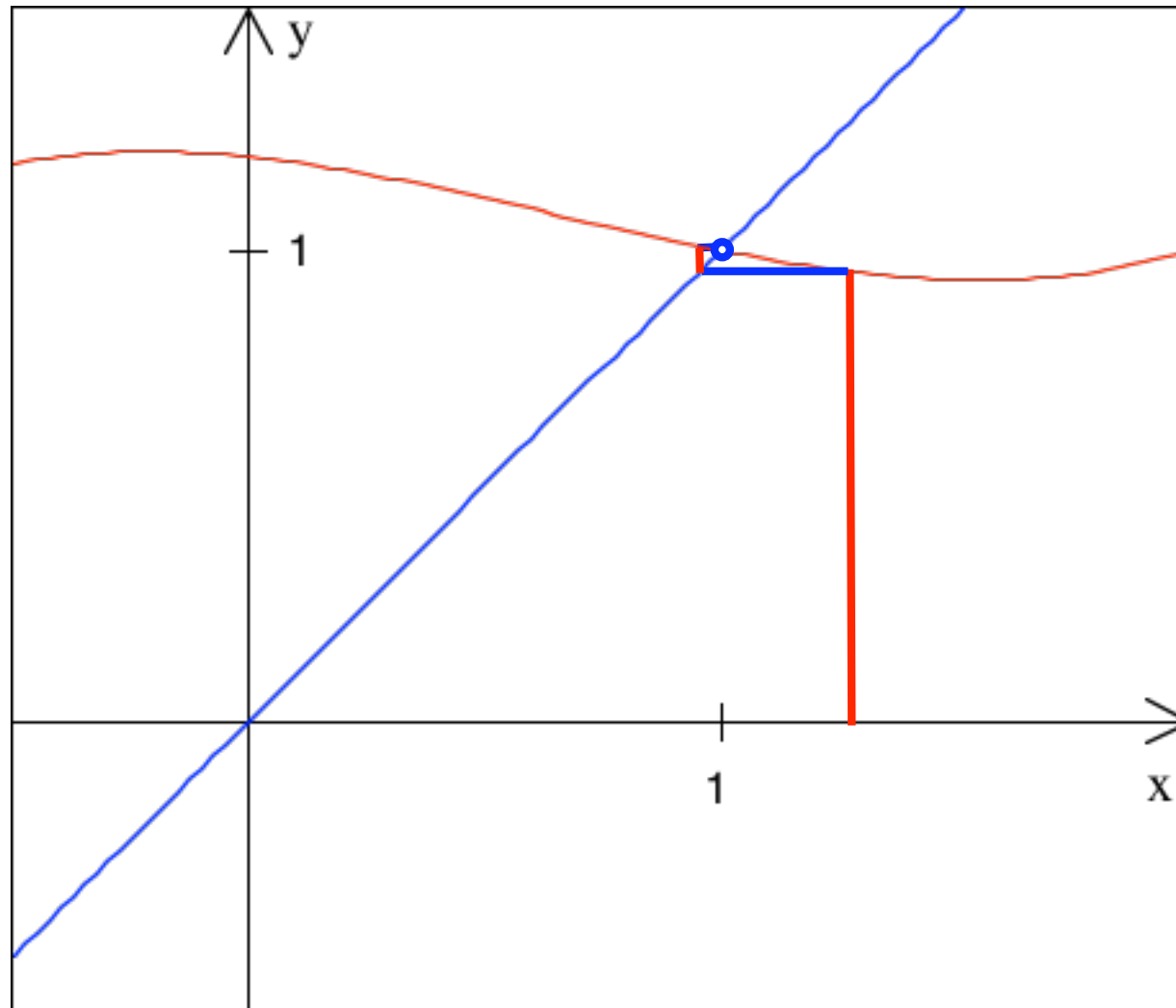


Ausschnitt

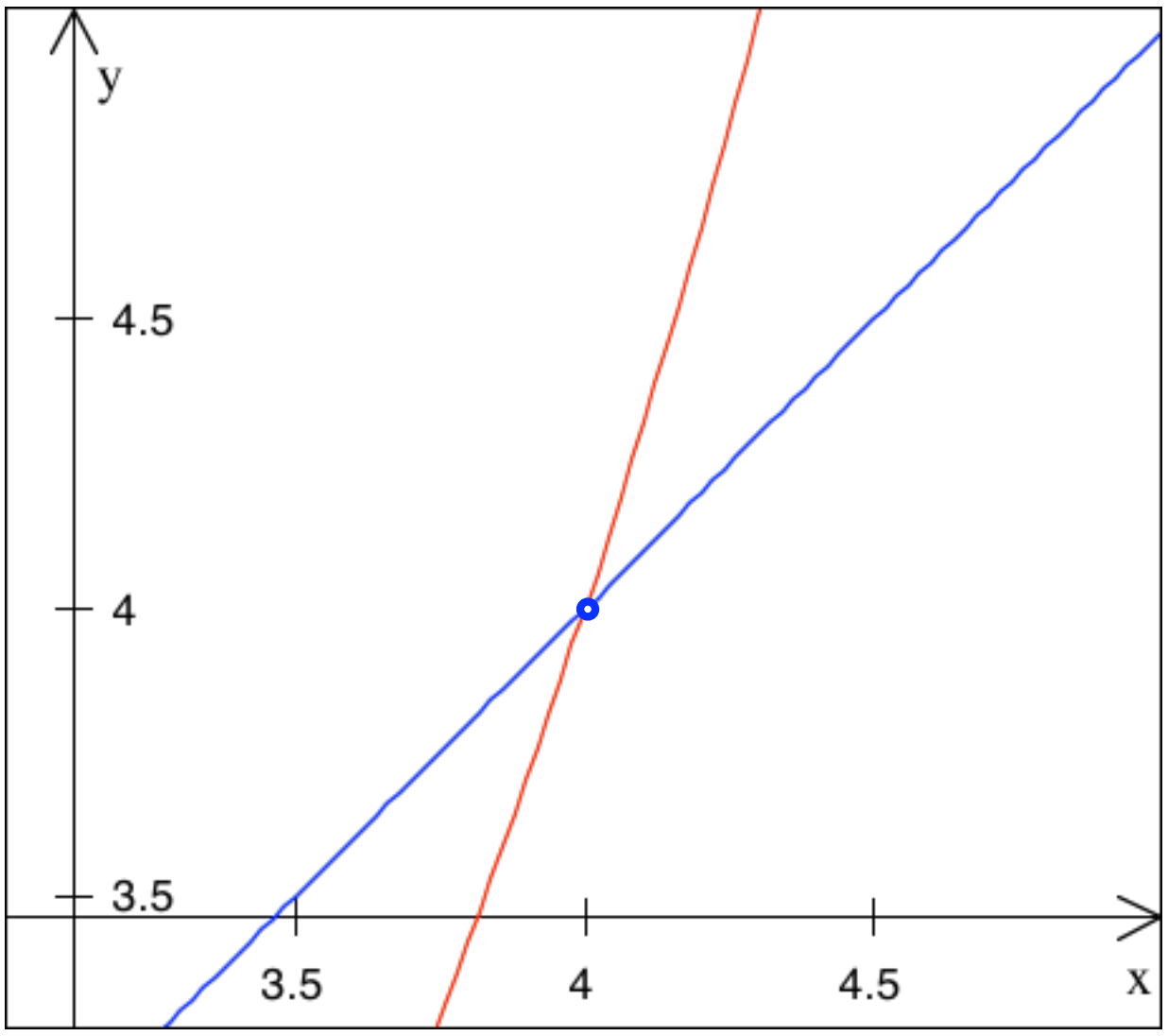






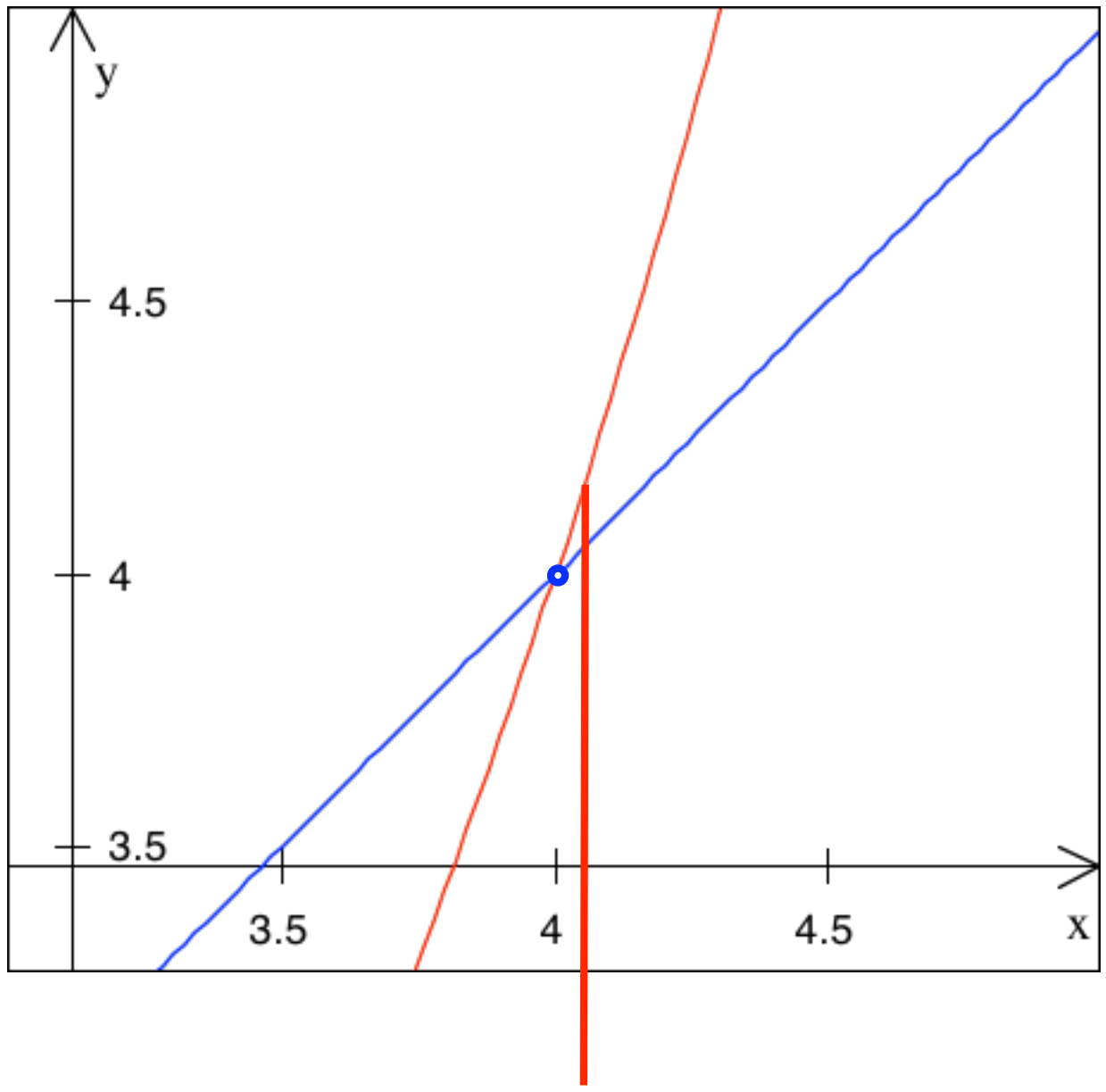


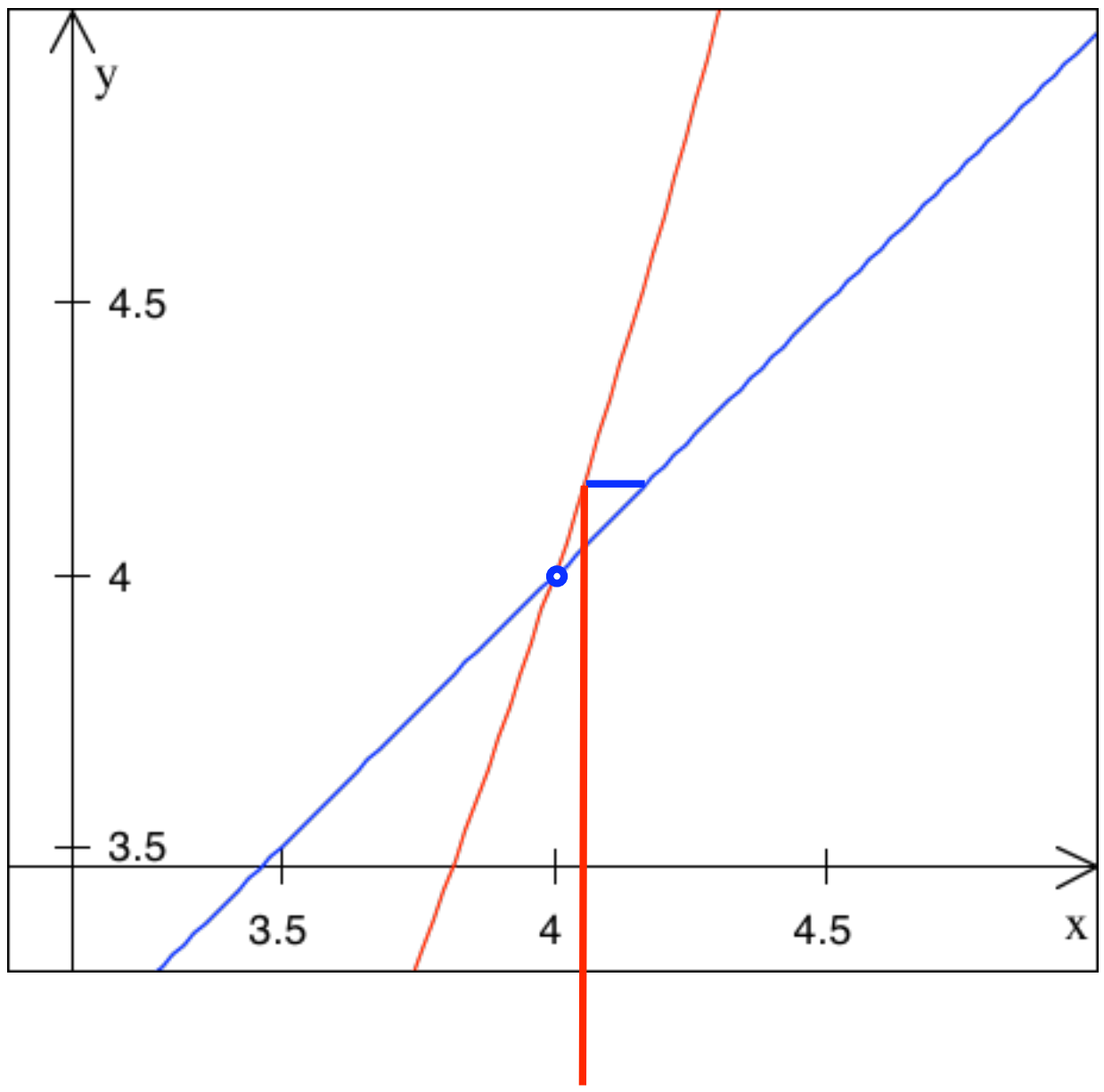
$x = 1$  ist ein stabiler Fixpunkt

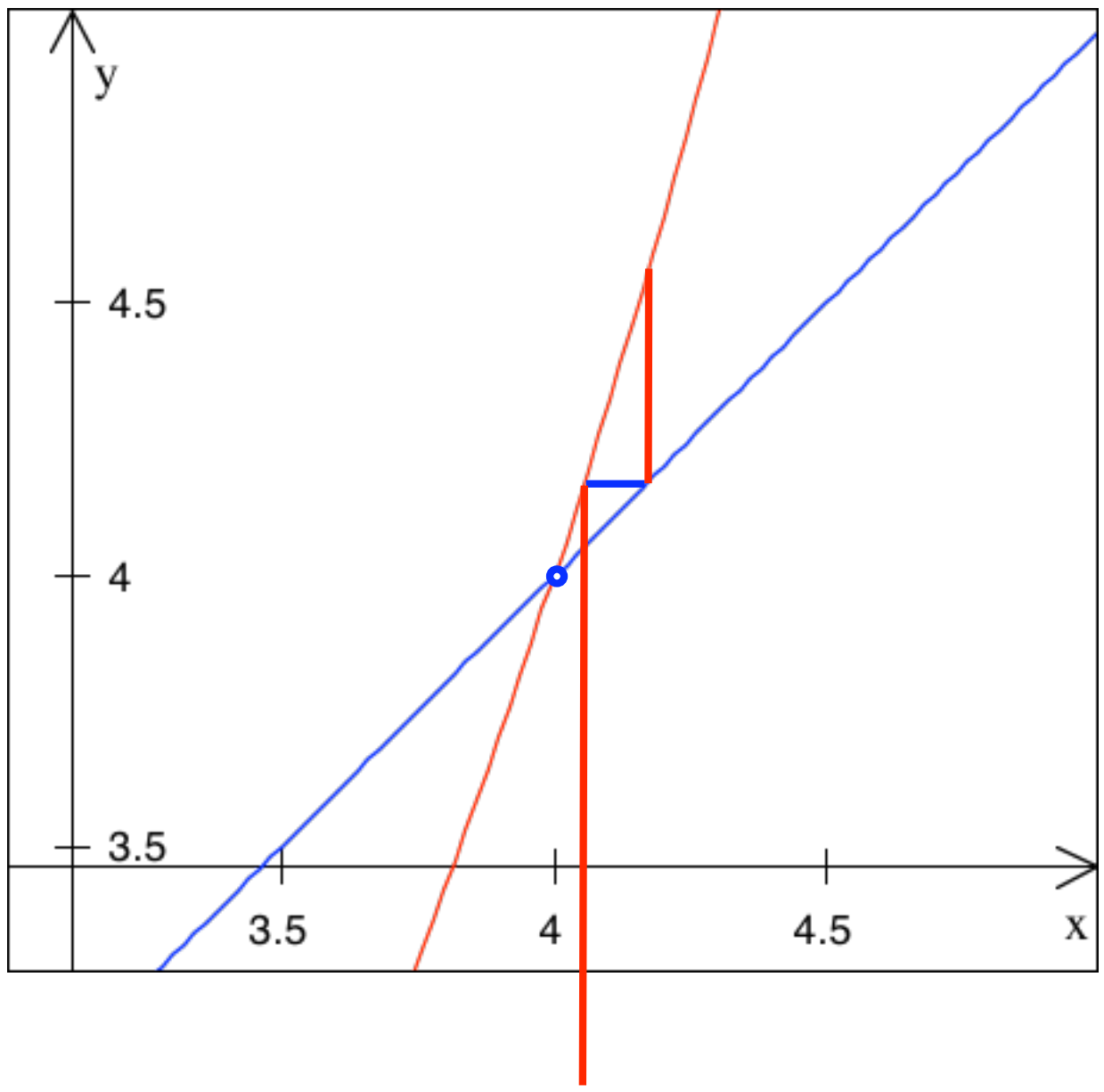


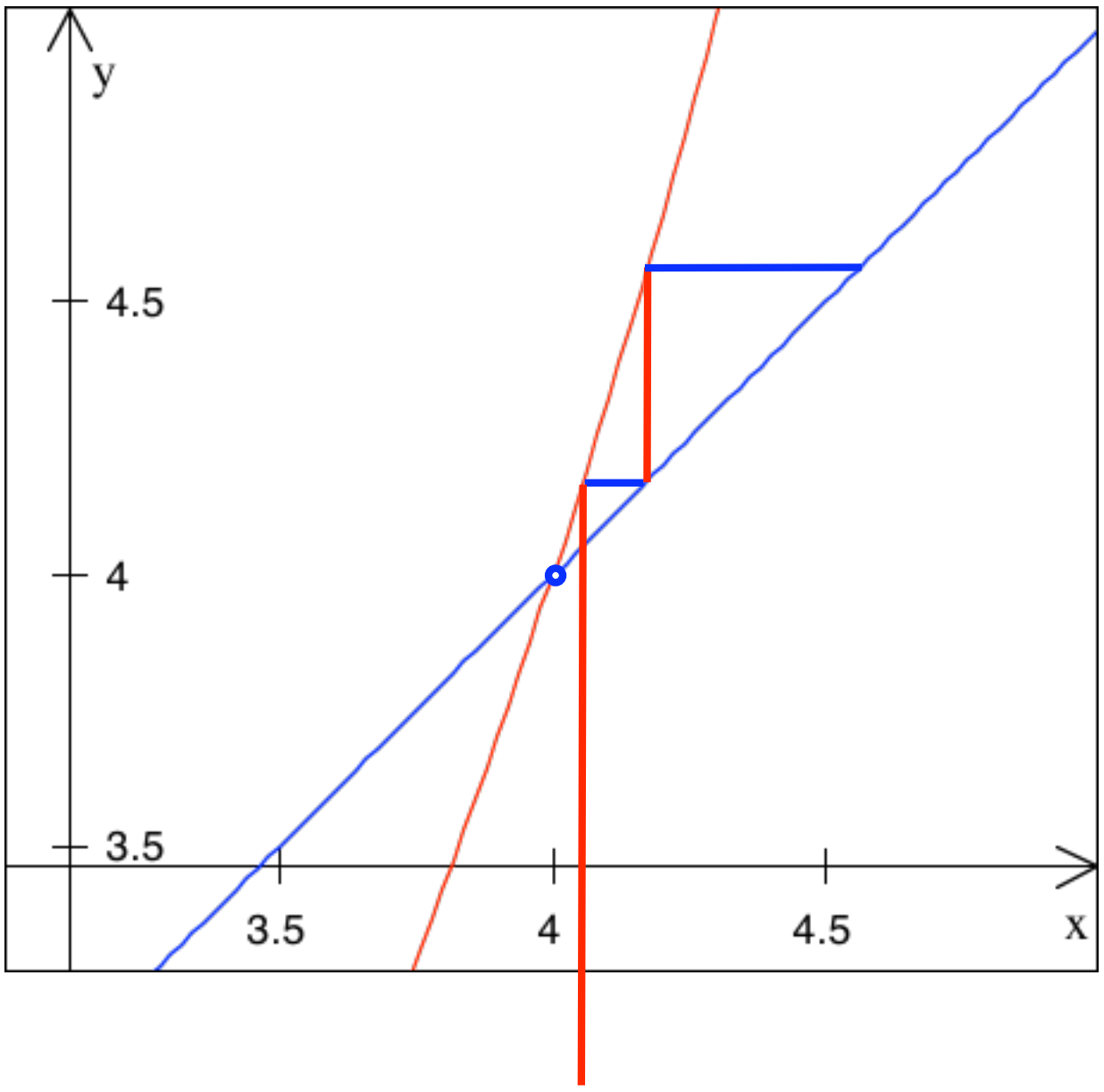
Anderer Ausschnitt

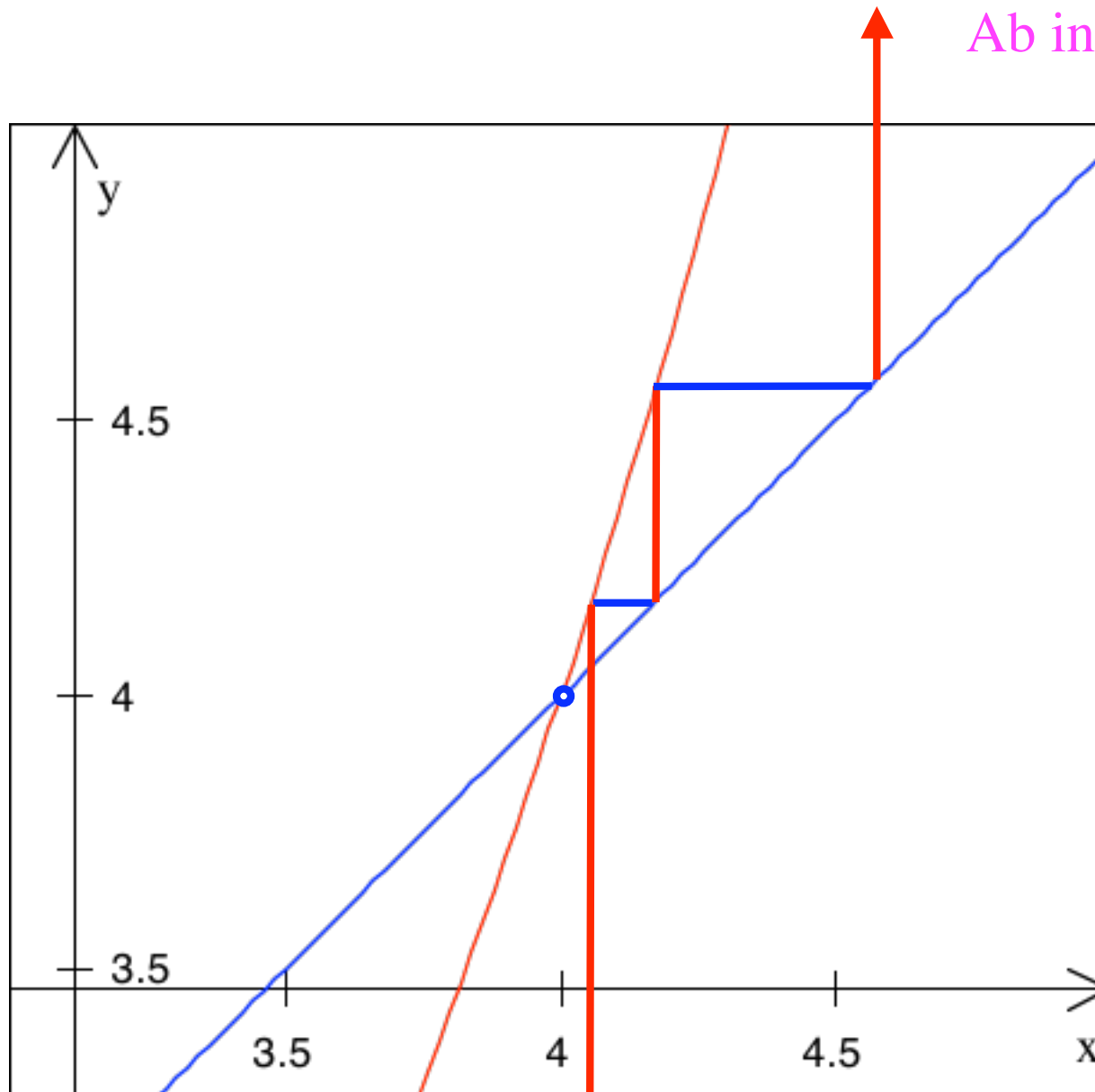












$x = 4$  ist ein labiler Fixpunkt

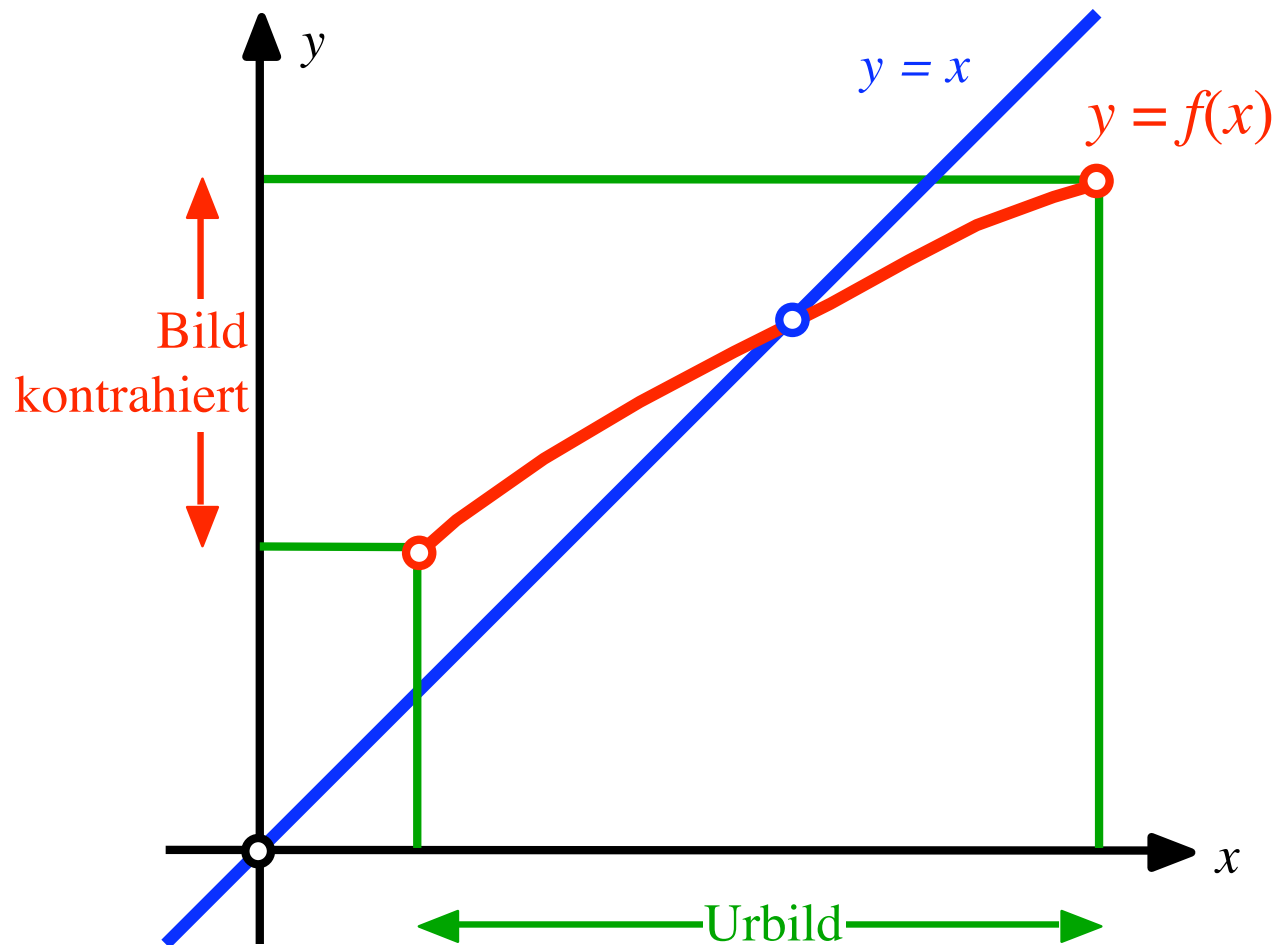
Steigung klein

$$-1 < f'(x) < 1$$

⇒ Stabiler Fixpunkt

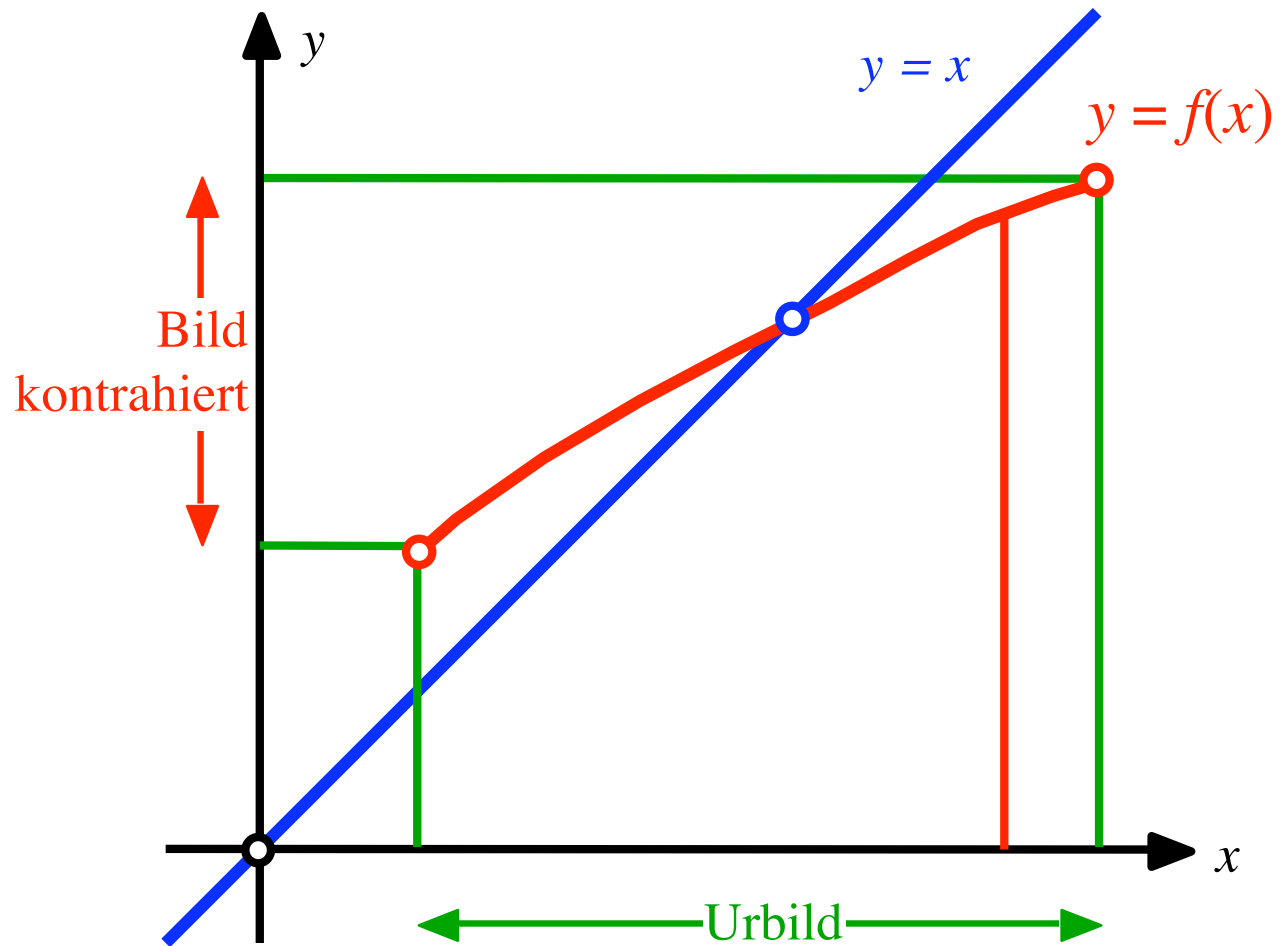
Kleine positive Steigung

$$0 < f'(x) < 1$$



Kleine positive Steigung

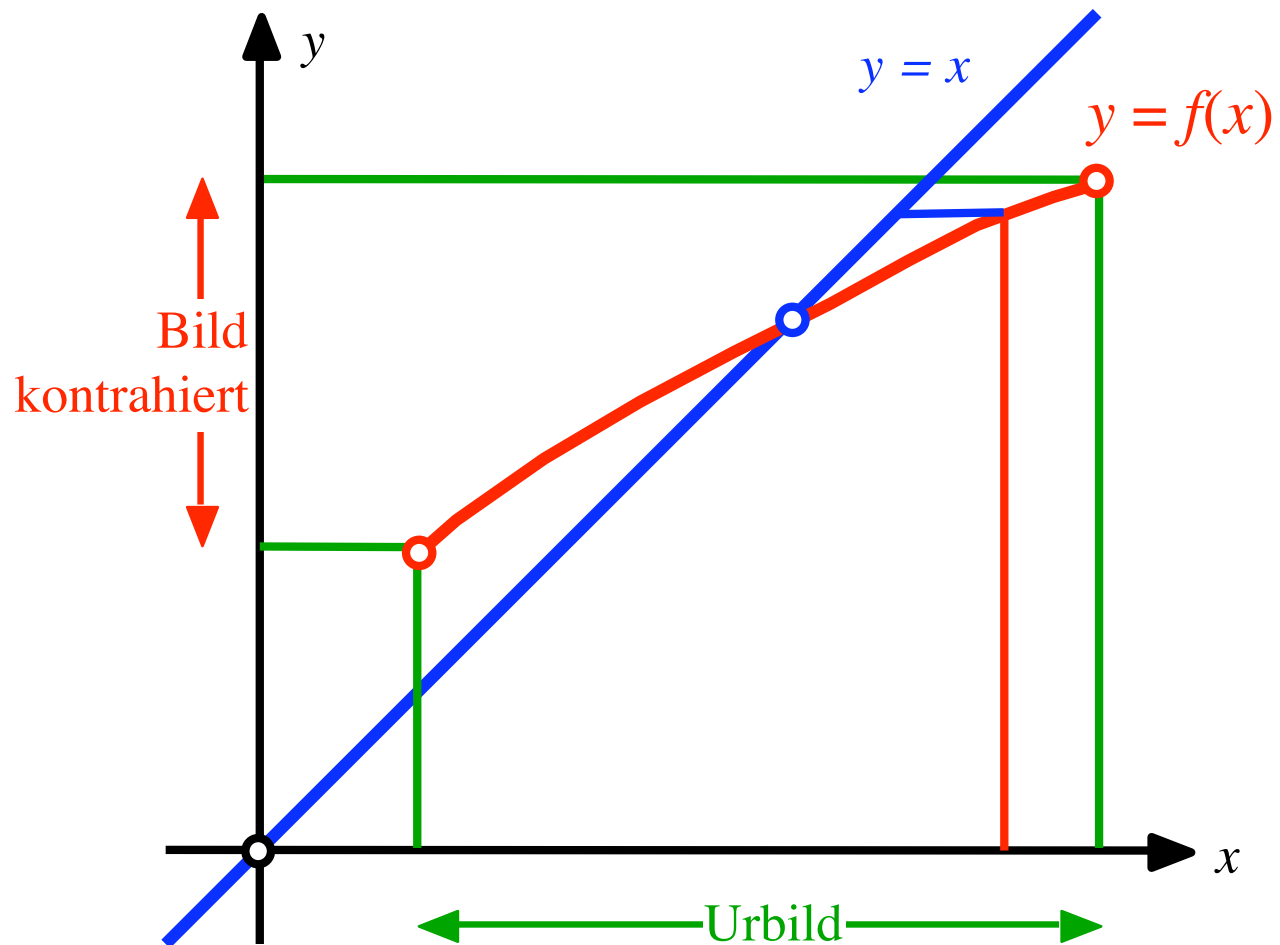
$$0 < f'(x) < 1$$





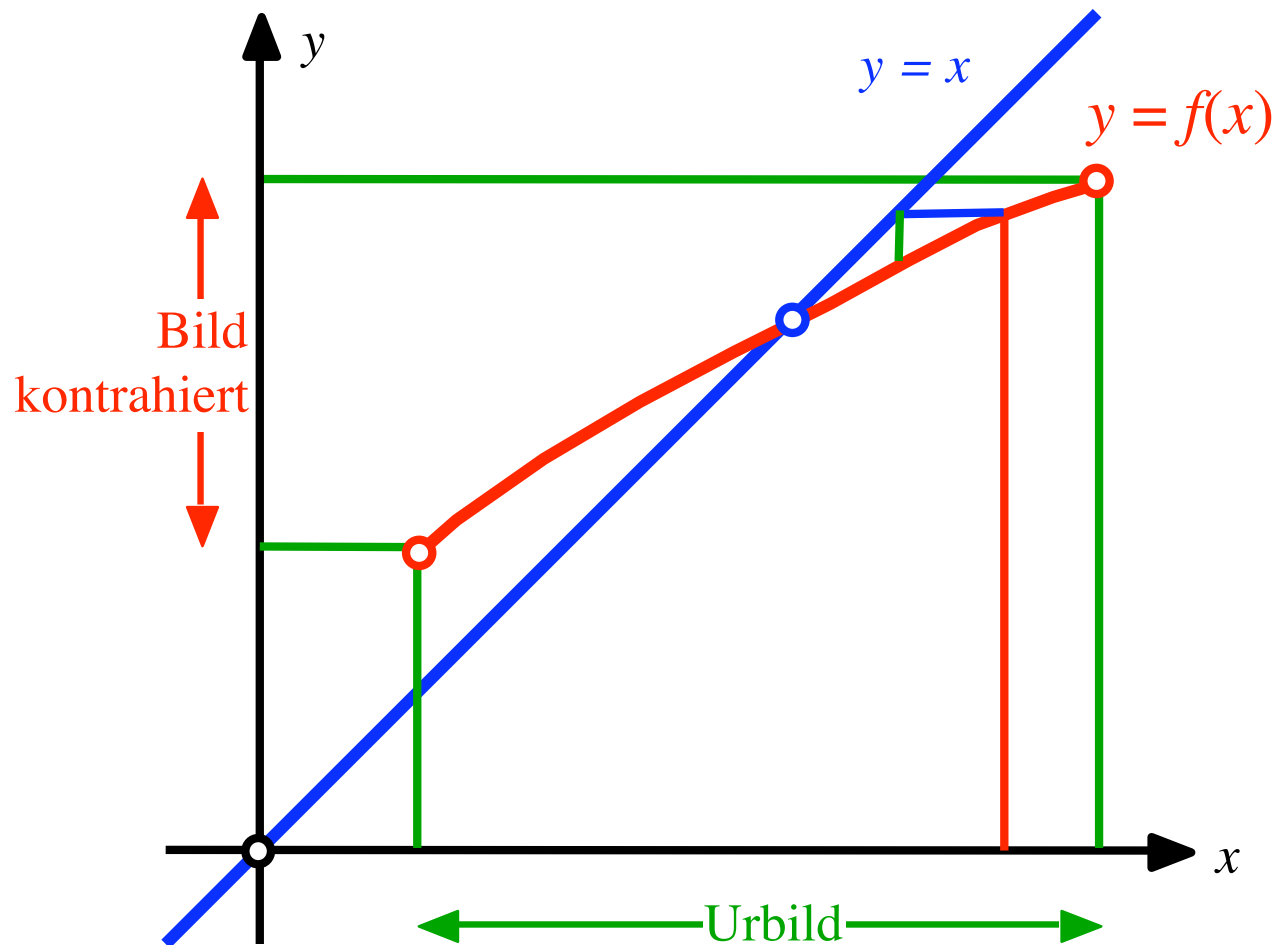
Kleine positive Steigung

$$0 < f'(x) < 1$$



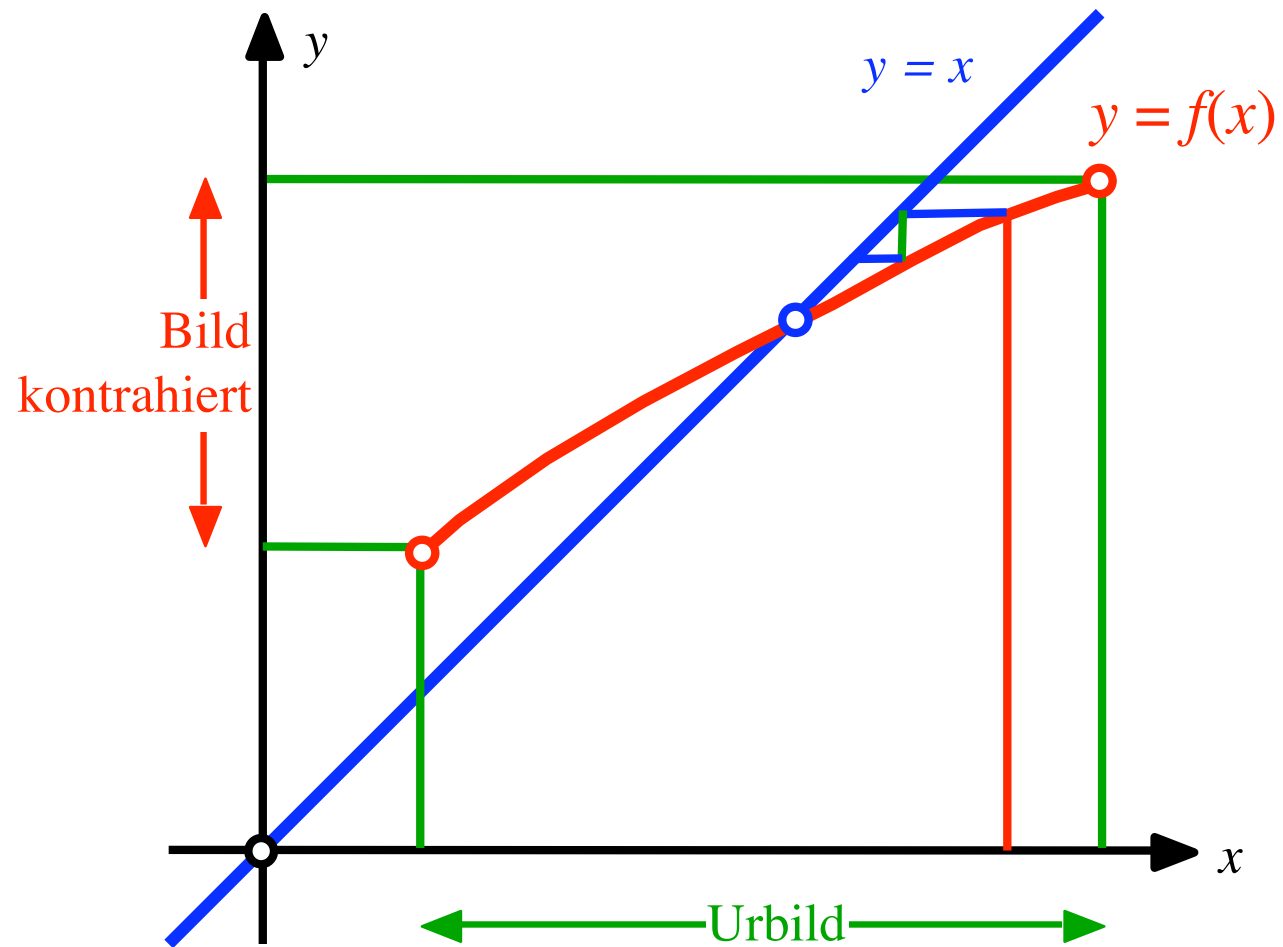
Kleine positive Steigung

$$0 < f'(x) < 1$$



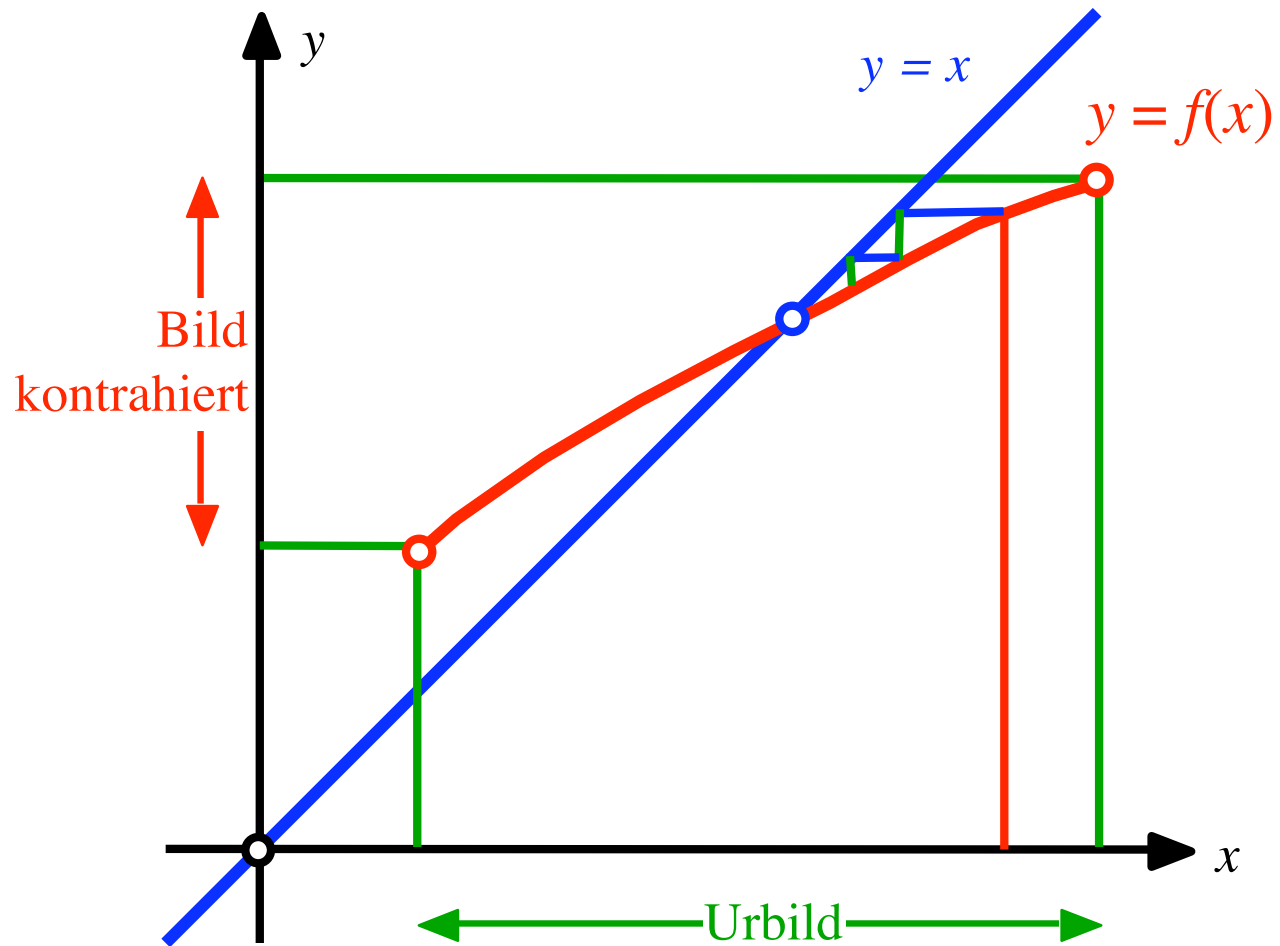
Kleine positive Steigung

$$0 < f'(x) < 1$$



Kleine positive Steigung

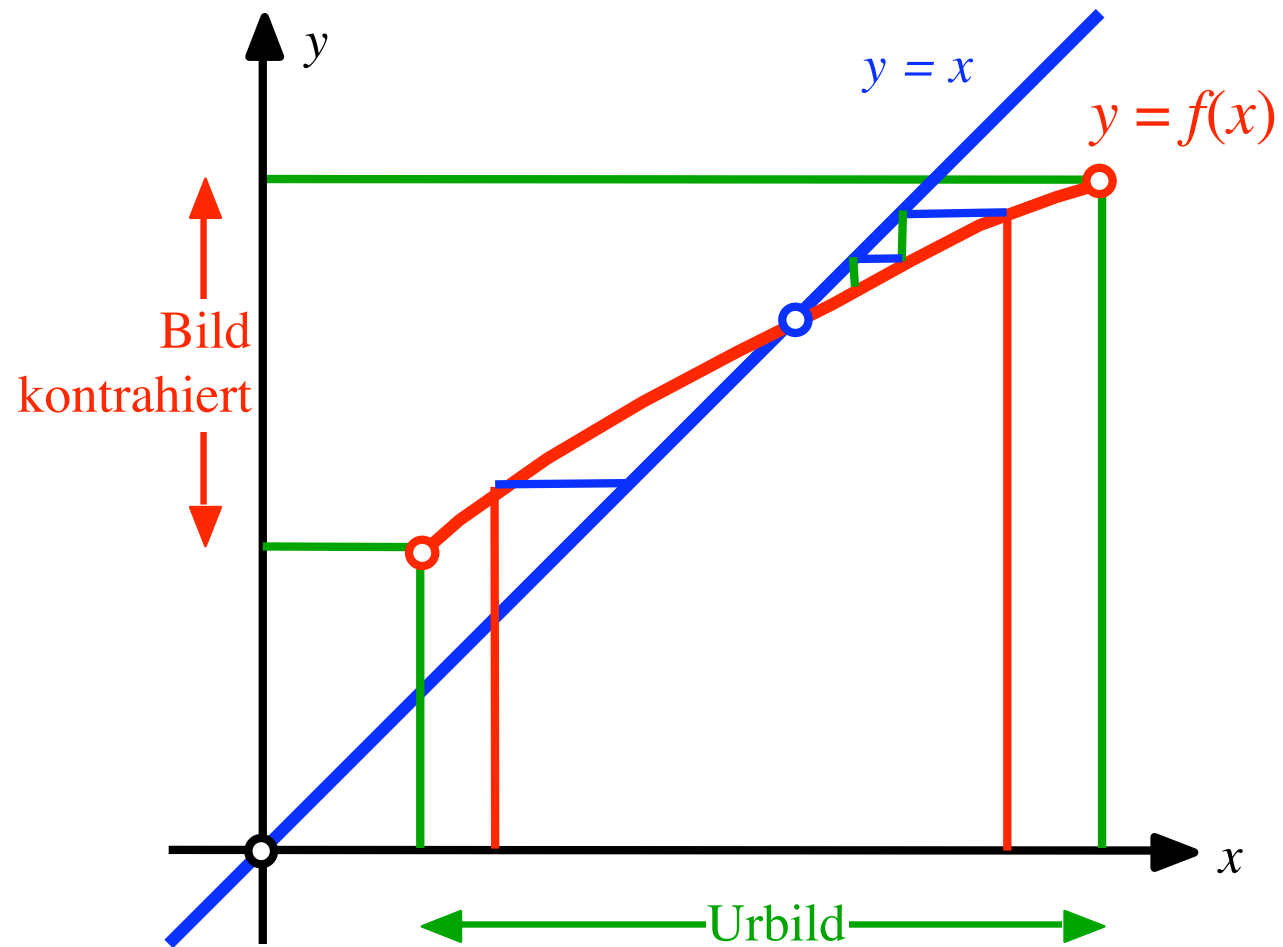
$$0 < f'(x) < 1$$





Kleine positive Steigung

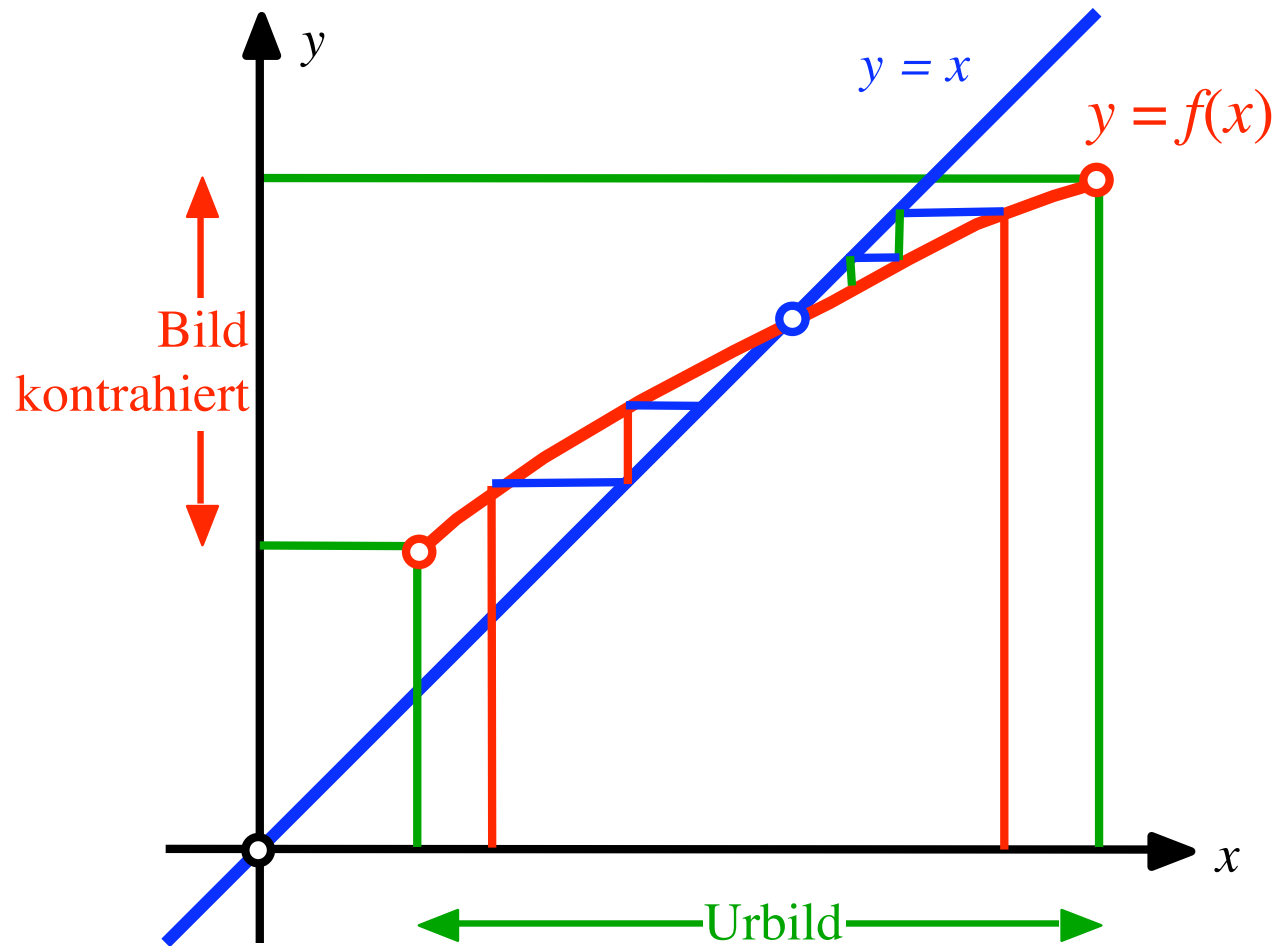
$$0 < f'(x) < 1$$





Kleine positive Steigung

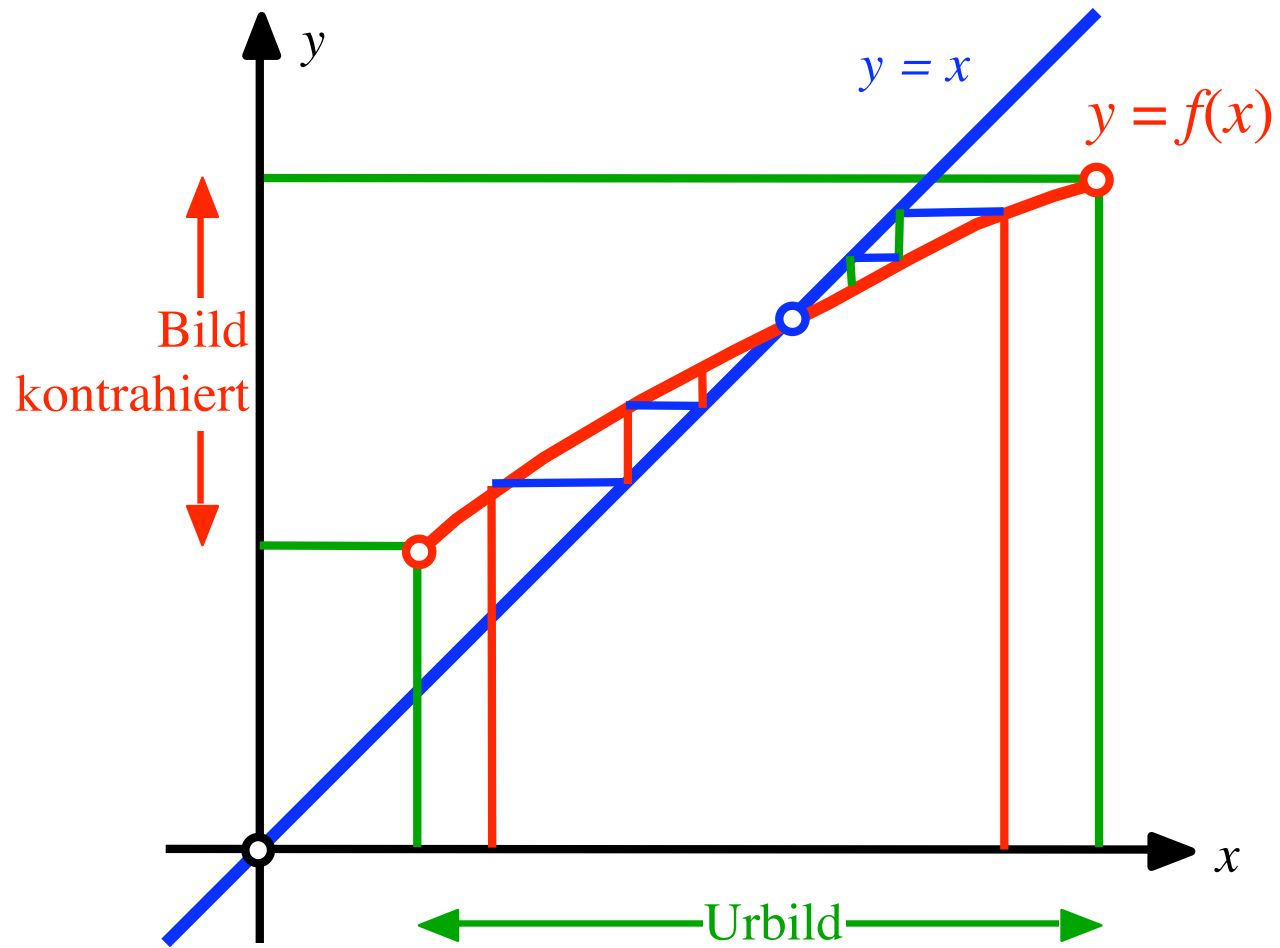
$$0 < f'(x) < 1$$





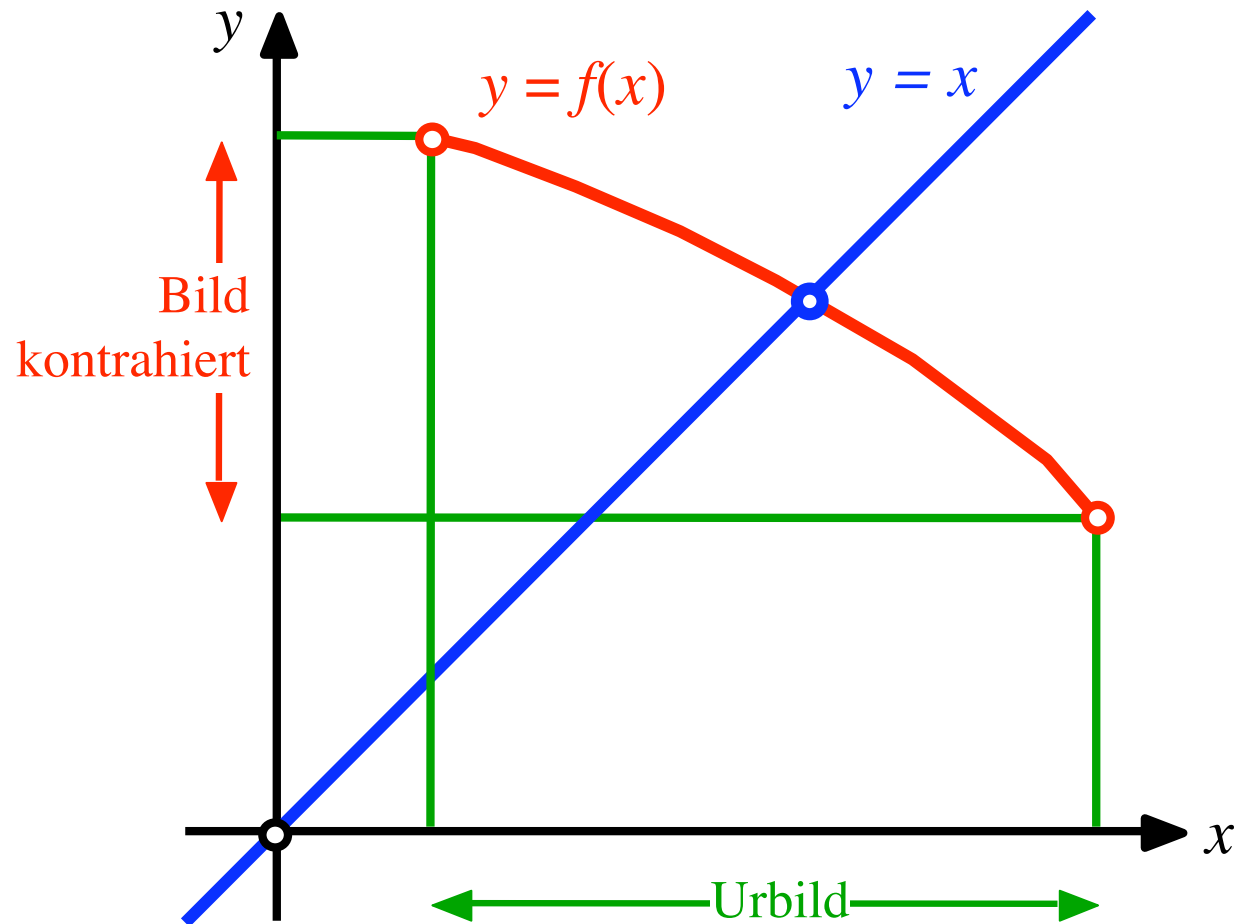
Kleine positive Steigung

$$0 < f'(x) < 1$$

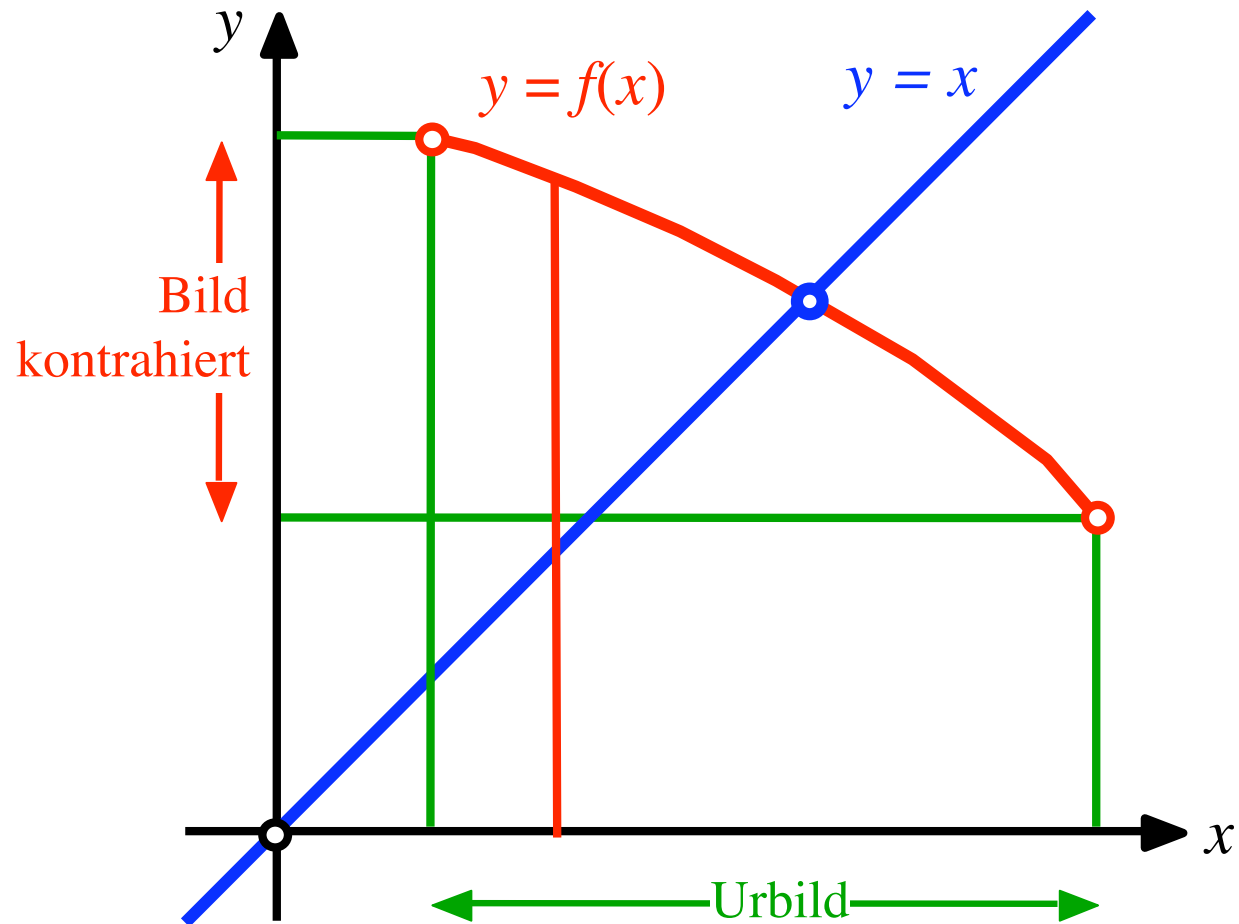


Treppen

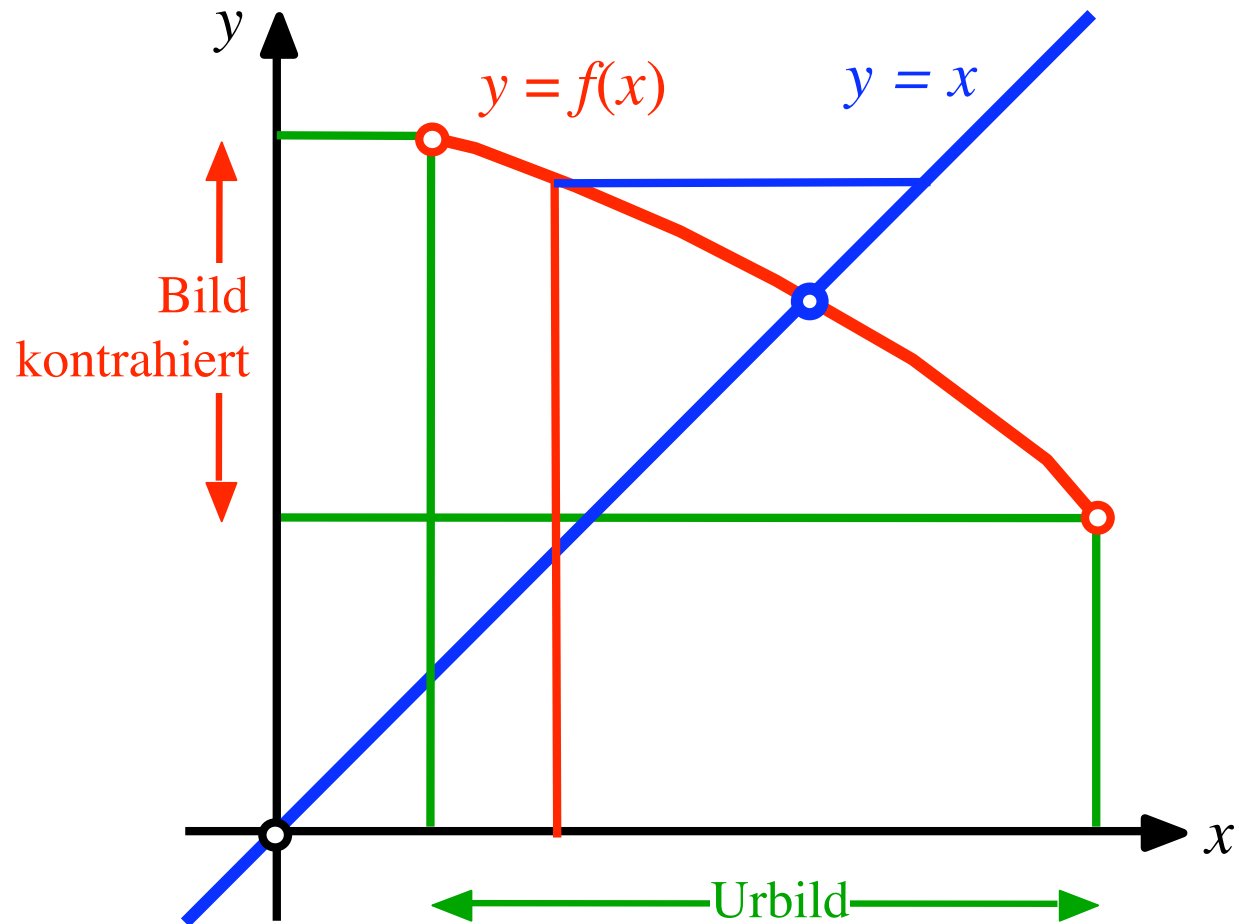
Kleine negative Steigung  $-1 < f'(x) < 0$



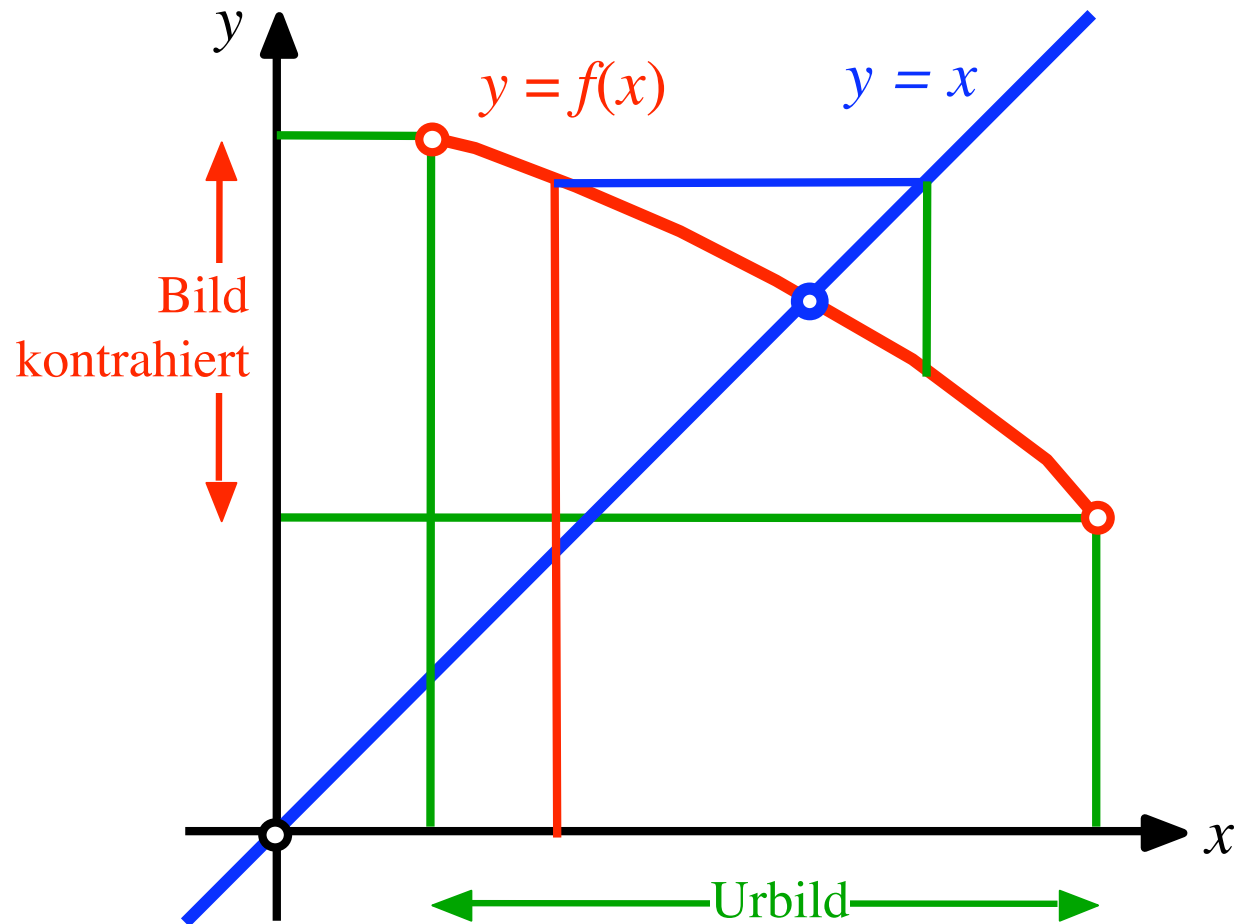
Kleine negative Steigung  $-1 < f'(x) < 0$



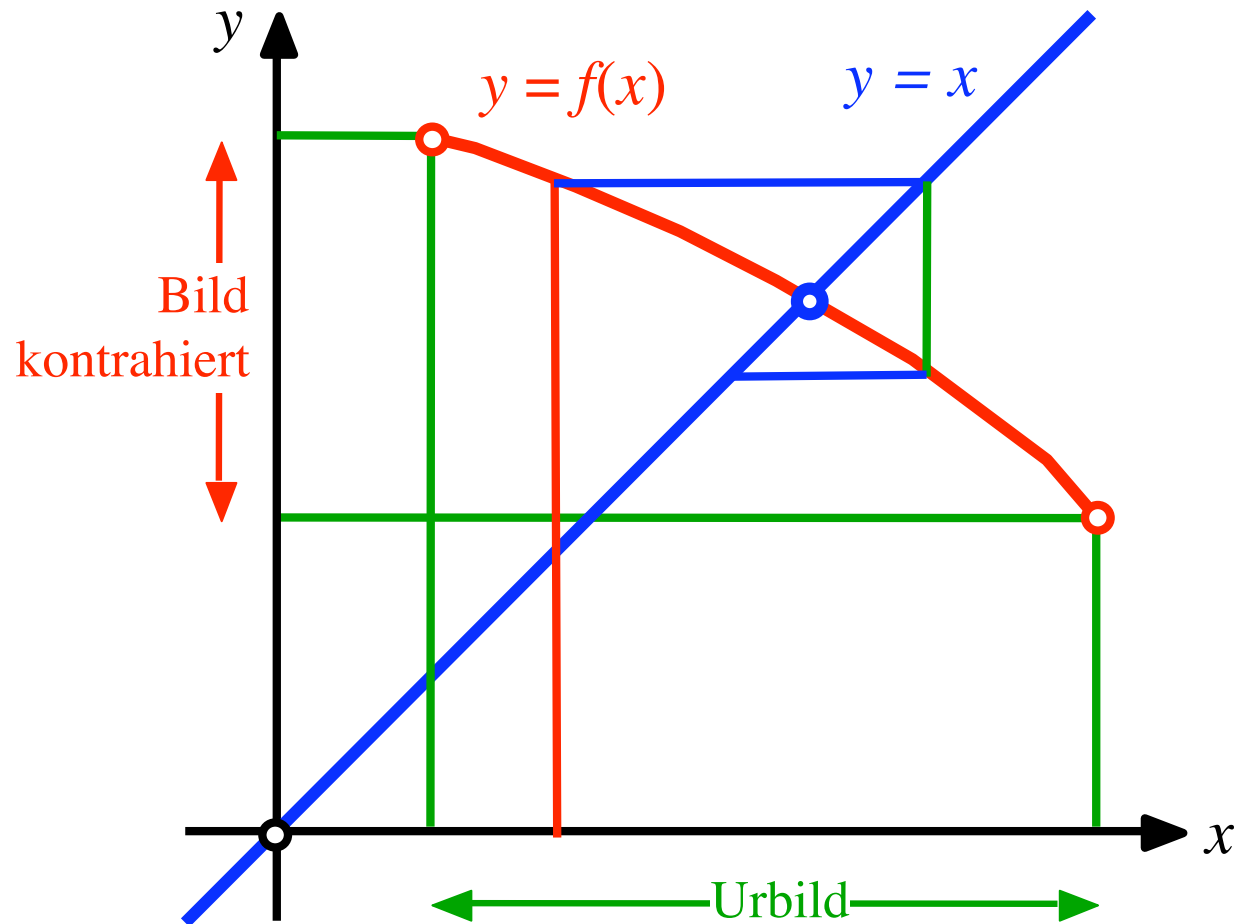
Kleine negative Steigung  $-1 < f'(x) < 0$



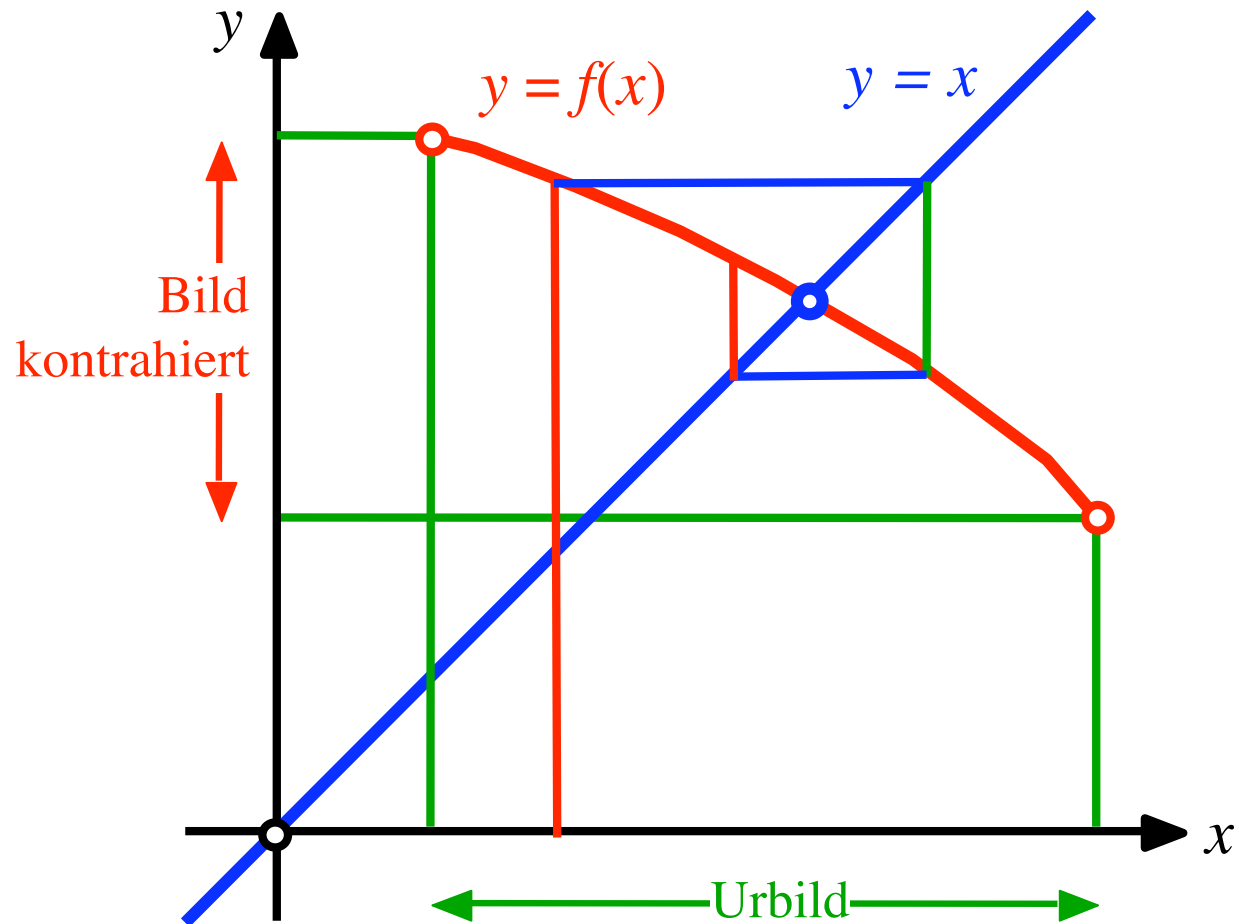
Kleine negative Steigung  $-1 < f'(x) < 0$



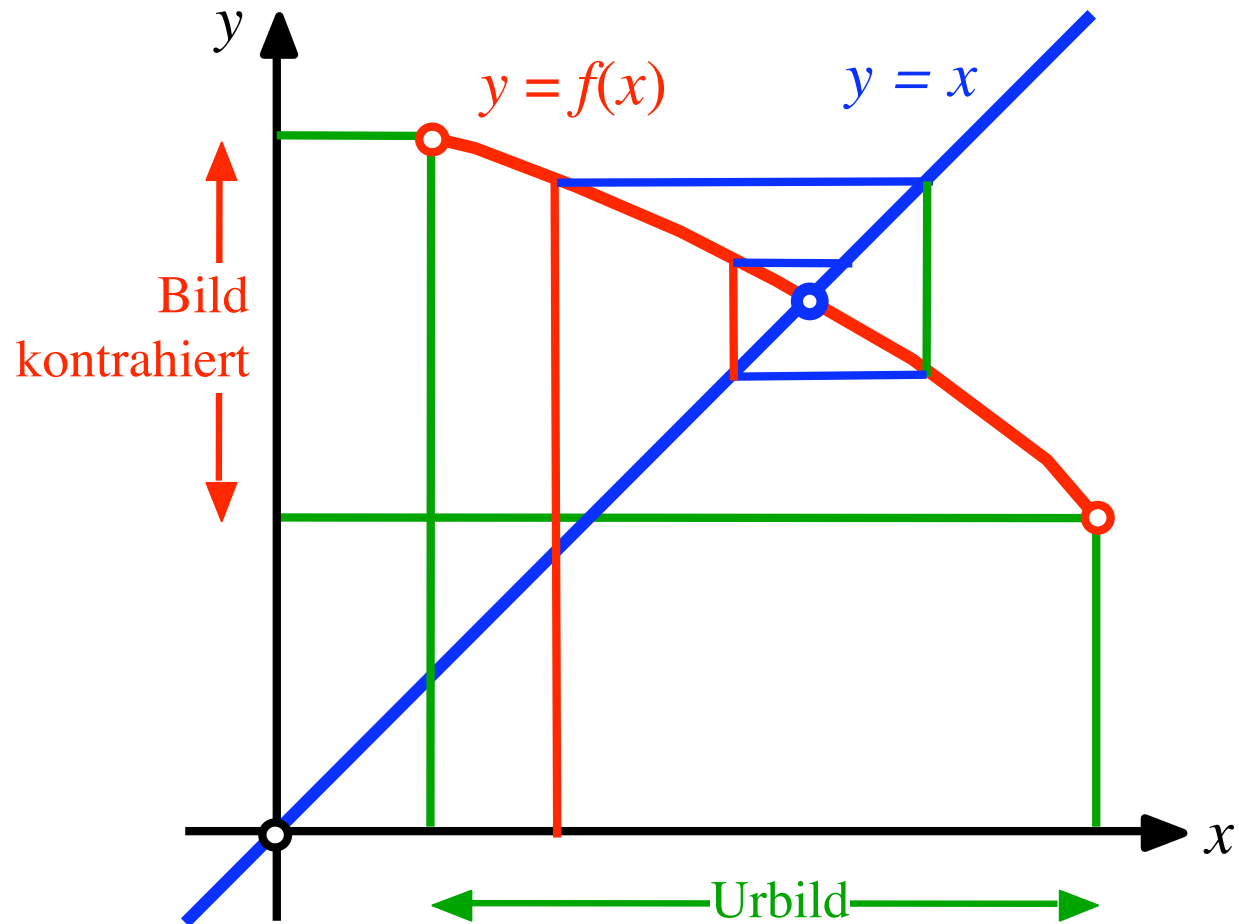
Kleine negative Steigung  $-1 < f'(x) < 0$



Kleine negative Steigung  $-1 < f'(x) < 0$

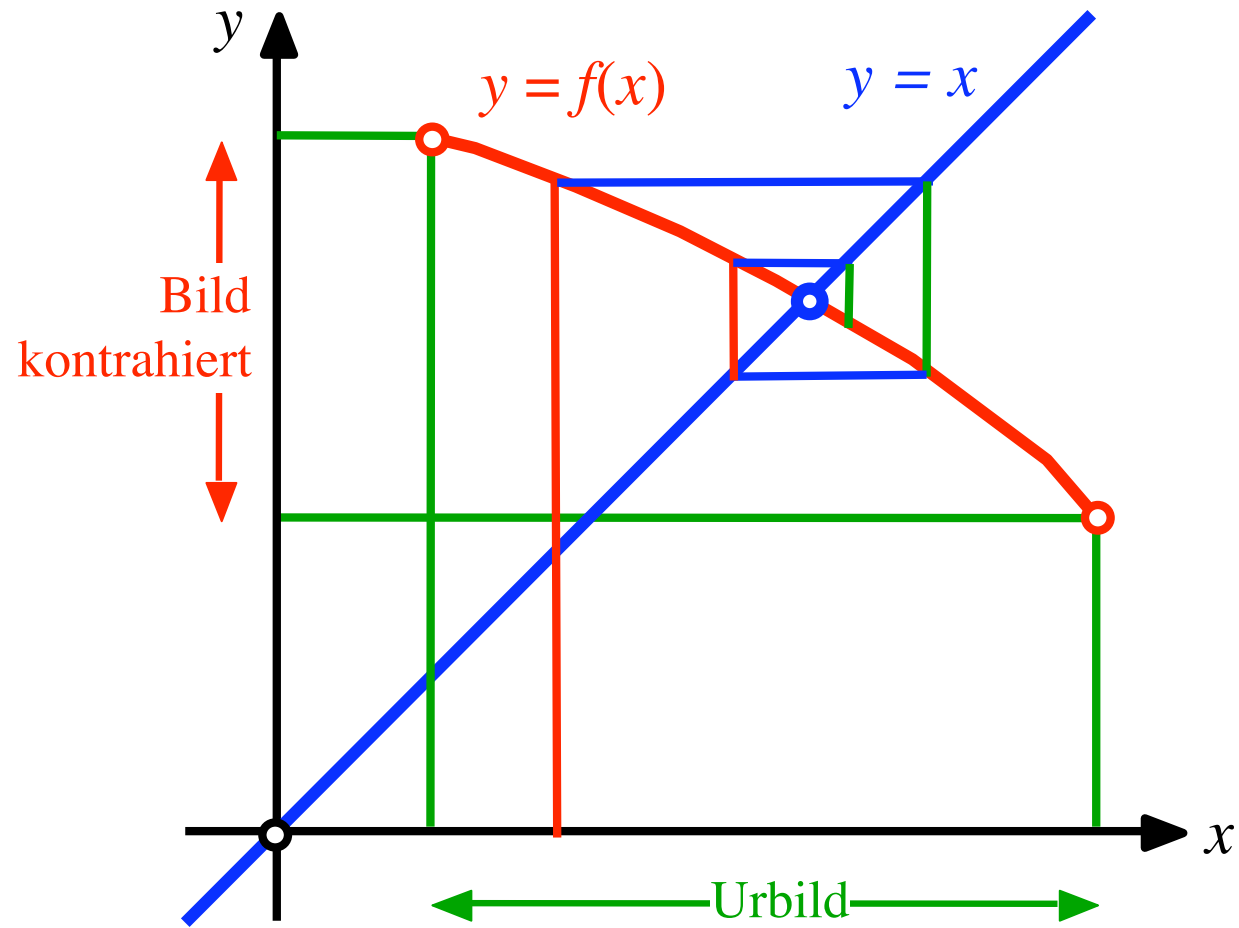


Kleine negative Steigung  $-1 < f'(x) < 0$



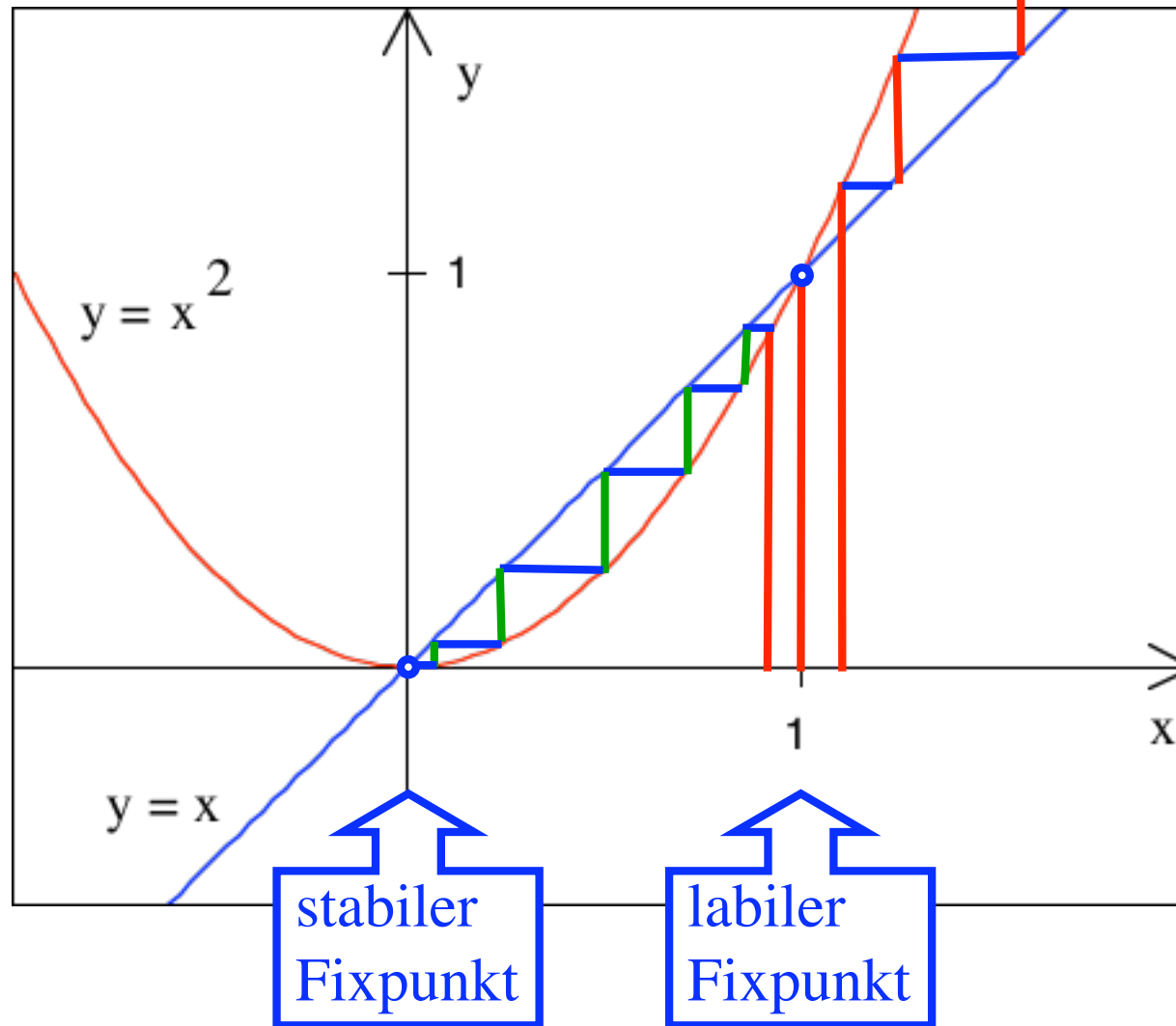


Kleine negative Steigung  $-1 < f'(x) < 0$



Spirale

Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$



Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$

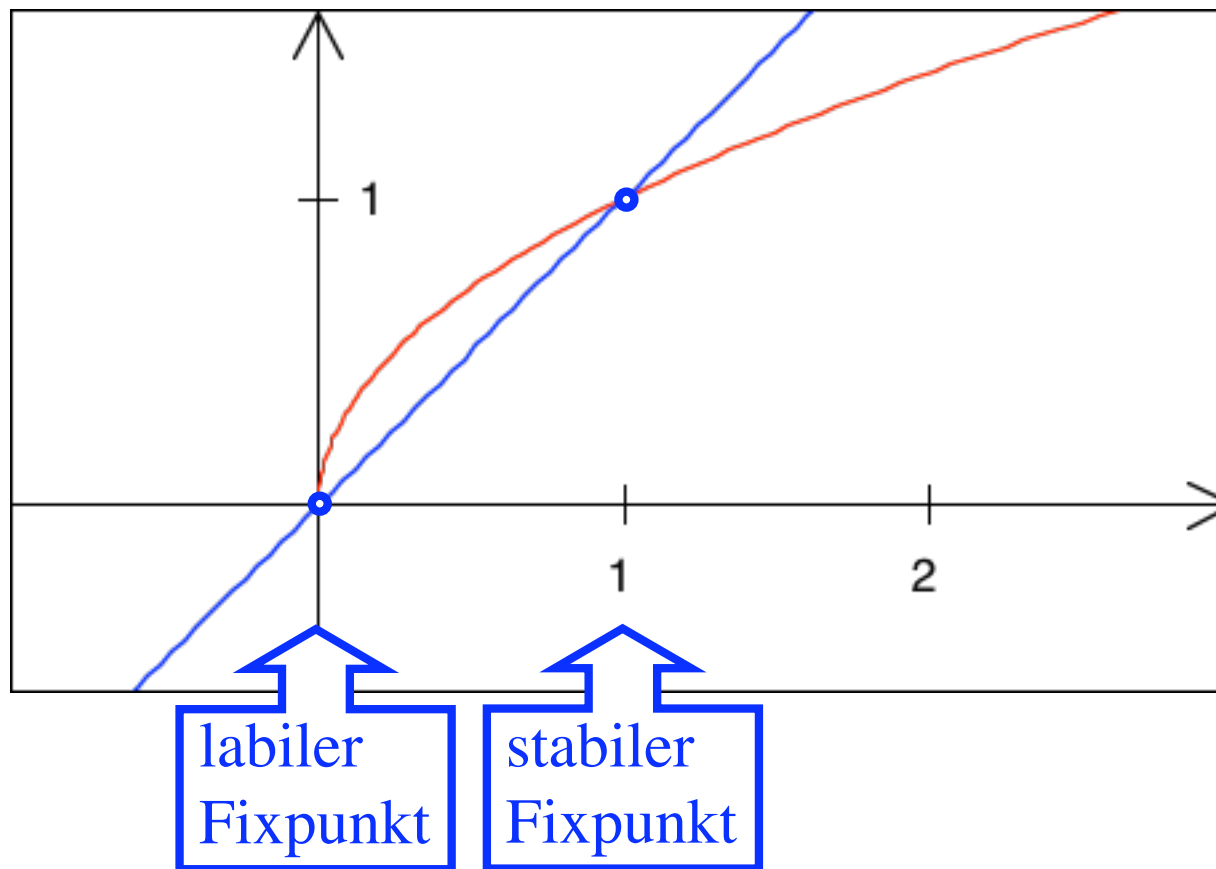
Umformung:  $x = \sqrt{x} \Rightarrow x = 0$  oder  $x = 1$

Übergang zur Umkehrfunktion



Beispiel:  $x^2 = x \Rightarrow x = 0$  oder  $x = 1$

Umformung:  $x = \sqrt{x} \Rightarrow x = 0$  oder  $x = 1$



# Vergleich

## **Newton-Raphson-Verfahren**

Schnell

Braucht Ableitung

Probleme mit  $f'(x) = 0$

—

## **Fixpunktverfahren**

Langsam

Braucht nur Funktion

—

Labile Fixpunkte

Allenfalls mit

Umkehrfunktion

arbeiten