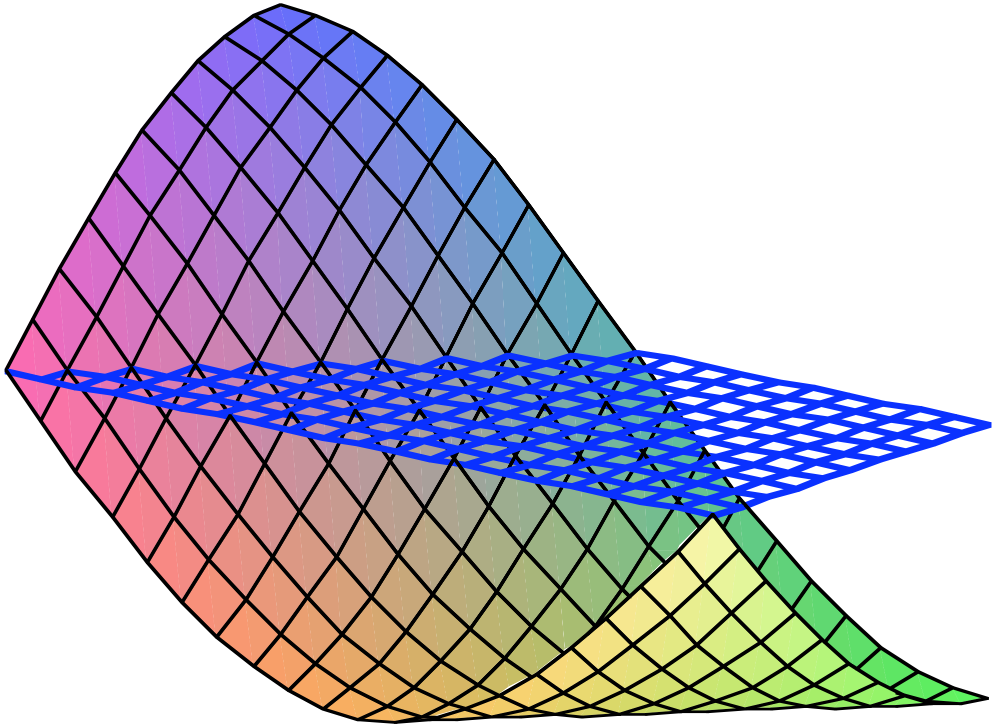

Modul 115 Extrema. Integration

Extrema

Extrema

Erinnerung : $y = f(x)$

Extrema

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Extremum $\Rightarrow \quad f'(x) = 0$

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$$\begin{array}{lcl} \text{Extremum} & \Rightarrow & f'(x) = 0 \\ & \xcancel{\Leftarrow} & \end{array}$$

Extrema

<u>Erinnerung</u>: $y = f(x)$

$$\begin{array}{lcl} \text{Extremum} & \Rightarrow & f'(x) = 0 \\ & \xcancel{\Leftarrow} & \end{array}$$

$$\begin{array}{lcl} f''(x) < 0 \,,\ f'(x) = 0 & \Rightarrow & \text{Maximum} \\ & \xcancel{\Leftarrow} & \\ f''(x) > 0 \,,\ f'(x) = 0 & \Rightarrow & \text{Minumum} \\ & \xcancel{\Leftarrow} & \end{array}$$

Extrema

<u>Erinnerung</u>: $y = f(x)$

$$\text{Extremum} \underset{\xcancel{\Leftarrow}}{\Rightarrow} f'(x) = 0$$

$$f''(x) < 0\,,\ f'(x) = 0 \underset{\xcancel{\Leftarrow}}{\Rightarrow} \text{Maximum}$$

$$f''(x) > 0\,,\ f'(x) = 0 \underset{\xcancel{\Leftarrow}}{\Rightarrow} \text{Minumum}$$

$$\text{Wendepunkt} \underset{\xcancel{\Leftarrow}}{\Rightarrow} f''(x) = 0$$

Neu: $z = f(x, y)$ Funktion zweier Variablen

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Extremum $\underset{\cancel{\Leftarrow}}{\Rightarrow}$ $\operatorname{grad}\left(f\right) = \vec{0}$ das heißt $f_x = 0$ und $f_y = 0$

Neu: $z = f(x, y)$ Funktion zweier Variablen

Extremum $\underset{\cancel{\Leftarrow}}{\Rightarrow}$ $\operatorname{grad}(f) = \vec{0}$ das heißt $f_x = 0$ <u>und</u> $f_y = 0$

$$\Delta = f_{xx} f_{yy} - f_{xy}^2 = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ \underbrace{f_{yx}}_{f_{xy}} & f_{yy} \end{bmatrix}\right)$$

Bezeichnung

Wird im Sommer
ausführlich besprochen.

Neu: $z = f(x, y)$ Funktion zweier Variablen

Extremum $\underset{\cancel{\Leftarrow}}{\Rightarrow}$ $\operatorname{grad}(f) = \vec{0}$ das heißt $f_x = 0$ und $f_y = 0$

$$\Delta = f_{xx} f_{yy} - f_{xy}^2 = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right)$$

$\operatorname{grad}(f) = \vec{0}$ und $\Delta > 0,\ f_{xx} > 0$ $\Rightarrow$ isoliertes Minimum

$\operatorname{grad}(f) = \vec{0}$ und $\Delta > 0,\ f_{xx} < 0$ $\Rightarrow$ isoliertes Maximum

$\operatorname{grad}(f) = \vec{0}$ und $\Delta < 0$ $\Rightarrow$ Sattelpunkt

$\operatorname{grad}(f) = \vec{0}$ und $\Delta = 0$ keine Aussage möglich

Beispiel: $f\left(x,y\right)=xy$

Beispiel: $f(x,y) = xy$

$$\operatorname{grad}(f) = \begin{bmatrix} y \\ x \end{bmatrix} \quad \Rightarrow \quad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Beispiel: $f(x,y) = xy$

$$\operatorname{grad}(f) = \begin{bmatrix} y \\ x \end{bmatrix} \quad \Rightarrow \quad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$f_{xx} = 0 \quad f_{xy} = 1$$

$$f_{yx} = 1 \quad f_{yy} = 0$$

Beispiel: $f(x,y) = xy$

$$\operatorname{grad}(f) = \begin{bmatrix} y \\ x \end{bmatrix} \quad \Rightarrow \quad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

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$$\Delta(0,0) = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) =$$

Beispiel: $f(x,y) = xy$

$$\operatorname{grad}(f) = \begin{bmatrix} y \\ x \end{bmatrix} \quad \Rightarrow \quad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$f_{xx} = 0 \quad f_{xy} = 1$$
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$$\Delta(0,0) = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1 < 0 \quad \Rightarrow \quad \text{Sattelpunkt}$$

Beispiel: $f\left(x,y\right)=xy$ Sattelpunkt im Ursprung

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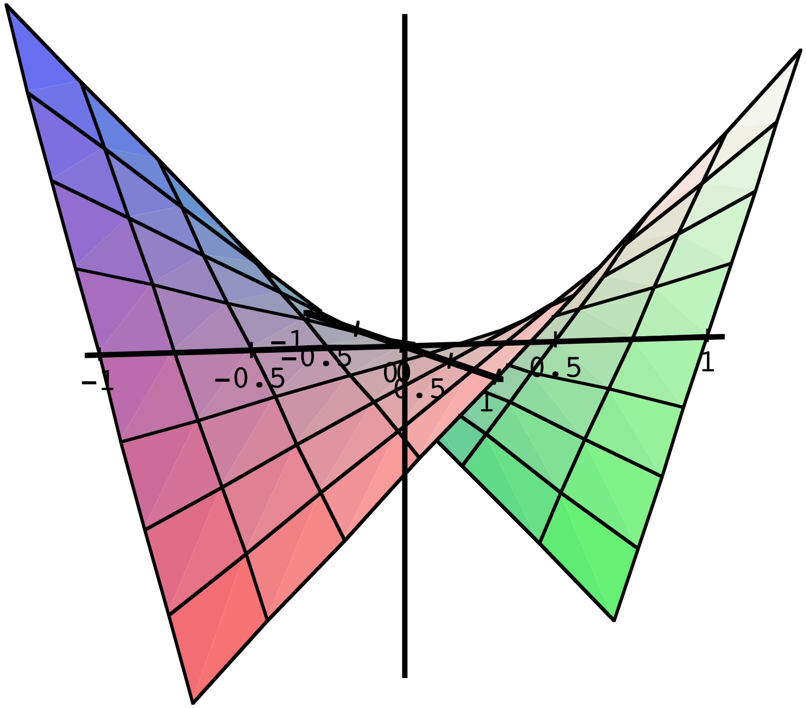
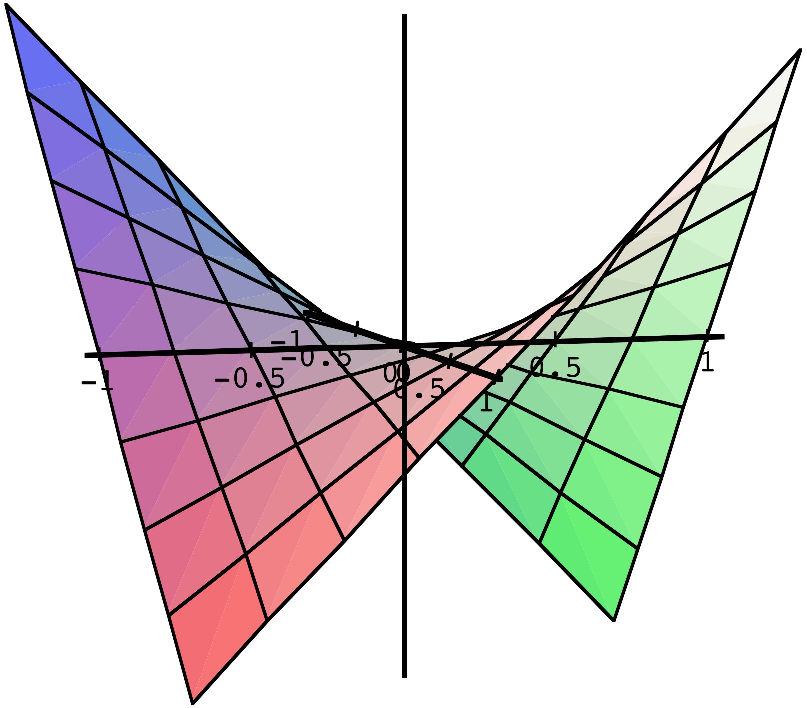
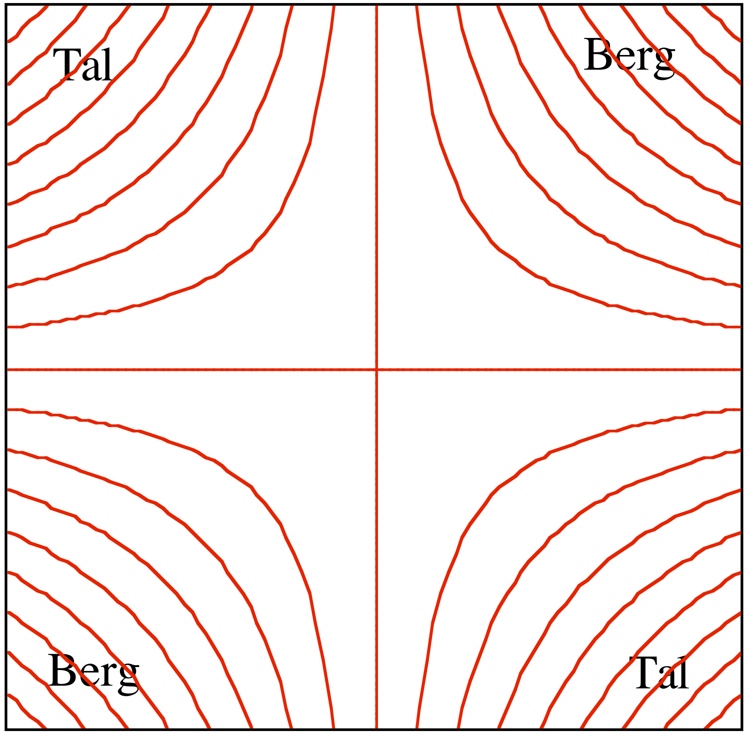


Beispiel: $f(x,y)=xy$ Sattelpunkt im Ursprung





Beispiel: $f(x,y) = x^3 - 3xy^2$

Beispiel: $f(x,y) = x^3 - 3xy^2$

$$\operatorname{grad}(f) = \begin{bmatrix} 3x^2 - 3y^2 \\ -6xy \end{bmatrix} \qquad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Beispiel: $f\left(x,y\right)=x^{3}-3xy^{2}$

$$\operatorname{grad}\left(f\right)=\begin{bmatrix}3x^{2}-3y^{2}\\ -6xy\end{bmatrix}\qquad \operatorname{grad}\left(f\right)\left(0,0\right)=\begin{bmatrix}0\\ 0\end{bmatrix}=\vec{0}$$

$$f_{xx}=6x\quad f_{xy}=-6y$$

$$f_{yx}=-6y\quad f_{yy}=-6x$$

Beispiel: $f(x,y) = x^3 - 3xy^2$

$$\operatorname{grad}(f) = \begin{bmatrix} 3x^2 - 3y^2 \\ -6xy \end{bmatrix} \qquad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$f_{xx} = 6x \qquad f_{xy} = -6y$$

$$f_{yx} = -6y \qquad f_{yy} = -6x$$

$$\Delta(x,y) = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 6x & -6y \\ -6y & -6x \end{bmatrix}\right) = -36x^2 - 36y^2$$

Beispiel: $f(x,y) = x^3 - 3xy^2$

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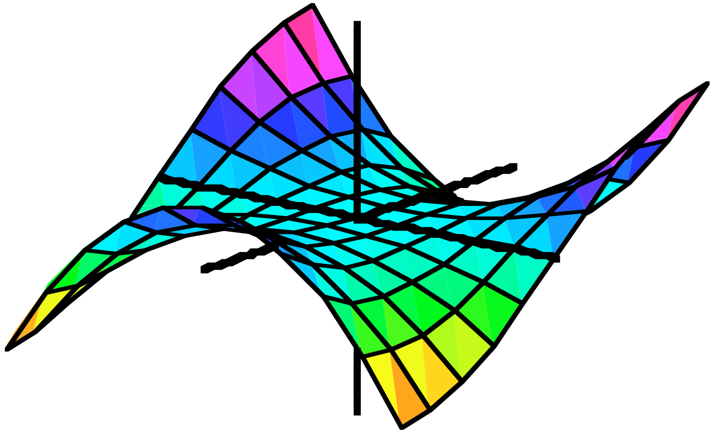
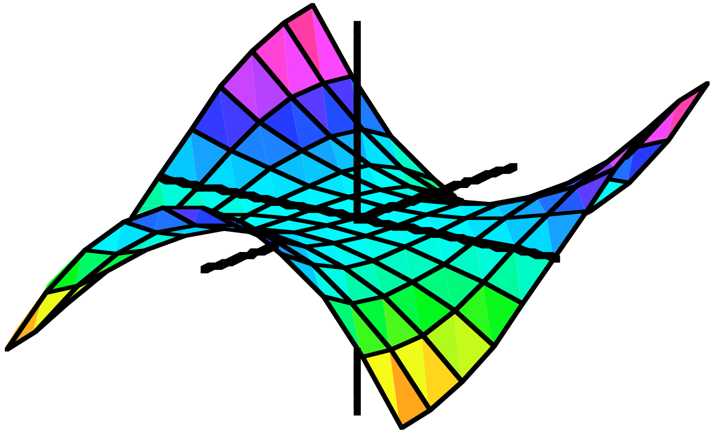
$\Delta(0,0) = 0$ Keine Aussage möglich

Beispiel: $f(x,y) = x^3 - 3xy^2$

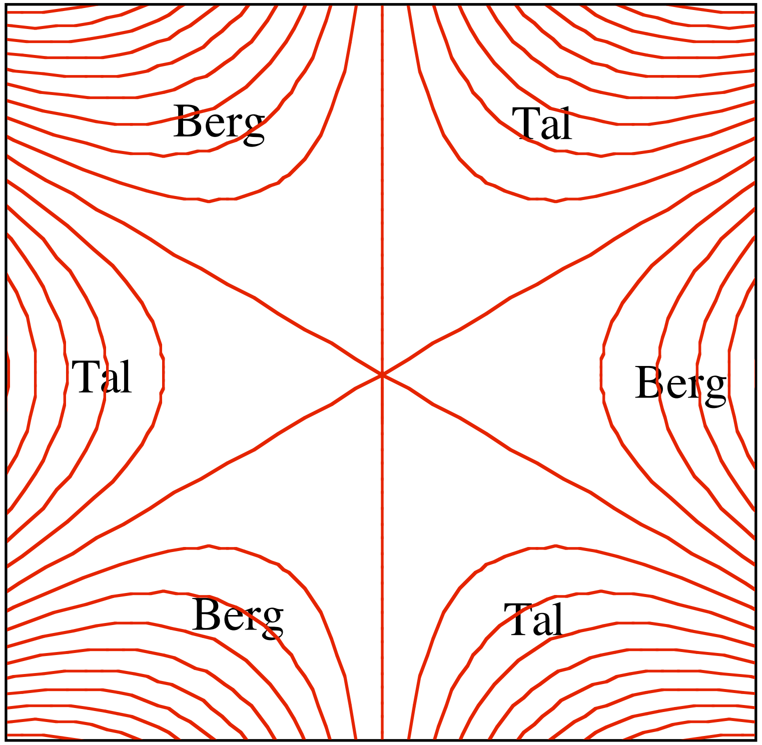


Affensattel

Beispiel: $f(x,y)=x^3-3xy^2$



Affensattel



Beispiel: $f\left(x,y\right)=x^2+y^2$

Beispiel: $f(x,y) = x^2 + y^2$

$$\operatorname{grad}(f) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

Beispiel: $f(x,y) = x^2 + y^2$

$$\operatorname{grad}(f) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$f_{xx} = 2 \quad f_{xy} = 0$$

$$f_{yx} = 0 \quad f_{yy} = 2$$

Beispiel: $f(x,y) = x^2 + y^2$

$$\operatorname{grad}(f) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$f_{xx} = 2 \quad f_{xy} = 0$$

$$f_{yx} = 0 \quad f_{yy} = 2$$

$$\Delta(x,y) = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4$$

Beispiel: $f\left(x,y\right)=x^2+y^2$

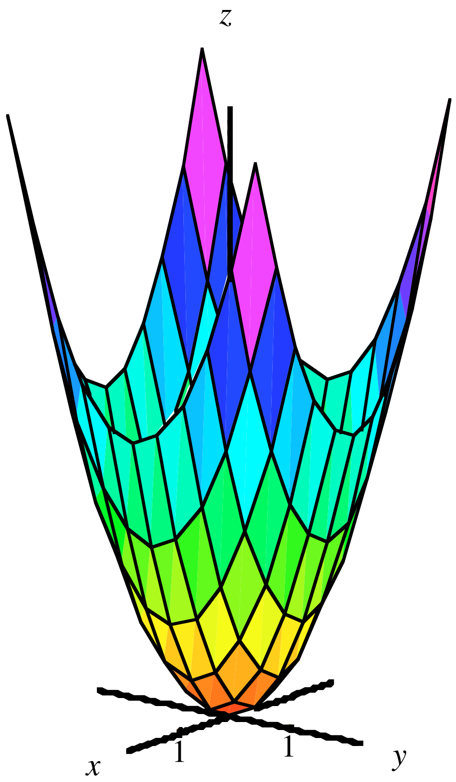
$$\operatorname{grad}\left(f\right)=\begin{bmatrix}2x\\2y\end{bmatrix}\qquad \operatorname{grad}\left(f\right)\left(0,0\right)=\begin{bmatrix}0\\0\end{bmatrix}=\vec{0}$$

$$f_{xx}=2\quad f_{xy}=0$$
$$f_{yx}=0\quad f_{yy}=2$$

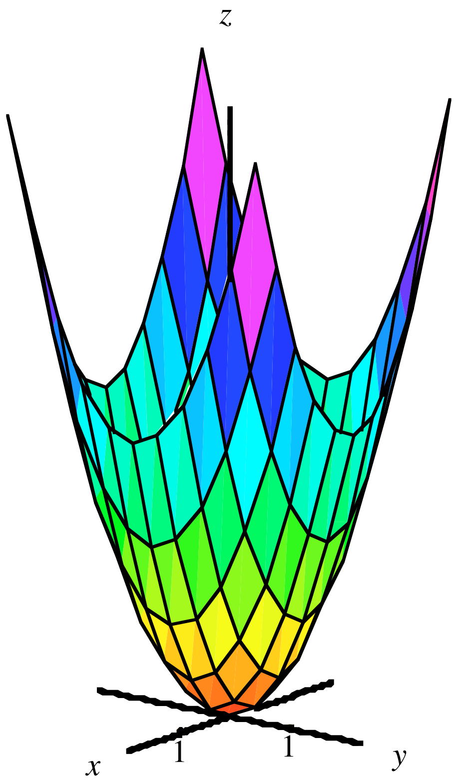
$$\Delta\left(x,y\right)=\det\left(\begin{bmatrix}f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{bmatrix}\right)=\det\left(\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\right)=4$$

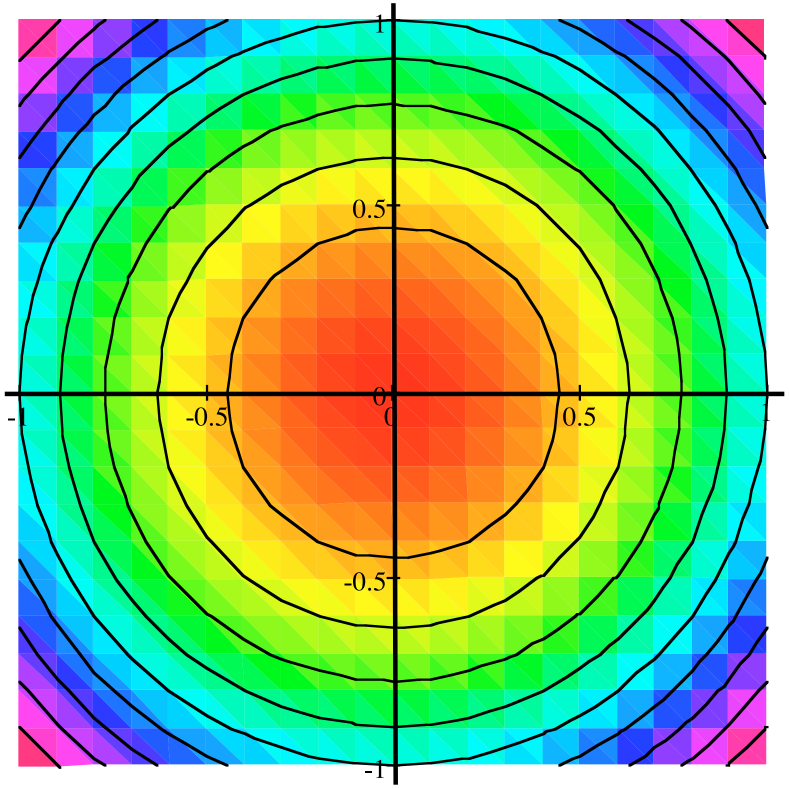
$$\Delta\left(0,0\right)=4>0\quad\Rightarrow\quad \text{Extremum}$$
$$f_{xx}\left(0,0\right)=2>0\Rightarrow\quad \text{isoliertes Minimum}$$

Beispiel: $f(x,y) = x^2 + y^2$



Beispiel: $f(x,y) = x^2 + y^2$





Beispiel: $f(x,y) = x^2 - 2xy + y^2 = (x-y)^2$

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$$f_{xx} = 2 \qquad f_{xy} = -2$$
$$f_{yx} = -2 \quad f_{yy} = 2$$

Beispiel: $f(x,y)=x^2-2xy+y^2=\underbrace{(x-y)^2}_{f(x,x)=0}$

$$\operatorname{grad}(f)=\begin{bmatrix}2x-2y\\-2x+2y\end{bmatrix}\qquad \operatorname{grad}(f)(0,0)=\begin{bmatrix}0\\0\end{bmatrix}=\vec{0}$$

$$\begin{array}{ll} f_{xx}=2 & f_{xy}=-2\\ f_{yx}=-2 & f_{yy}=2\end{array}$$

$$\Delta(x,y)=\det\left(\begin{bmatrix}f_{xx}&f_{xy}\\f_{yx}&f_{yy}\end{bmatrix}\right)=\det\left(\begin{bmatrix}2&-2\\-2&2\end{bmatrix}\right)=0$$

Beispiel: $f(x,y) = x^2 - 2xy + y^2 = \underbrace{(x-y)^2}_{f(x,x)=0}$

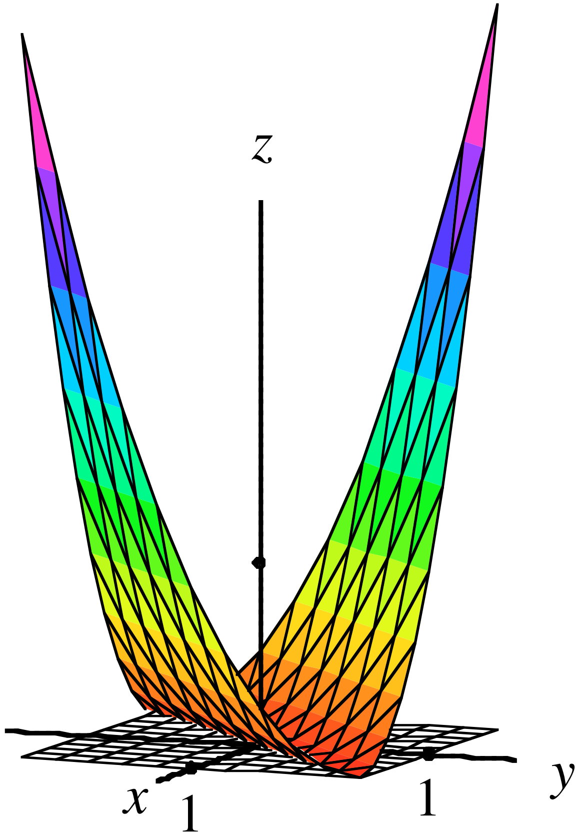
$$\operatorname{grad}(f) = \begin{bmatrix} 2x - 2y \\ -2x + 2y \end{bmatrix} \qquad \operatorname{grad}(f)(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

$$f_{xx} = 2 \qquad f_{xy} = -2$$
$$f_{yx} = -2 \quad f_{yy} = 2$$

$$\Delta(x,y) = \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\right) = 0$$

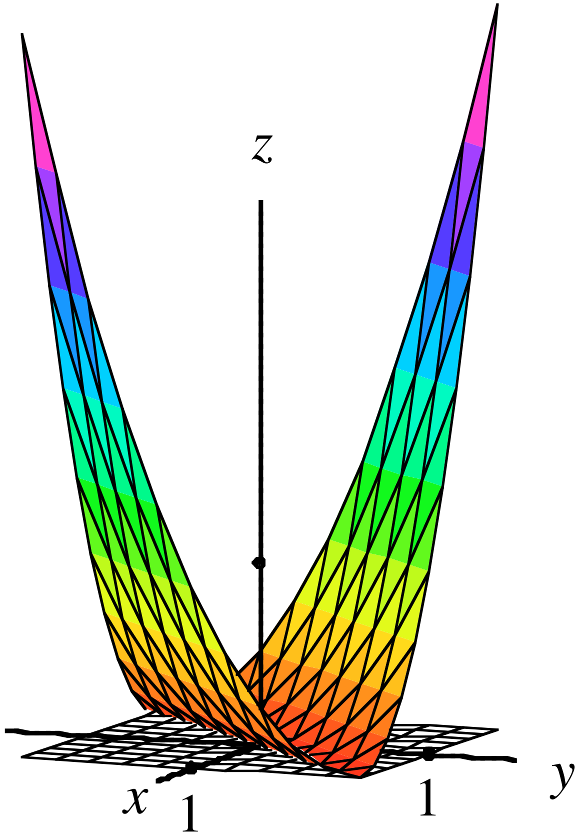
$\Delta = 0$ Keine Aussage möglich

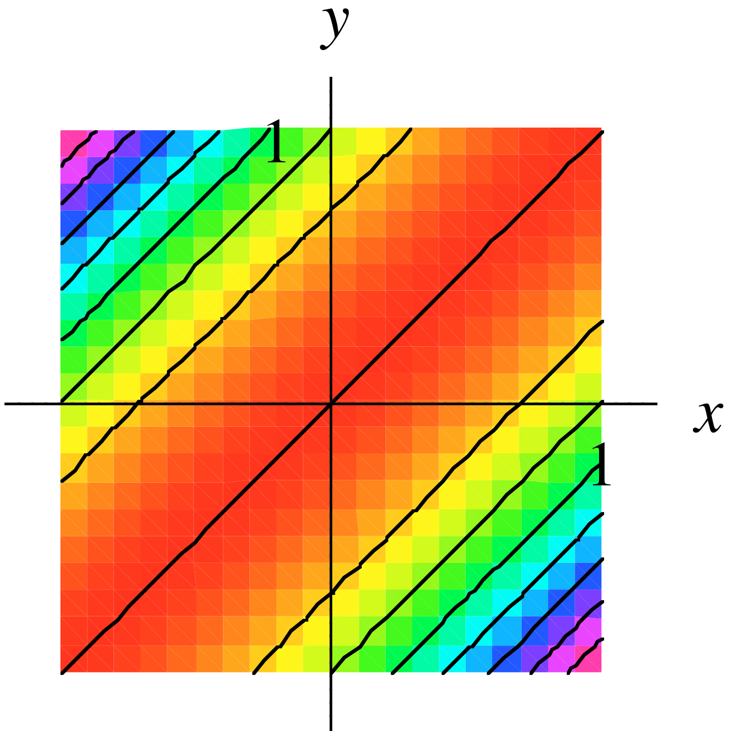
Beispiel: $f(x,y) = x^2 - 2xy + y^2 = \underbrace{(x-y)^2}_{f(x,x)=0}$



Fällt er in den Graben,
fressen ihn die Raben.

$$\text{Beispiel: } f(x,y) = x^2 - 2xy + y^2 = \underbrace{(x-y)^2}_{f(x,x)=0}$$

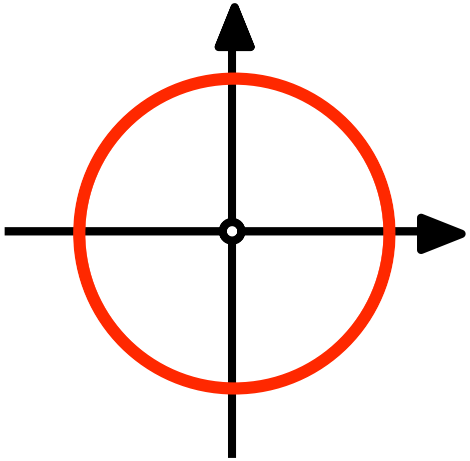




Nicht isoliertes Minimum für $y = x$

Implizite Darstellungen

$$x^2 + y^2 - 1 = 0$$

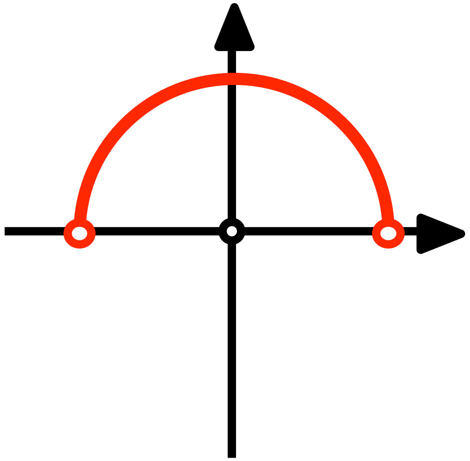


Explizit als Funktion darstellen?

Nach *y* auflösen:

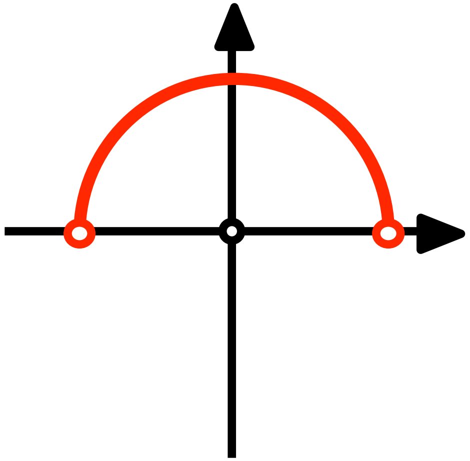
$$y = g_{+}(x) = \sqrt{1 - x^{2}}$$

Wurzel ist positiv

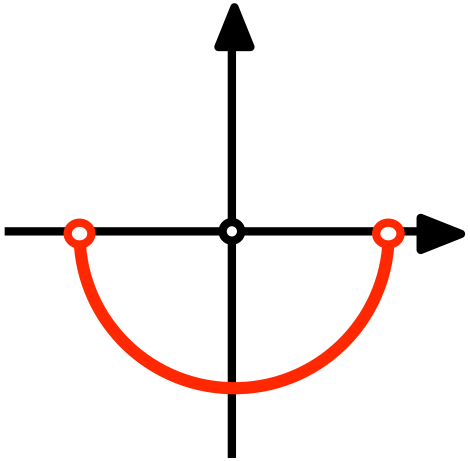


Nach *y* auflösen:

$$y = g_+(x) = \sqrt{1 - x^2}$$

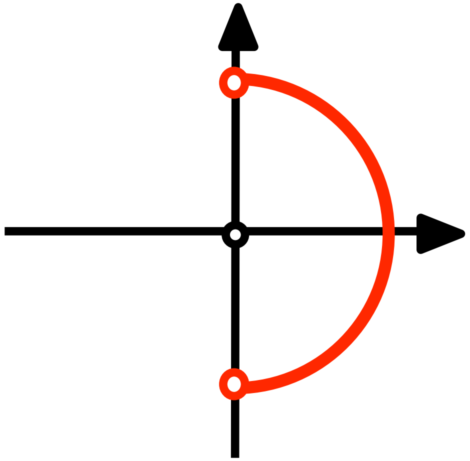


$$y = g_-(x) = -\sqrt{1 - x^2}$$



Nach x auflösen:

$$x = h_+(y) = \sqrt{1 - y^2}$$



Nach *x* auflösen:

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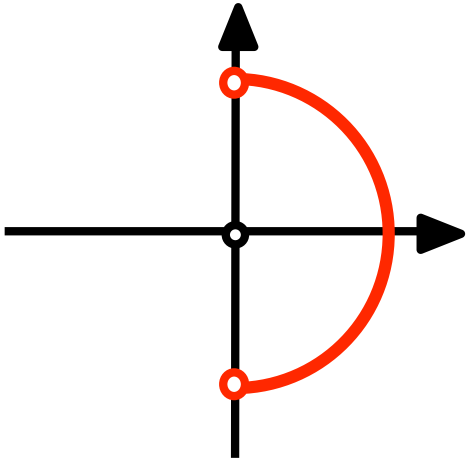

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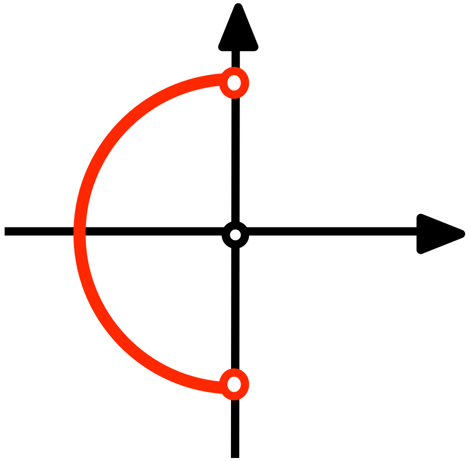
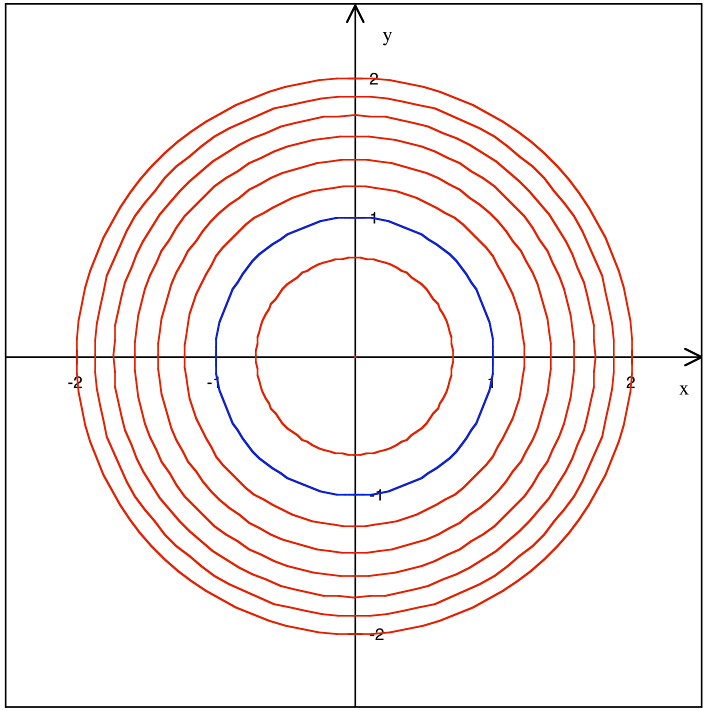
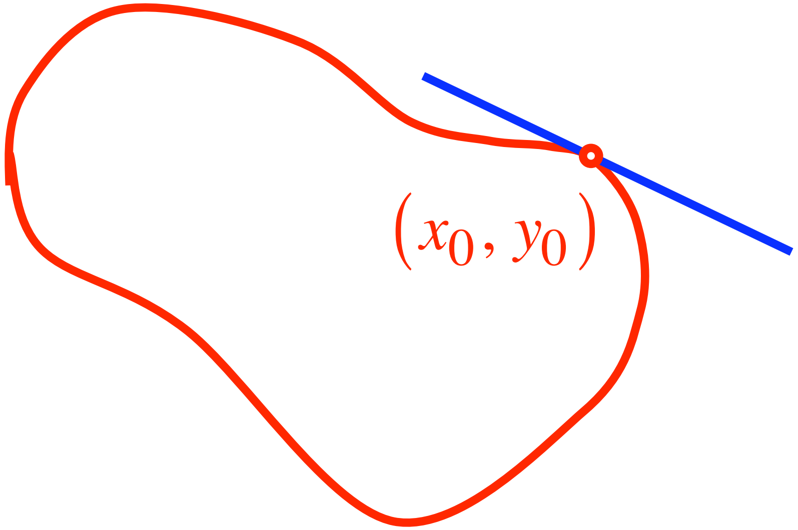


Niveaulinien von $f(x,y) = x^2 + y^2 - 1$

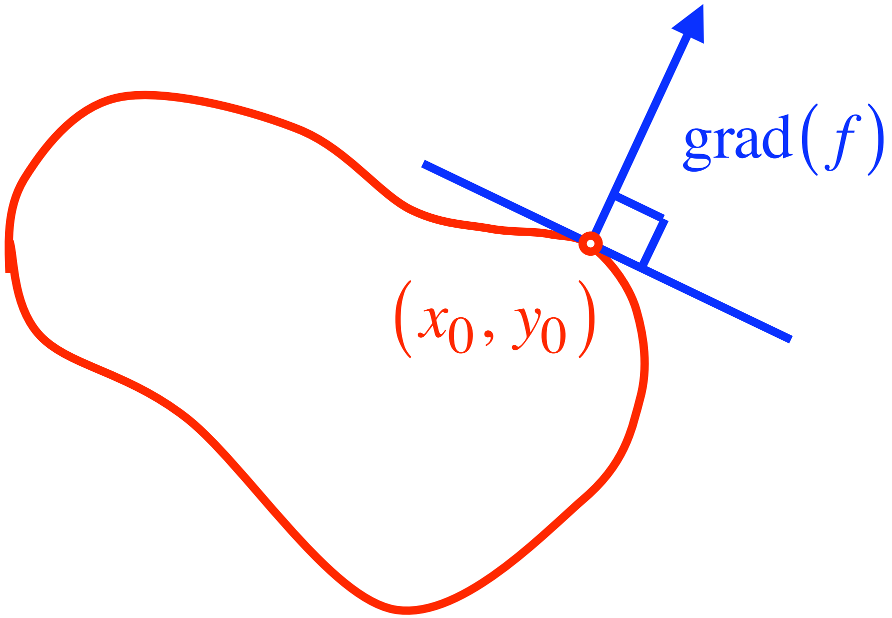


Einheitskreis: Niveau Null

Tangenten an Niveaulinien $f(x,y) = c$



Tangenten an Niveaulinien $f(x,y)=c$



Trick/Erinnerung:

$\mathrm{grad}(f) \perp$ Niveaulinie

$\Rightarrow \quad \mathrm{grad}(f) \perp$ Tangente

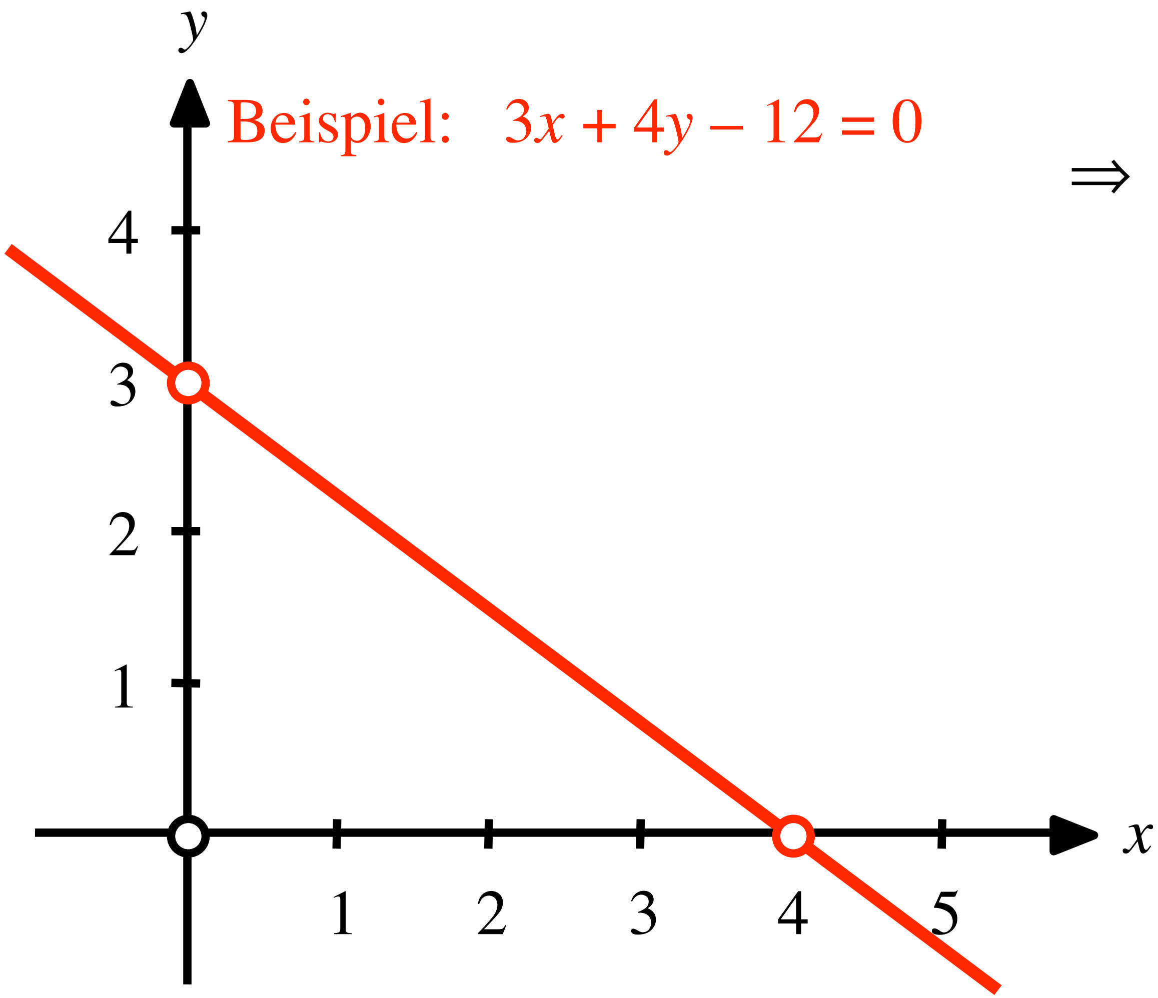
Erinnerung Schule:

Implizite Geradengleichung: $ax + by + d = 0$

$$\Rightarrow \qquad \underbrace{\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Normalenvektor}}$$

Erinnerung Schule:
Implizite Geradengleichung:

$$ax + by + d = 0$$

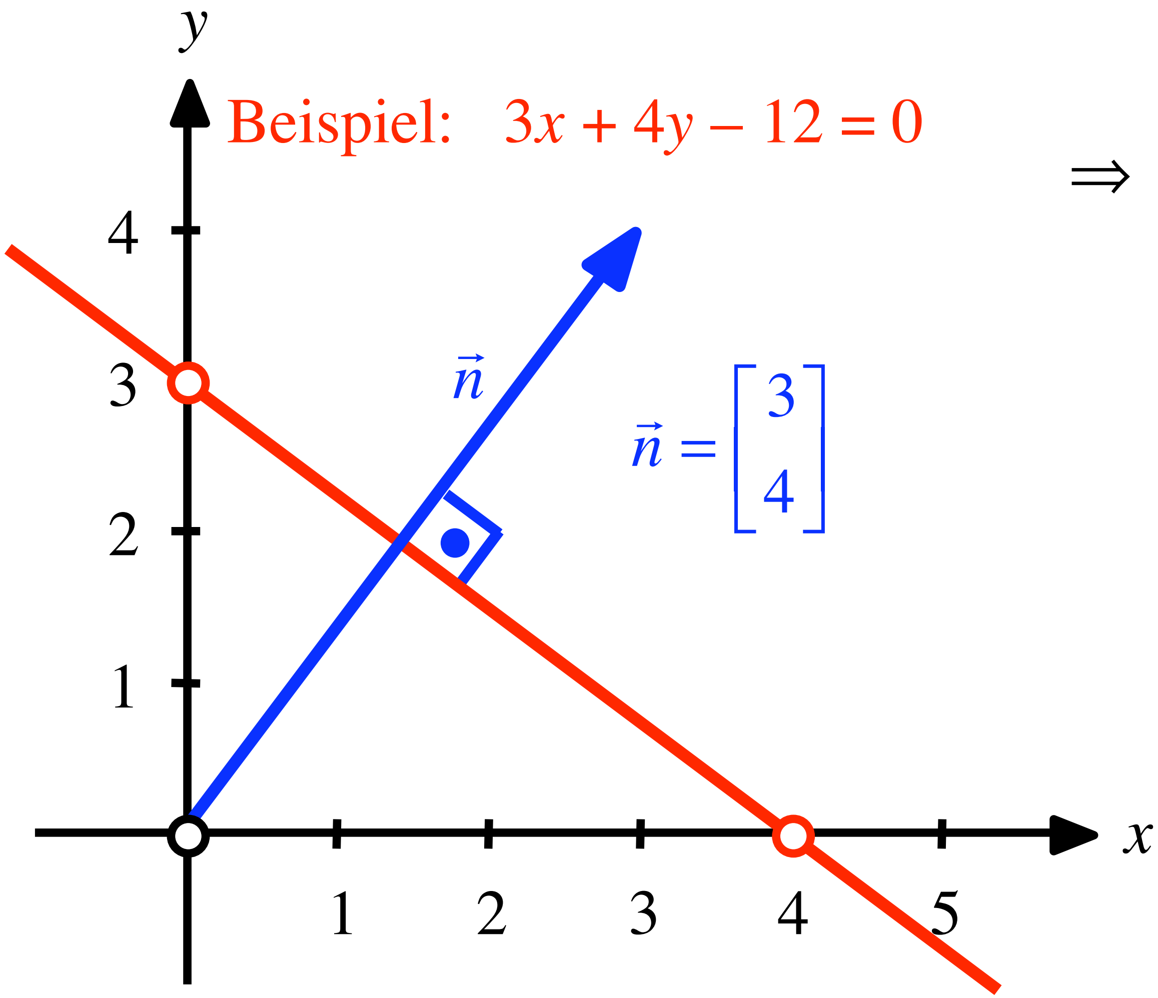


Beispiel: $3x + 4y - 12 = 0$

$$\Rightarrow \qquad \underbrace{\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Normalenvektor}}$$

Erinnerung Schule:
Implizite Geradengleichung:

$$ax + by + d = 0$$



Beispiel: $3x + 4y - 12 = 0$

$$\vec{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\Rightarrow \qquad \underbrace{\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Normalenvektor}}$$

Erinnerung Schule:
Implizite Geradengleichung:

$$ax + by + d = 0$$

$$\Rightarrow \qquad \underbrace{\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Normalenvektor}}$$

Erinnerung Schule:
Implizite Geradengleichung:

$$ax + by + d = 0$$

$$\Rightarrow \qquad \underbrace{\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Normalenvektor}}$$

Somit:

$$\vec{n} = \operatorname{grad}(f) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

Erinnerung Schule:
Implizite Geradengleichung:

$$ax + by + d = 0$$

$$\Rightarrow \qquad \underbrace{\vec{n} = \begin{bmatrix} a \\ b \end{bmatrix}}_{\text{Normalenvektor}}$$

Somit:

$$\vec{n} = \text{grad}(f) = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

Tangente:

$$x\ f_x + y\ f_y + \underset{\overset{\uparrow}{?}}{d} = 0$$

$$x\, f_x\left(x_0, y_0\right) + y\, f_y\left(x_0, y_0\right) + \underset{\overset{\uparrow}{?}}{d} = 0$$

$$x\ f_x\left(x_0,y_0\right)+y\ f_y\left(x_0,y_0\right)+\underset{\underset{?}{\uparrow}}{d}=0$$

$$\Rightarrow \quad d=-\left(x_0\ f_x\left(x_0,y_0\right)+y_0\ f_y\left(x_0,y_0\right)\right)$$

$$x\,\underline{f_x(x_0,y_0)}+y\,f_y(x_0,y_0)+\underset{\substack{\uparrow\\ ?}}{d}=0$$

$$\Rightarrow \quad d=-\left(x_0\,\underline{f_x(x_0,y_0)}+y_0\,f_y(x_0,y_0)\right)$$

Somit Tangente im Punkt (x_0,y_0)

$$(x-x_0)\underline{f_x(x_0,y_0)}+(y-y_0)f_y(x_0,y_0)=0$$

$$x\,\underline{f_x(x_0,y_0)} + y\,f_y(x_0,y_0) + \underset{\overset{\uparrow}{?}}{d} = 0$$

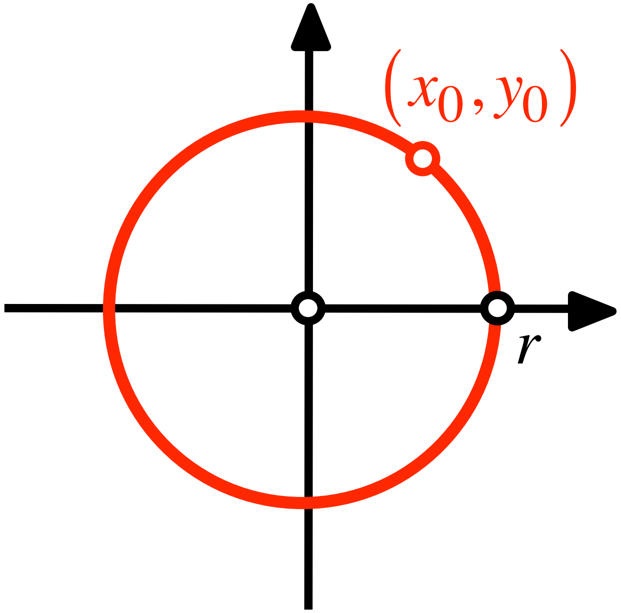
$$\Rightarrow \quad d = -\left(x_0\,\underline{f_x(x_0,y_0)} + y_0\,f_y(x_0,y_0)\right)$$

Somit Tangente im Punkt (x_0,y_0)

$$\boxed{(x-x_0)\,f_x(x_0,y_0) + (y-y_0)\,f_y(x_0,y_0) = 0}$$

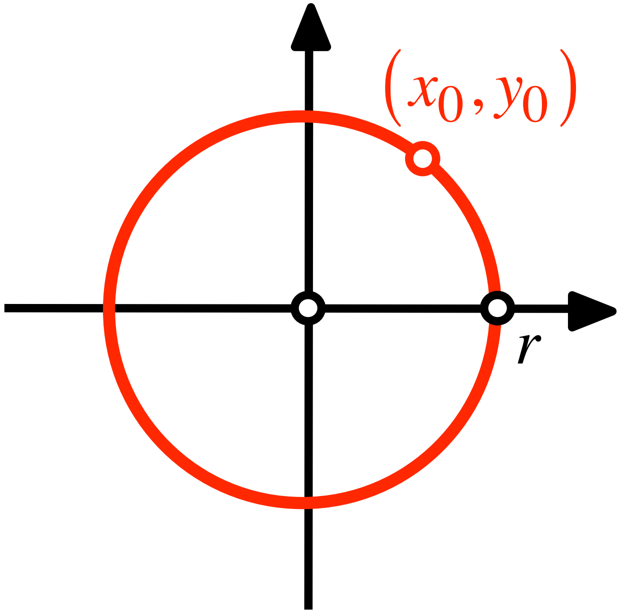
Beispiel:

Kreis mit dem Radius r:



Beispiel:

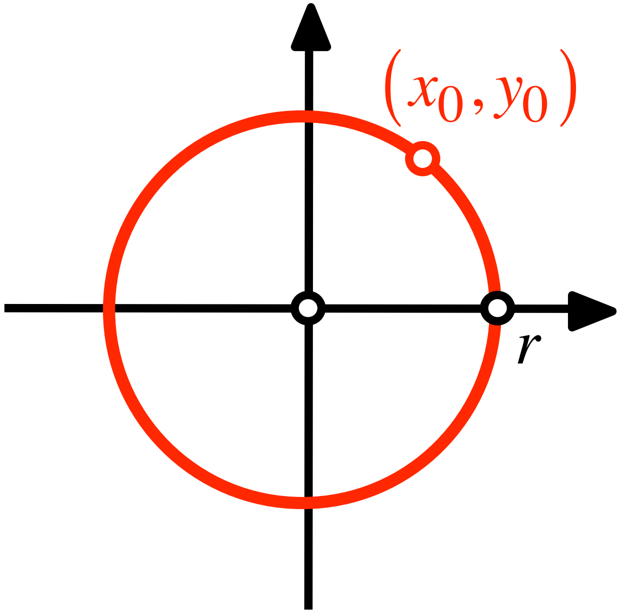
Kreis mit dem Radius r: $x^2 + y^2 - r^2 = 0$



Beispiel:

Kreis mit dem Radius r: $x^2 + y^2 - r^2 = 0$

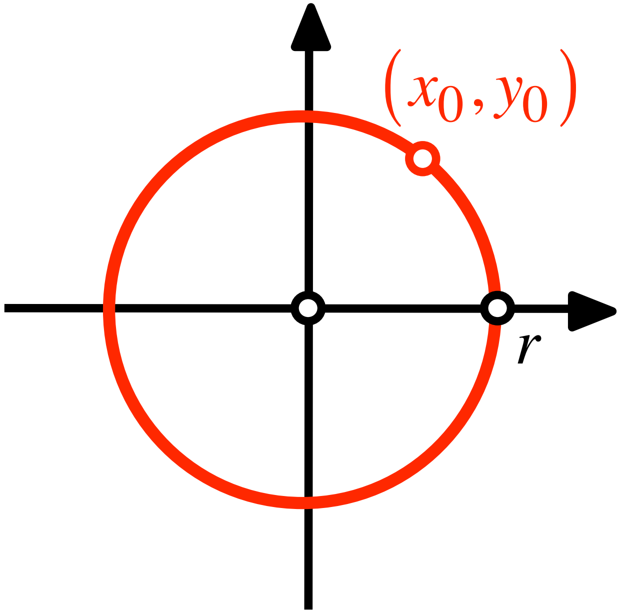
$$f(x,y) = x^2 + y^2 - r^2$$



Kreis: Niveau Null

Beispiel:

Kreis mit dem Radius r: $x^2 + y^2 - r^2 = 0$



$$f(x,y) = x^2 + y^2 - r^2$$

$$f_x = 2x \quad ; \quad f_x(x_0, y_0) = 2x_0$$

Beispiel:

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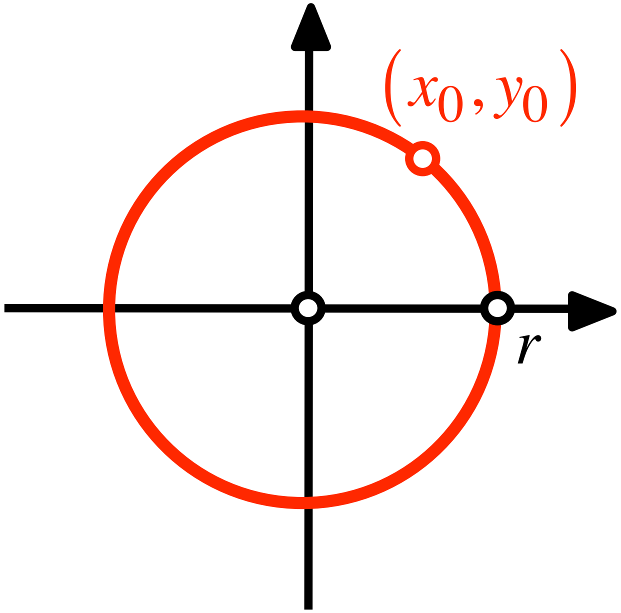

$$f(x,y) = x^2 + y^2 - r^2$$

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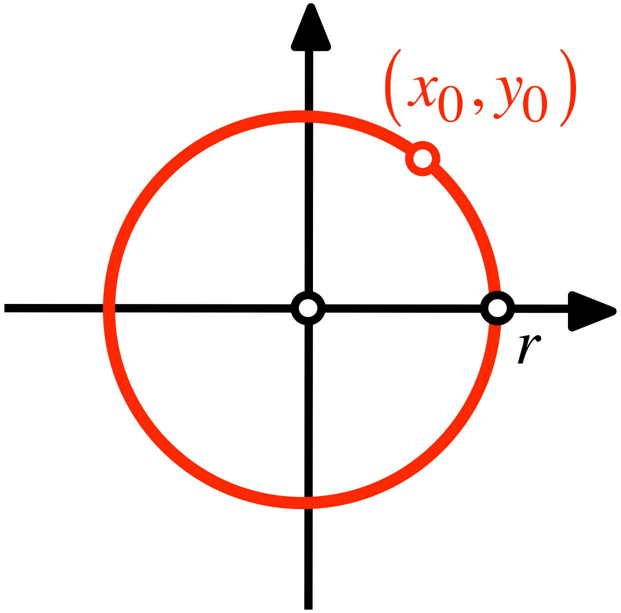

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$$(x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) = 0$$

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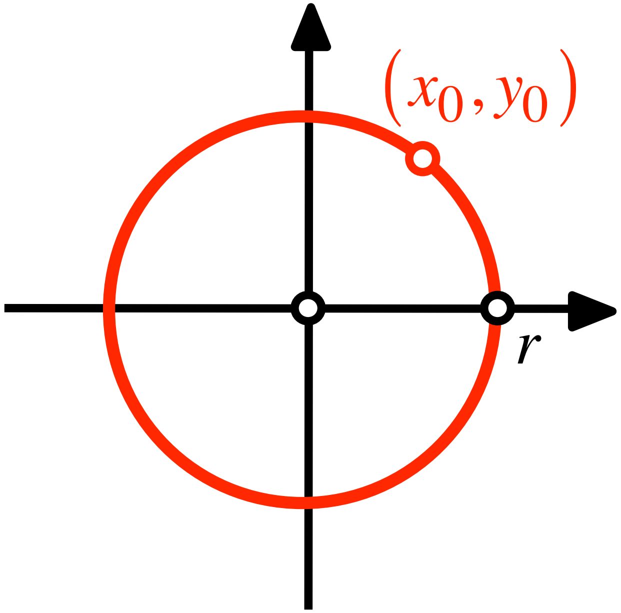
Kreis mit dem Radius r: $x^2 + y^2 - r^2 = 0$



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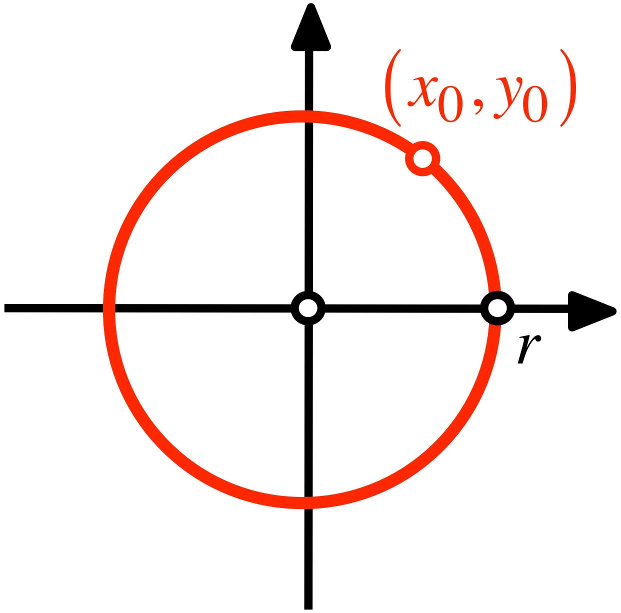
$$(x - x_0) 2x_0 + (y - y_0) 2y_0 = 0$$

Beispiel:

Kreis mit dem Radius r:

$$\left(x-x_0\right)2x_0+\left(y-y_0\right)2y_0=0$$



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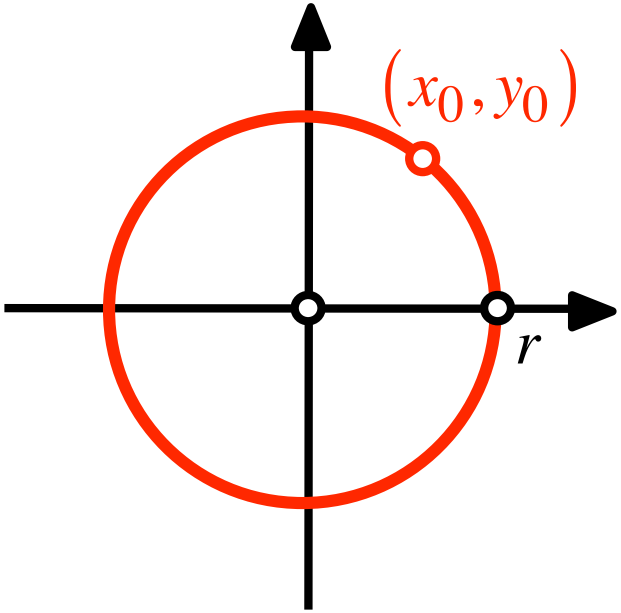
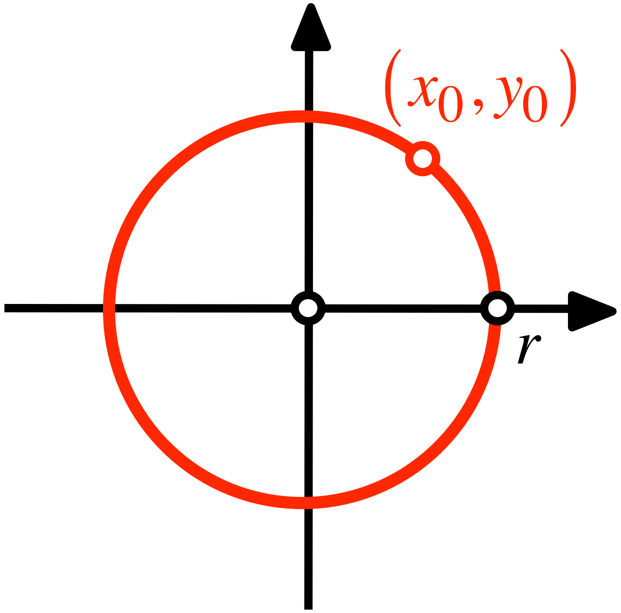


$$(x - x_0)2x_0 + (y - y_0)2y_0 = 0$$

$$xx_0 + yy_0 = \underbrace{x_0^2 + y_0^2}_{r^2}$$

Beispiel:

Kreis mit dem Radius r:



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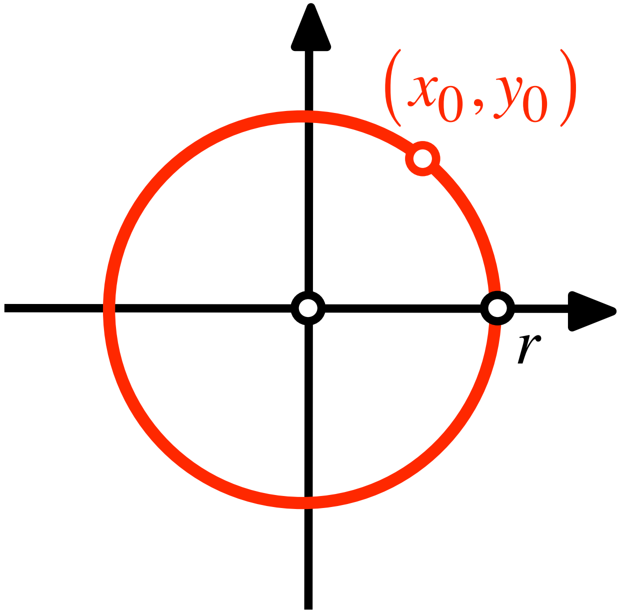
Tangentengleichung

Beispiel:

Kreis mit dem Radius r:



$$x\,x_0 + y\,y_0 = r^2$$

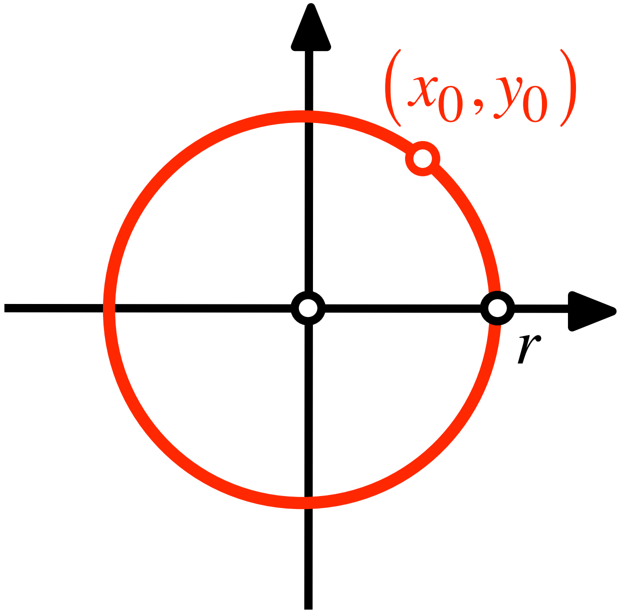
Tangentengleichung

Beispiel:

Kreis mit dem Radius r:

$$x\,x_0 + y\,y_0 = r^2$$

Tangentengleichung

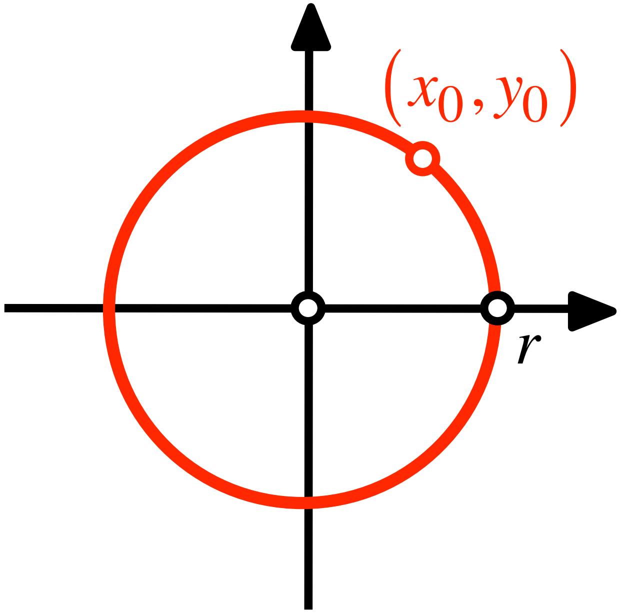


Beispiel: $r = 5$

$$\left(x_0, y_0\right) = \left(3, 4\right)$$

Beispiel:

Kreis mit dem Radius r:



$$\boxed{x\,x_0 + y\,y_0 = r^2}$$

Tangentengleichung

Beispiel: $r = 5$

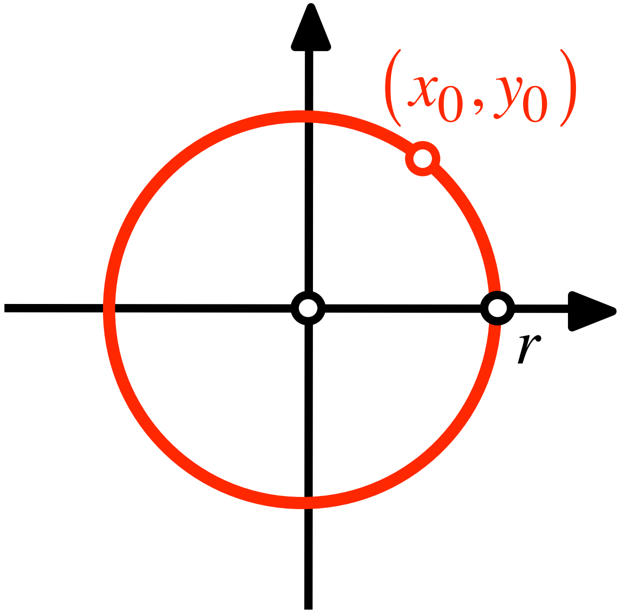
$$(x_0, y_0) = (3,4)$$

Tangentengleichung

$$3x + 4y = 25$$

Beispiel:

Kreis mit dem Radius r:



$$x\,x_0 + y\,y_0 = r^2$$

Tangentengleichung

Beispiel: $r = 5$

$$(x_0, y_0) = (3,4)$$

Tangentengleichung

$$3x + 4y = 25$$

$$4y = -3x + 25$$

$$y = -\frac{3}{4}x + \frac{25}{4} = -0.75x + 6.25$$

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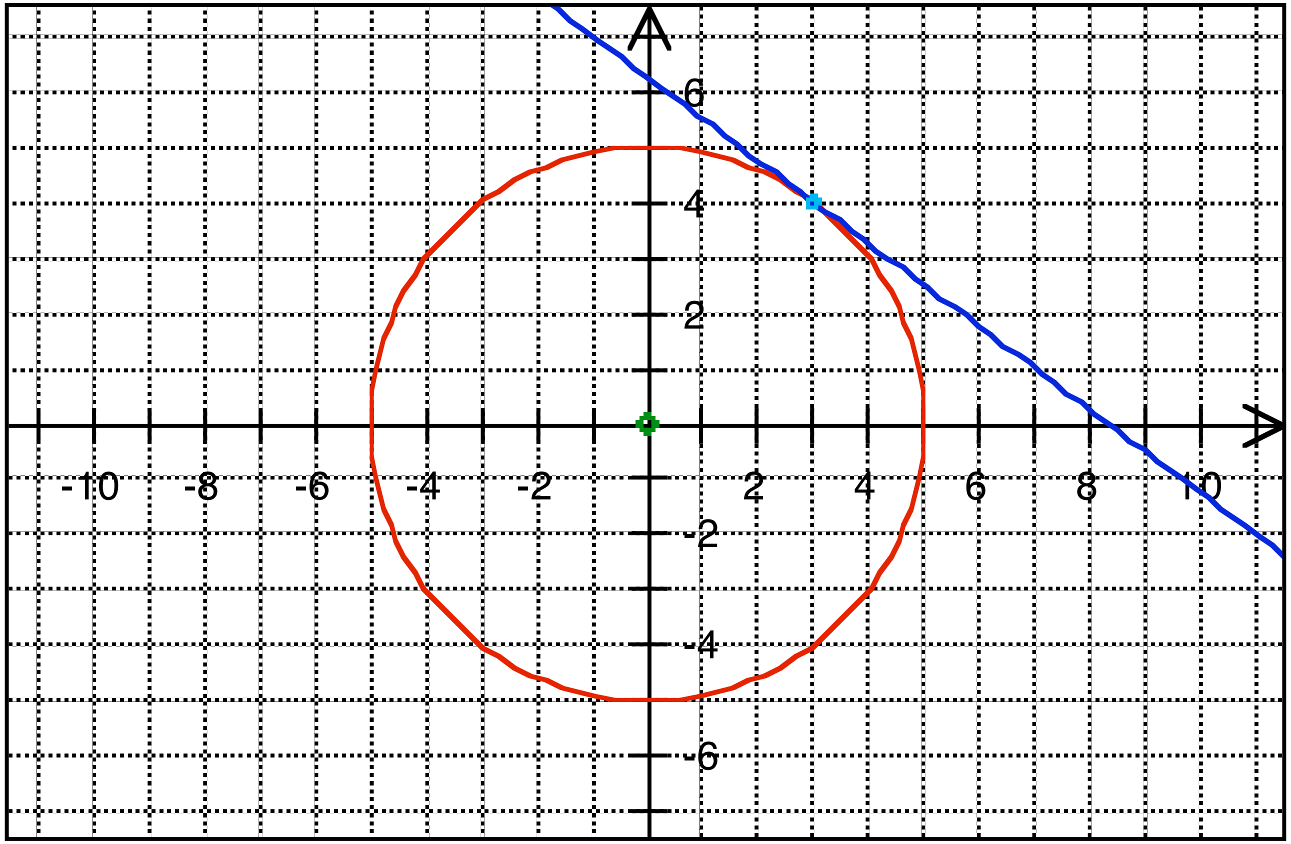

Parabel : $y = x^2 \quad \Rightarrow \quad f(x,y) = y - x^2$ Niveau Null

$(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{4}\right)$

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$(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{4}\right)$

$f_x = -2x \qquad f_x\left(\frac{1}{2}, \frac{1}{4}\right) = -1$

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Tangentengleichung : $(x - x_0) f_x(x_0, y_0) + (y - y_0) f_y(x_0, y_0) = 0$

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$\left(x - \frac{1}{2}\right) \cdot (-1) + \left(y - \frac{1}{4}\right) \cdot 1 = 0$

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$\left(x - \tfrac{1}{2}\right) \cdot (-1) + \left(y - \tfrac{1}{4}\right) \cdot 1 = 0$

$-x + \tfrac{1}{2} + y - \tfrac{1}{4} = 0$

$y = x - \tfrac{1}{4}$

$$y = x^2 \quad (x_0, y_0) = \left(\tfrac{1}{2}, \tfrac{1}{4}\right)$$

Tangentengleichung: $y = x - \frac{1}{4}$

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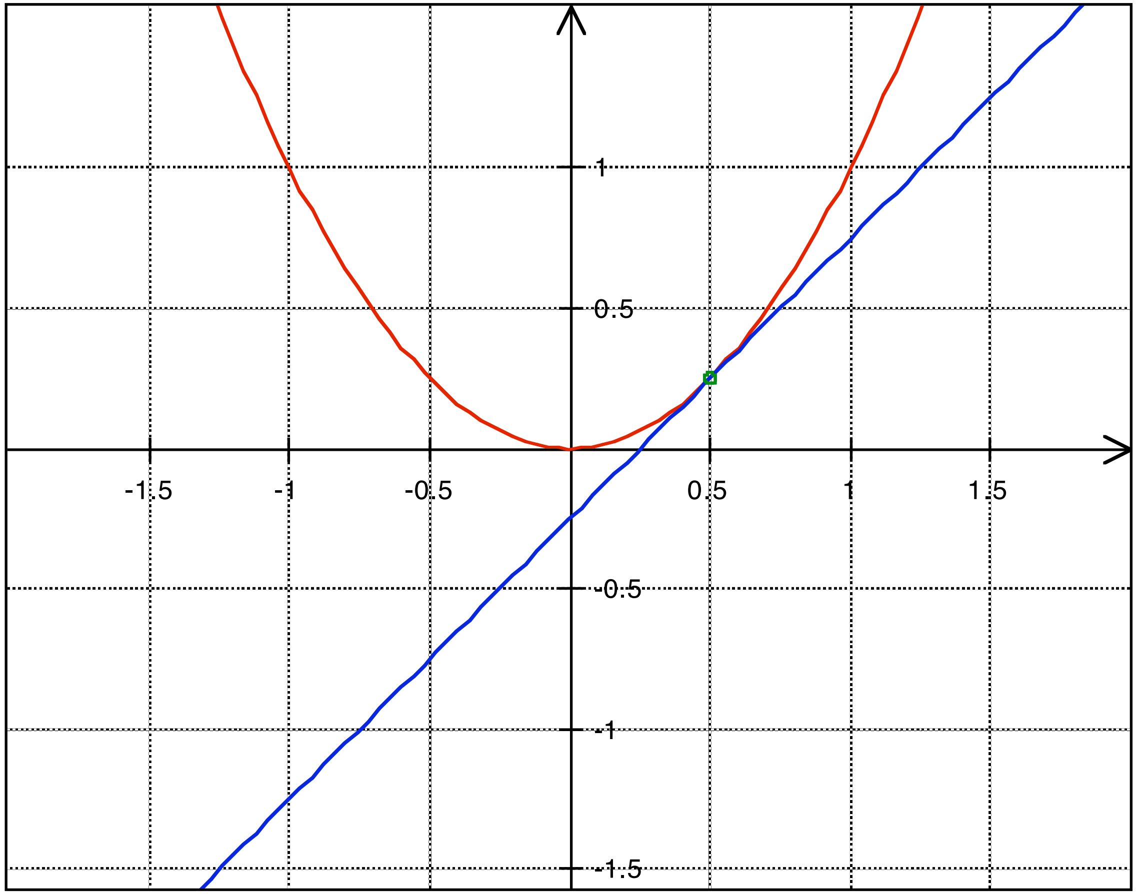
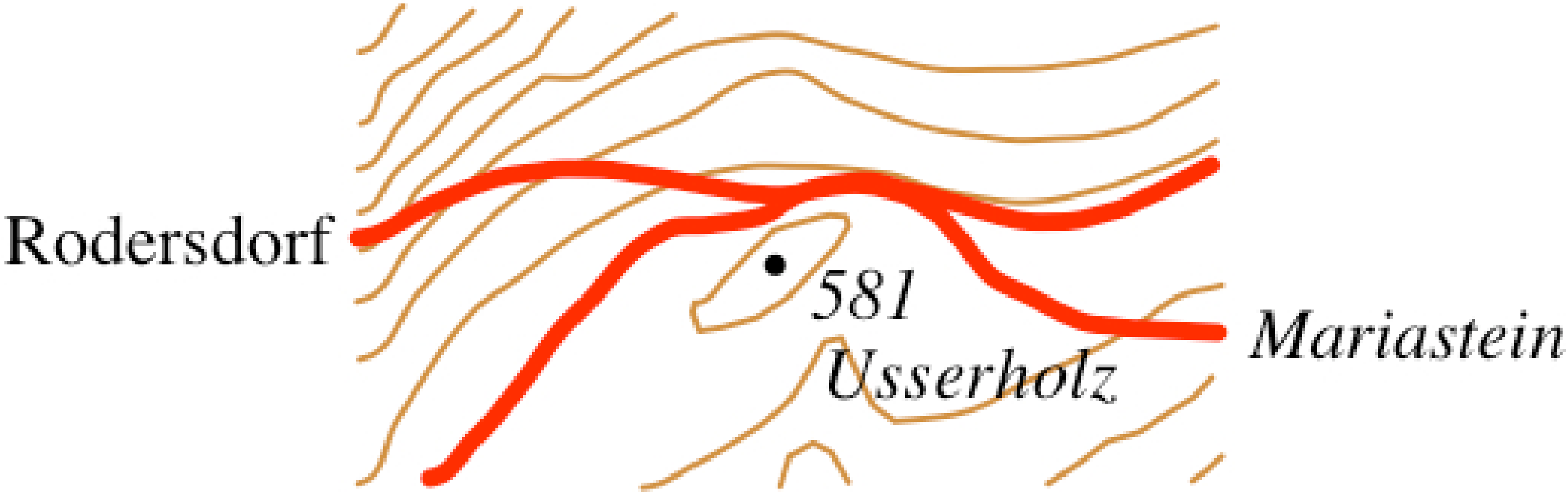


Extrema mit Nebenbedingungen

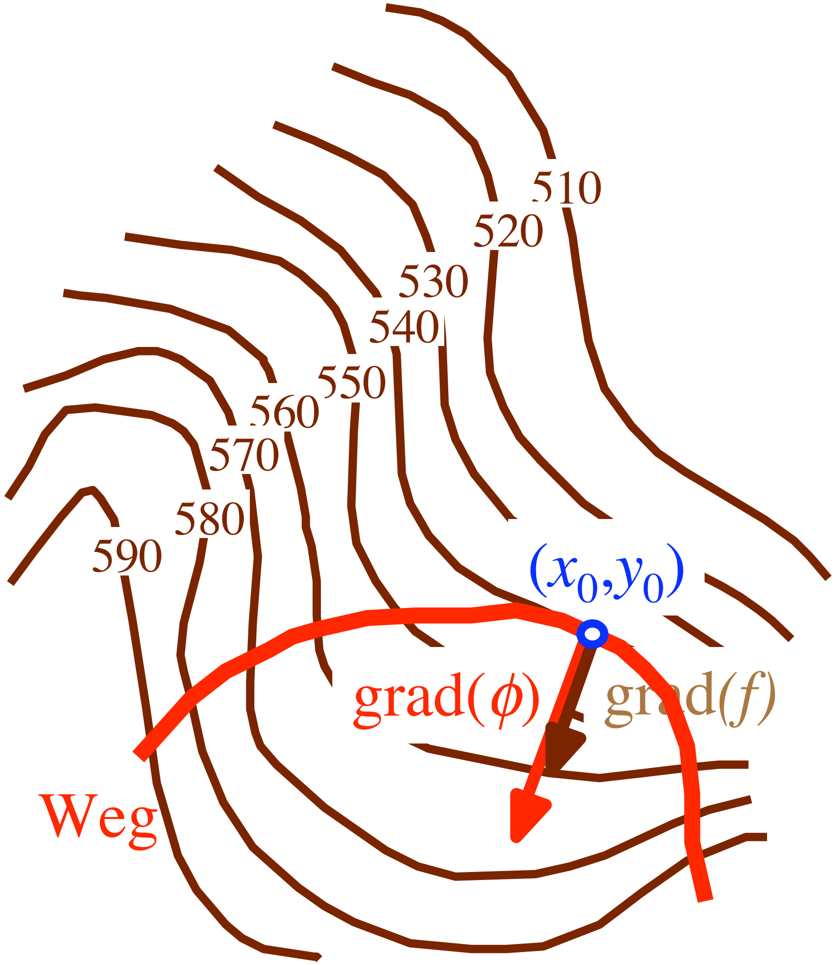


Rodersdorf: Endstation Tram 10

Extrema mit Nebenbedingungen

$f(x, y)$	„Gelände“
$\phi(x, y) = 0$	Nebenbedingung implizite Darstellung eines Weges
Gesucht:	Extrema von $f(x, y)$ auf diesem Weg Höchster Wegpunkt? Tiefster Wegpunkt?

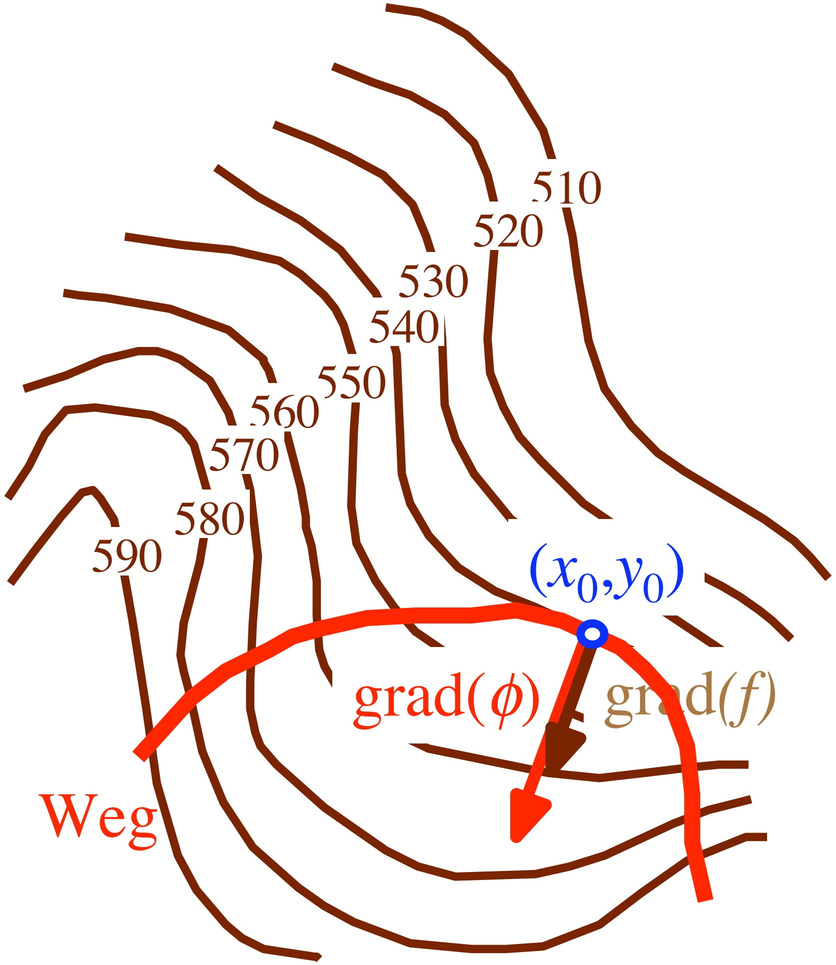
Extrema mit Nebenbedingungen



(lokal) tiefster Wegpunkt

Vergleich mit Niveaulinien des Geländes

Extrema mit Nebenbedingungen



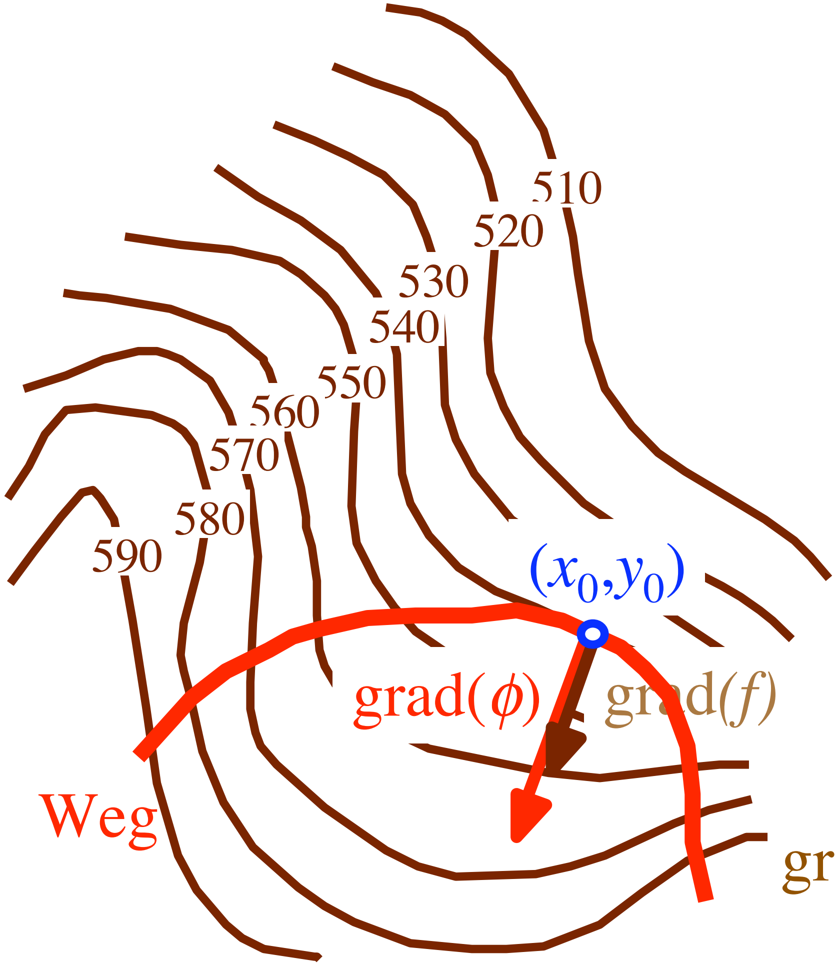
(lokal) tiefster Wegpunkt

Vergleich mit Niveaulinien des Geländes

Weg im tiefsten Punkt tangential an Niveaulinie

⇨ Parallele Gradienten

Extrema mit Nebenbedingungen



(lokal) tiefster Wegpunkt

Vergleich mit Niveaulinien des Geländes

Weg im tiefsten Punkt tangential an Niveaulinie

⇨ Parallele Gradienten

$$\operatorname{grad}(f)(x_0,y_0)=\lambda\cdot\operatorname{grad}(\phi)(x_0,y_0)$$

Gesucht: $\left(x_0, y_0\right)$ so dass

$$\operatorname{grad}(f)\left(x_0, y_0\right) = \lambda \cdot \operatorname{grad}(\phi)\left(x_0, y_0\right)$$

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Methode:

$$\underbrace{F(x, y, \lambda)}_{\substack{\text{Hilfsfunktion} \\ \text{in } x, y, \lambda}} = f(x, y) - \lambda\phi(x, y)$$

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$$\underbrace{F(x,y,\lambda)}_{\substack{\text{Hilfsfunktion} \\ \text{in } x,y,\lambda}} = f(x,y) - \lambda\phi(x,y)$$

$$\operatorname{grad}(F) = \begin{bmatrix} F_x \\ F_y \\ F_\lambda \end{bmatrix} = \begin{bmatrix} f_x - \lambda\phi_x \\ f_y - \lambda\phi_y \\ -\phi \end{bmatrix} \overset{\substack{\text{soll} \\ \downarrow}}{=} \vec{0}$$

Gesucht: $\left(x_0, y_0\right)$ so dass

$$\operatorname{grad}\left(f\right)\left(x_0, y_0\right) = \lambda \cdot \operatorname{grad}\left(\phi\right)\left(x_0, y_0\right)$$

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$$\underbrace{F\left(x, y, \lambda\right)}_{\substack{\text{Hilfsfunktion} \\ \text{in } x, y, \lambda}} = f\left(x, y\right) - \lambda\phi\left(x, y\right)$$

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$$\left.\begin{aligned} f_x - \lambda\phi_x &= 0 \\ f_y - \lambda\phi_y &= 0 \end{aligned}\right\} \Leftrightarrow \quad \operatorname{grad}\left(f\right) = \lambda \cdot \operatorname{grad}\left(\phi\right)$$

Gesucht: $\left(x_0, y_0\right)$ so dass

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Methode:

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$-\phi = 0$ $\Leftrightarrow$ Nebenbedingung erfüllt

Beispiel : $f(x, y) = x^2 + y^2$

Nebenbedingung : $\phi(x, y) = \frac{x^2}{4} + y^2 - 1 = 0$

$$\text{Beispiel}: \; f(x,y) = x^2 + y^2$$

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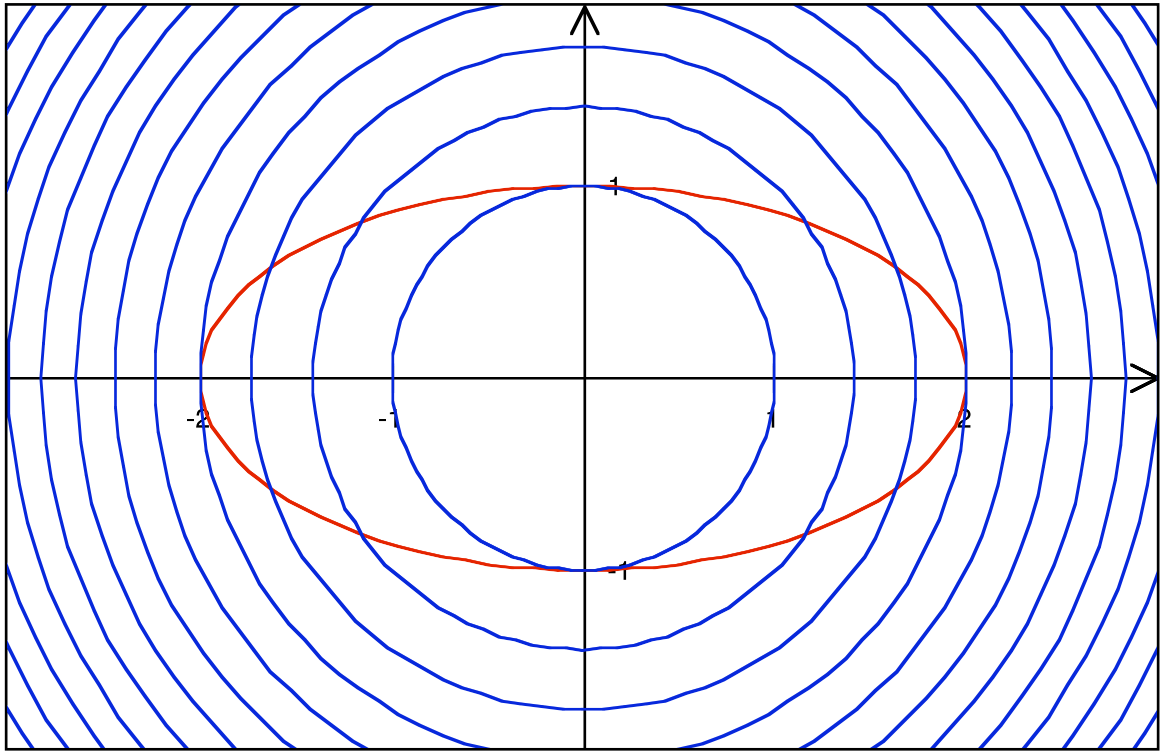

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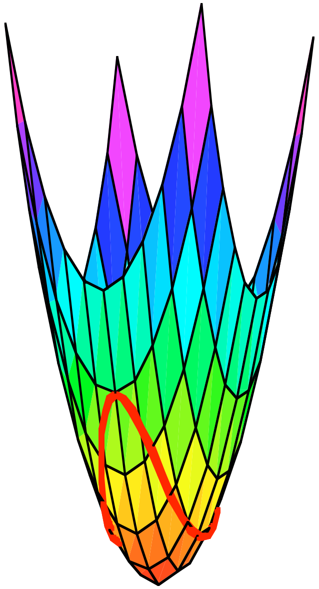

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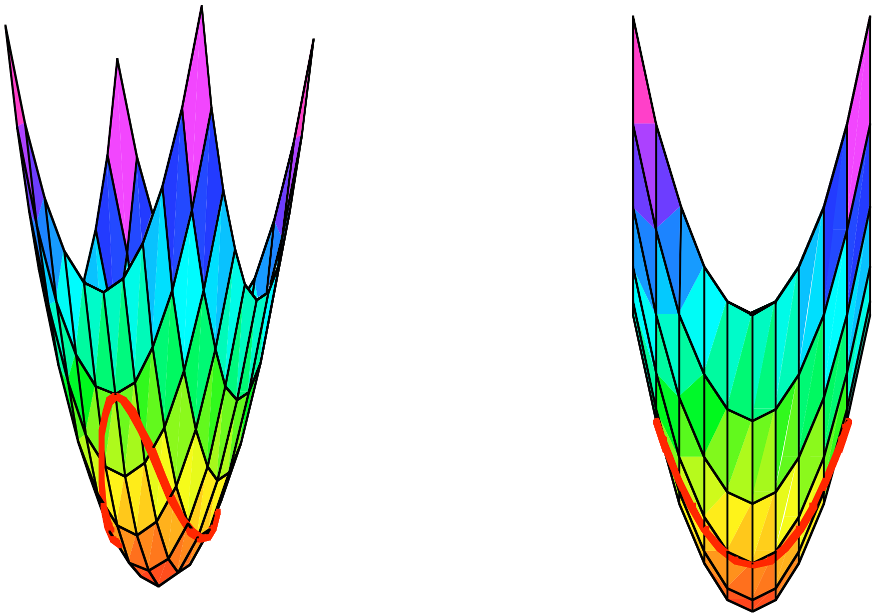

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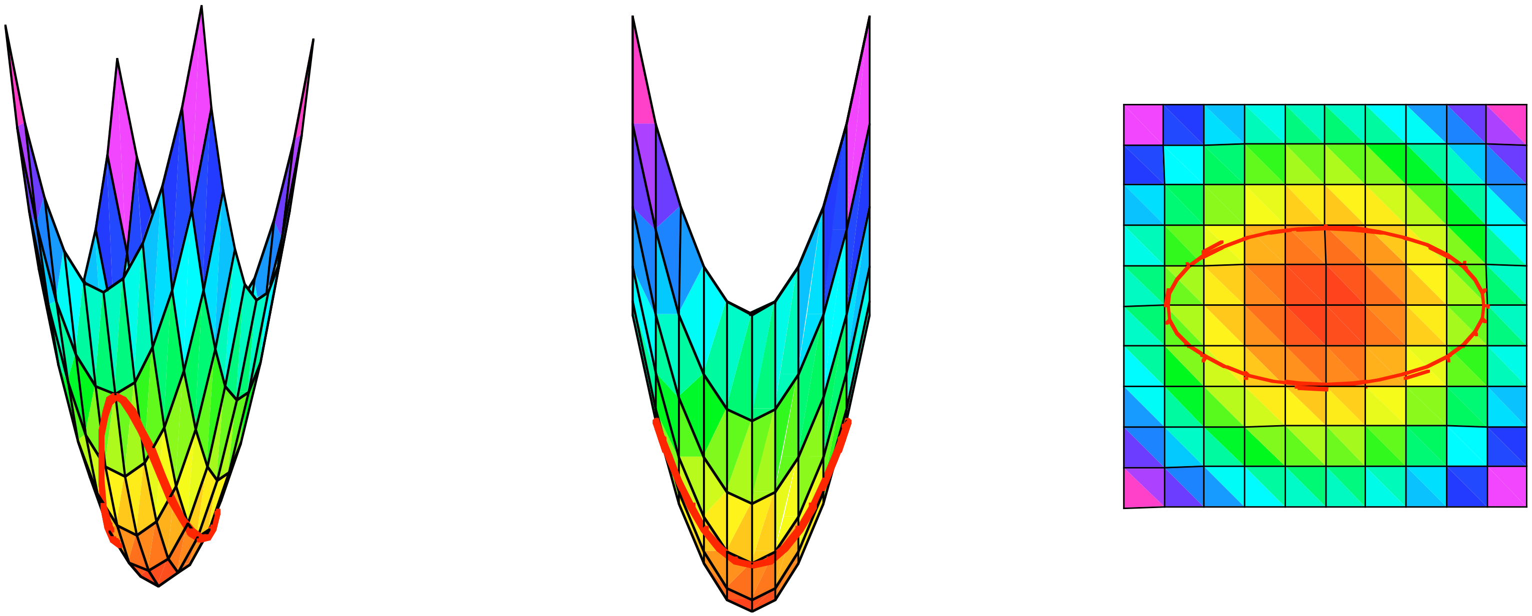

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Beispiel : $f(x,y)=x^2+y^2$

Nebenbedingung : $\phi(x,y)=\frac{x^2}{4}+y^2-1=0$

$$F(x,y,\lambda)=f(x,y)-\lambda\phi(x,y)=x^2+y^2-\lambda\left(\frac{x^2}{4}+y^2-1\right)$$

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$$F_x = 2x - \tfrac{1}{2}x\lambda \overset{!}{=} 0$$

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$x = 0$ oder $2 - \frac{1}{2}\lambda = 0$, das heißt $\lambda = 4$

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Beispiel : $f(x,y)=x^2+y^2$

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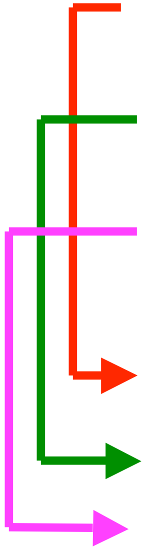

$$F_x=2x-\tfrac{1}{2}x\lambda\overset{!}{=}0$$

$$F_y=2y-2y\lambda\overset{!}{=}0$$

$$F_\lambda=-\tfrac{x^2}{4}-y^2+1\overset{!}{=}0$$

$x=0$ oder $2-\frac{1}{2}\lambda=0,$ das heißt $\lambda=4$

$y=0$ oder $2-2\lambda=0,$ das heißt $\lambda=1$

Ellipsengleichung (Weg)

Beispiel : $f(x,y) = x^2 + y^2$

Nebenbedingung : $\phi(x,y) = \frac{x^2}{4} + y^2 - 1 = 0$

$$F(x,y,\lambda) = f(x,y) - \lambda\phi(x,y) = x^2 + y^2 - \lambda\left(\frac{x^2}{4} + y^2 - 1\right)$$

$$F_x = 2x - \tfrac{1}{2}x\lambda \overset{!}{=} 0$$

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$$F_\lambda = -\frac{x^2}{4} - y^2 + 1 \overset{!}{=} 0$$

$x = 0$ oder $2 - \frac{1}{2}\lambda = 0$, das heißt $\lambda = 4$

$y = 0$ oder $2 - 2\lambda = 0$, das heißt $\lambda = 1$

Ellipsengleichung (Weg)

Lösungen : $\boxed{x = \pm 2,\ y = 0,\ \lambda = 4}$ Maximum

$\boxed{x = 0,\ y = \pm 1,\ \lambda = 1}$ Minimum

Integration in mehreren Variablen

$$\iint\limits_{A} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

2-dim Bereich

Integration in mehreren Variablen

$$\iint\limits_{A} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

2-dim Bereich

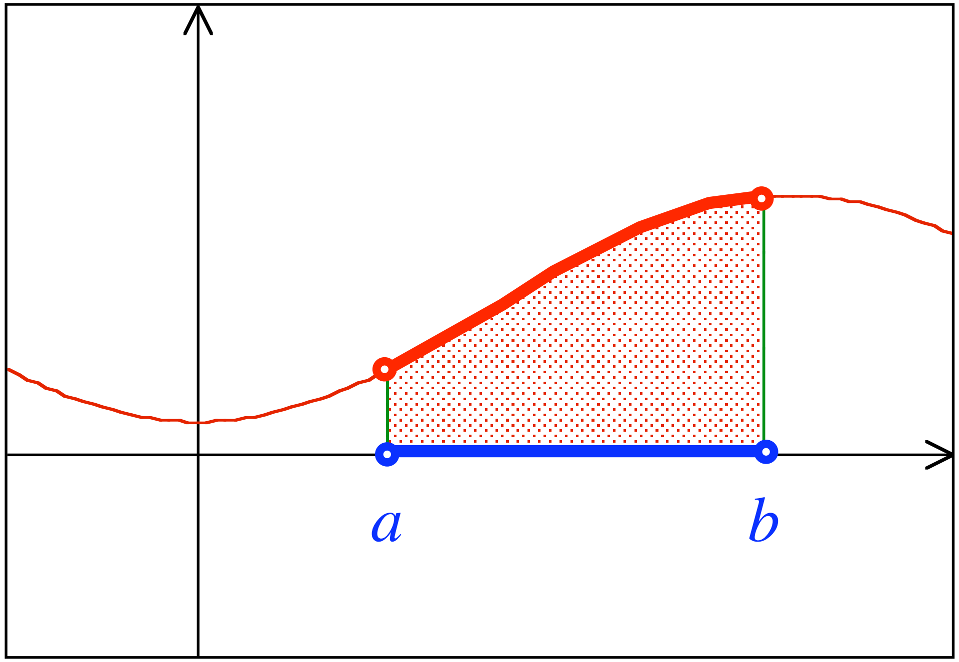
$$\iiint\limits_{A} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

3-dim Bereich

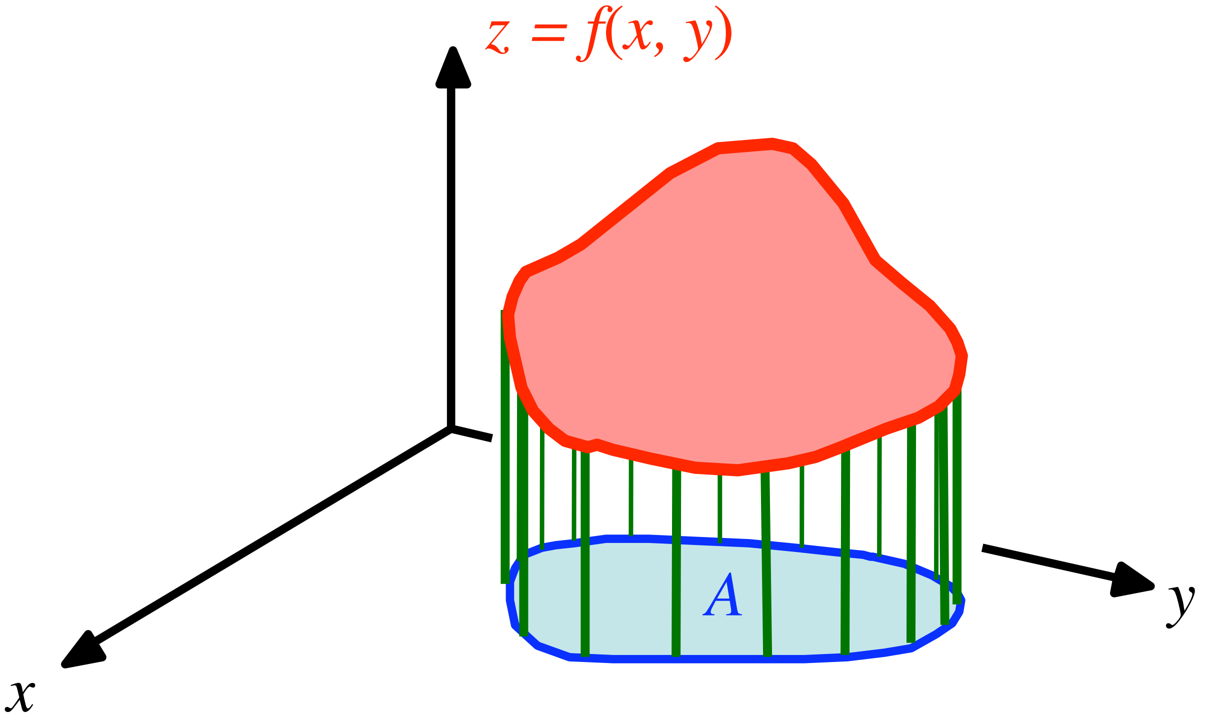
Erinnerung

$$\int_a^b f(x)\,\mathrm{d}x = \text{"Fläche unter der Kurve"}$$

Integrationsbereich $[a,b]$ eindimensional

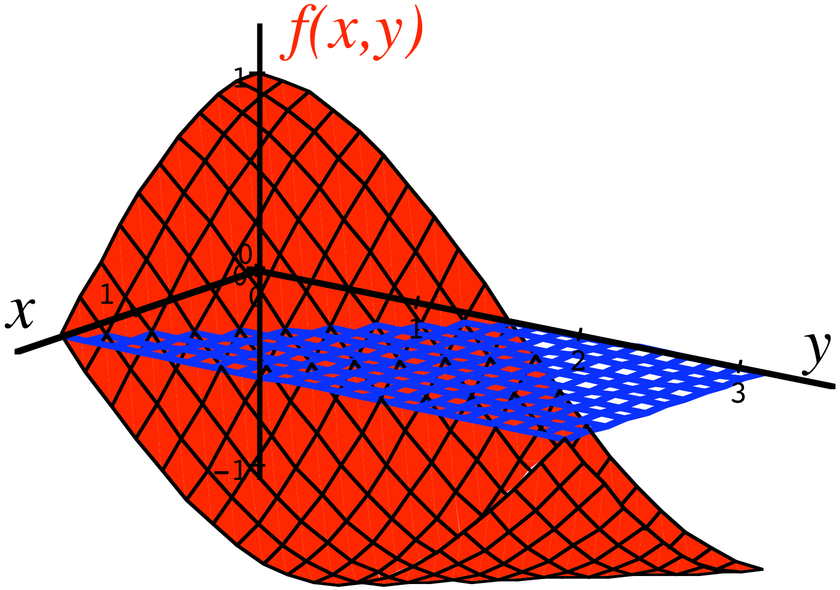


$$\iint_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \text{"Volumen unter der Fläche" mit Grundfläche } A$$



Beispiel: $f(x,y) = \cos(x+y)$

$$A = \left[0, \frac{\pi}{2}\right] \times [0, \pi] = \left\{ (x,y) \,\middle|\, x \in \left[0, \frac{\pi}{2}\right], y \in [0, \pi] \right\}$$



$$I = \iint\limits_{A} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\pi} \left[\underbrace{\int\limits_{0}^{\frac{\pi}{2}} \cos\left(x + y\right)\mathrm{d}x}_{I_1} \right] \mathrm{d}y$$

$$I = \iint\limits_{A} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\pi} \left[\underbrace{\int\limits_{0}^{\frac{\pi}{2}} \cos\left(x+y\right)\mathrm{d}x}_{I_1} \right] \mathrm{d}y$$

$$I_1 = \int\limits_{0}^{\frac{\pi}{2}} \cos\left(x+y\right)\mathrm{d}x = \sin\left(x+y\right) \Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$I = \iint\limits_{A} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\pi} \left[\underbrace{\int\limits_{0}^{\frac{\pi}{2}} \cos(x+y)\,\mathrm{d}x}_{I_1} \right] \mathrm{d}y$$

$$I_1 = \int\limits_{0}^{\frac{\pi}{2}} \cos(x+y)\,\mathrm{d}x = \sin(x+y) \Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \sin\left(\frac{\pi}{2} + y\right) - \sin(y) = \cos(y) - \sin(y)$$

$$I = \iint_A f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi}\left[\underbrace{\int_0^{\frac{\pi}{2}} \cos(x+y)\,\mathrm{d}x}_{I_1}\right]\mathrm{d}y$$

$$I_1 = \int_0^{\frac{\pi}{2}} \cos(x+y)\,\mathrm{d}x = \sin(x+y)\Big|_{x=0}^{x=\frac{\pi}{2}}$$

$$= \sin\left(\frac{\pi}{2}+y\right) - \sin(y) = \cos(y) - \sin(y)$$

Nebenrechnung:

$$\sin\left(\frac{\pi}{2}+y\right) = \underbrace{\sin\left(\frac{\pi}{2}\right)}_{1}\cos(y) + \underbrace{\cos\left(\frac{\pi}{2}\right)}_{0}\sin(y) = \cos(y)$$

$$I = \iint\limits_{A} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\pi} \left[\underbrace{\int\limits_{0}^{\frac{\pi}{2}} \cos\left(x + y\right)\mathrm{d}x}_{I_1 = \cos(y) - \sin(y)} \right] \mathrm{d}y$$

$$I = \iint\limits_{A} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\pi} \left[\underbrace{\int\limits_{0}^{\frac{\pi}{2}} \cos\left(x + y\right)\mathrm{d}x}_{I_1 = \cos(y) - \sin(y)} \right] \mathrm{d}y$$

$$= \int\limits_{0}^{\pi} \left(\cos\left(y\right) - \sin\left(y\right)\right)\mathrm{d}y = \left(\sin\left(y\right) + \cos\left(y\right)\right)\Big|_{0}^{\pi}$$

$$I = \iint\limits_{A} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{\pi}\left[\underbrace{\int\limits_{0}^{\frac{\pi}{2}} \cos\left(x+y\right)\mathrm{d}x}_{I_1=\cos(y)-\sin(y)}\right]\mathrm{d}y$$

$$= \int\limits_{0}^{\pi}\left(\cos\left(y\right)-\sin\left(y\right)\right)\mathrm{d}y = \left(\sin\left(y\right)+\cos\left(y\right)\right)\Bigg|_{0}^{\pi}$$

$$= \left(0-1\right)-\left(0+1\right) = -2$$

```
> Int( Int(cos(x+y), x=0..Pi/2 ), y=0..Pi)
= int( int(cos(x+y), x=0..Pi/2 ), y=0..Pi);
```

$$\int_0^{\pi}\int_0^{1/2\,\pi}\cos(x+y)\,dx\,dy = -2$$

```
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```

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```

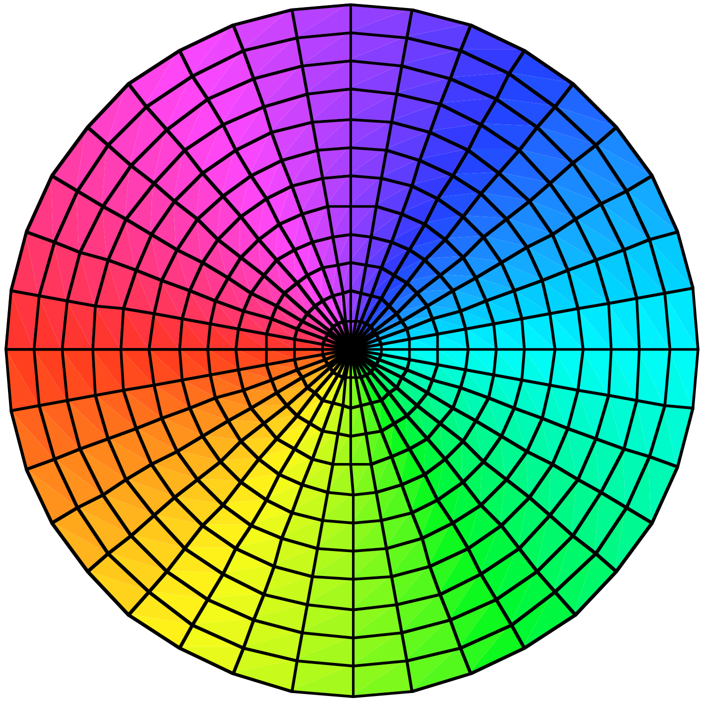
$$\int_0^{1/2\,\pi}\int_0^{\pi}\cos(x+y)\,dy\,dx = -2$$

Kreisscheibe mit Radius r

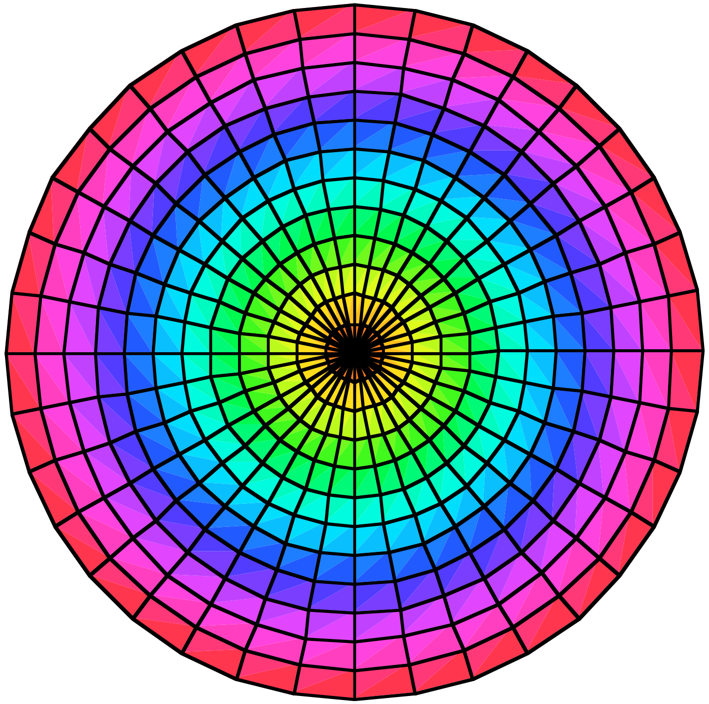
$$A = \{(\rho,\phi) | \rho \in [0,r], \phi \in [0,2\pi]\}$$

Kreisscheibe mit Radius *r*

$$A = \{(\rho,\phi) | \rho \in [0,r], \phi \in [0,2\pi]\}$$

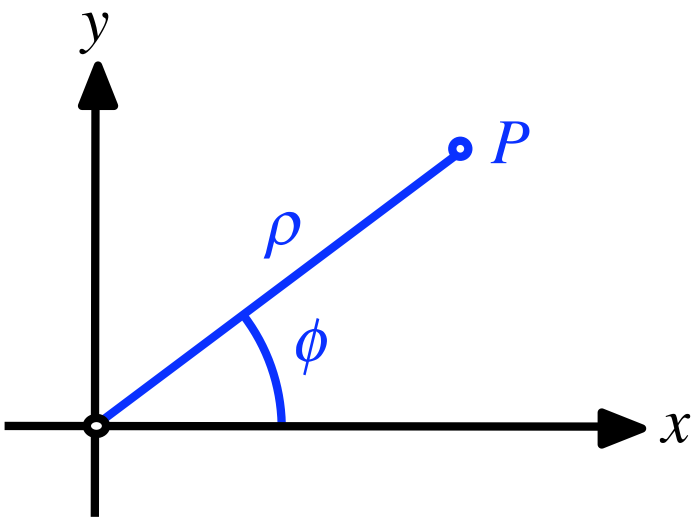


Sektoren



Ringe

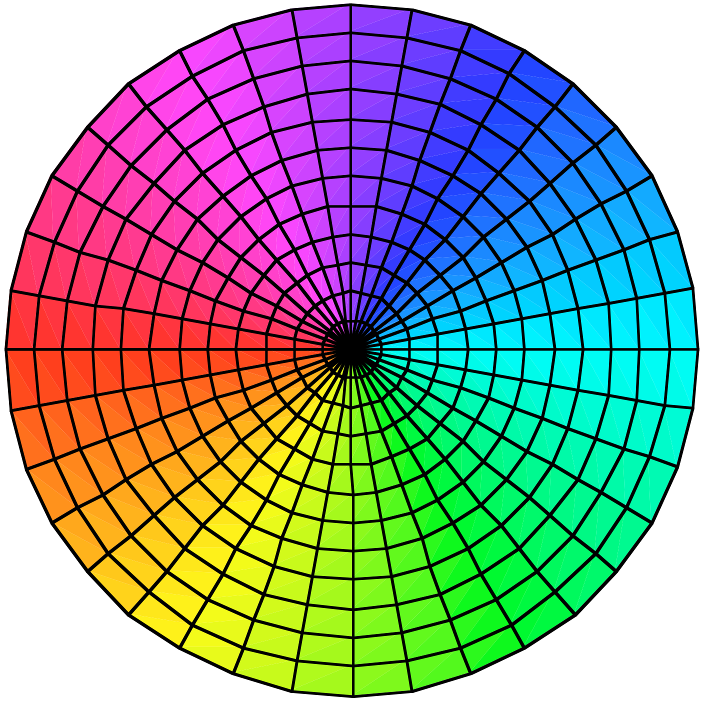
Polarkoordinaten



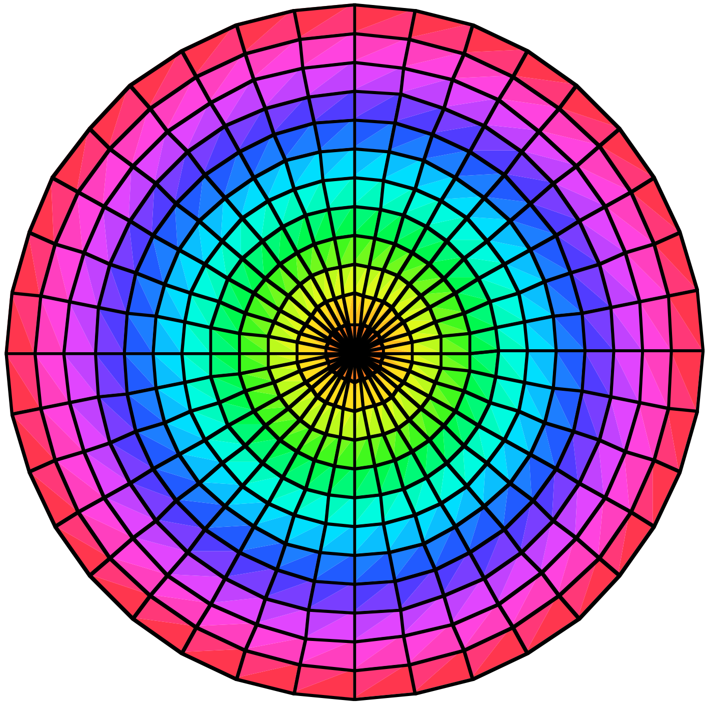
$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$

Kreisscheibe mit Radius r

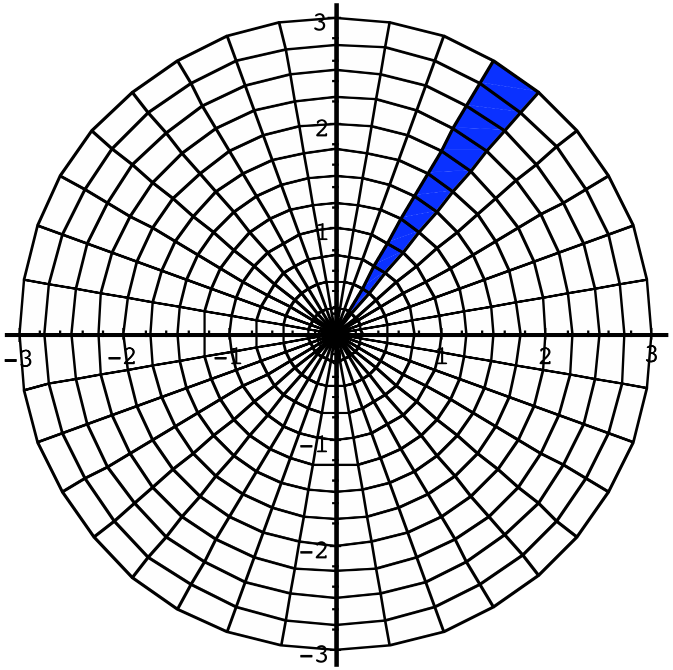
$$A = \{(\rho, \phi) | \rho \in [0, r], \phi \in [0, 2\pi]\}$$



Sektoren

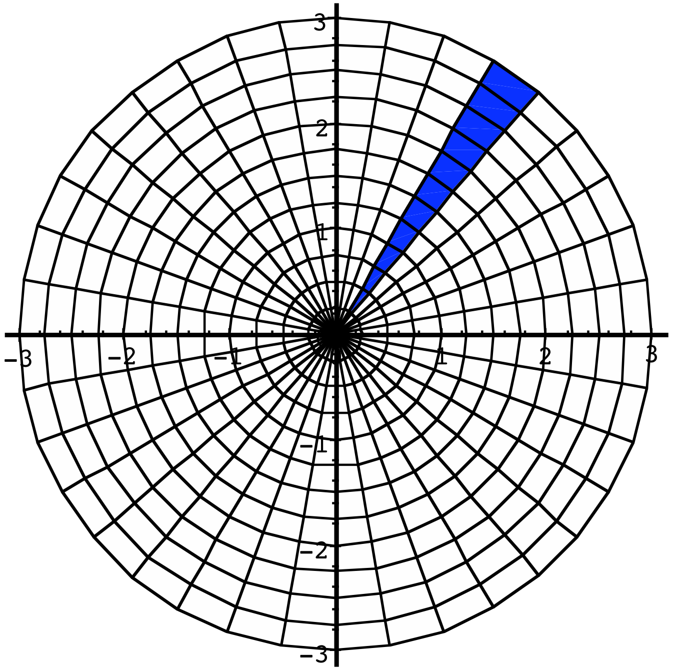


Ringe



Sektorfläche (“Dreieck”)

$$\frac{1}{2} \underset{\substack{\uparrow \\ \text{Radius,} \\ \text{"Höhe"}}}{r} \cdot \underset{\substack{\uparrow \\ \text{Bogen,} \\ \text{"Grund-} \\ \text{linie"}}}{r\mathrm{d}\phi} = \frac{1}{2} r^2 \mathrm{d}\phi$$



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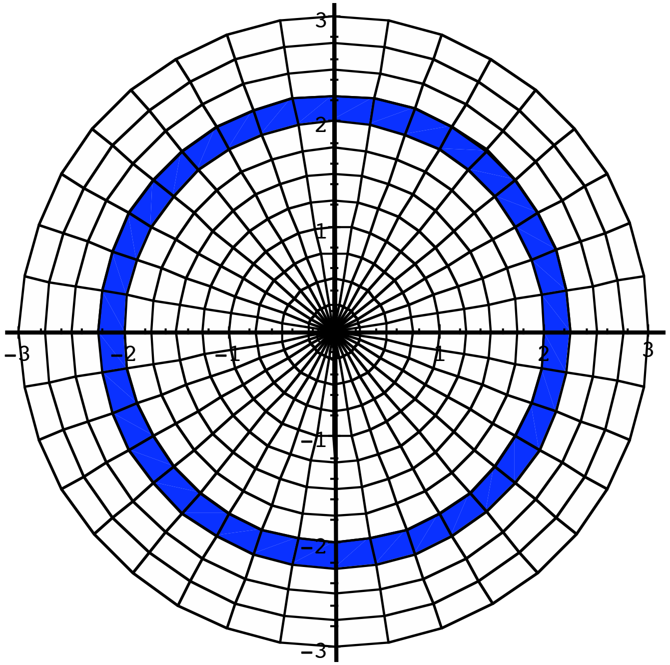
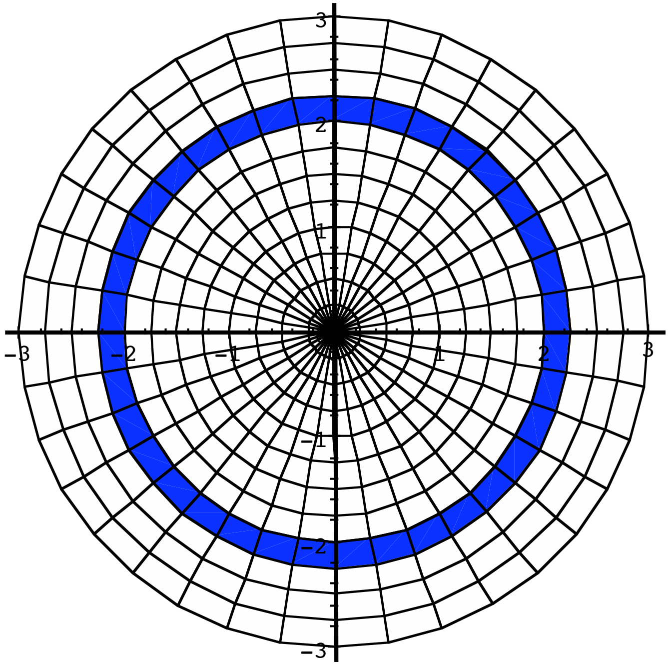
Kreisfläche:

$$\boxed{\int_0^{2\pi}\frac{1}{2}r^2\mathrm{d}\phi = \frac{1}{2}r^2\underbrace{\int_0^{2\pi}\mathrm{d}\phi}_{2\pi} = r^2\pi}$$



Ringfläche (“Straße”)

$$\underset{\substack{\uparrow \\ \text{Länge}}}{2\pi\rho}\cdot\ \underset{\substack{\uparrow \\ \text{Breite}}}{d\rho} = 2\pi\rho\, d\rho$$

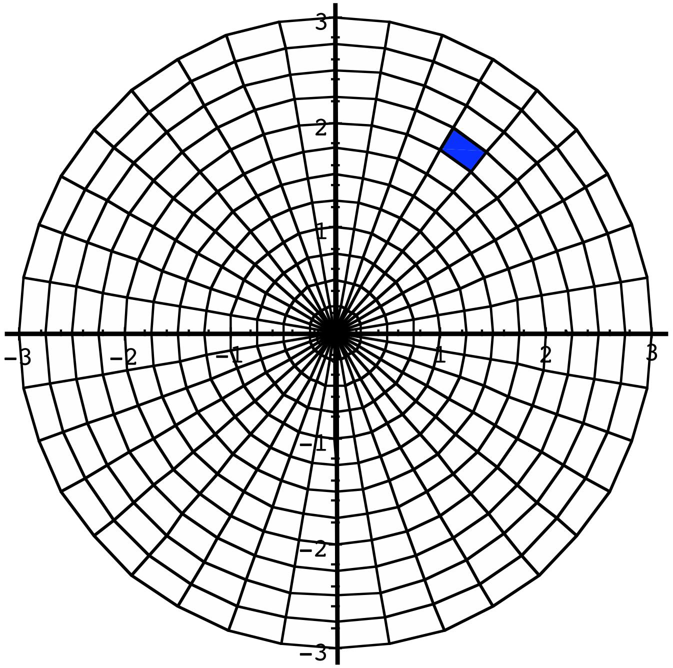


Ringfläche (“Straße”)

$$\underset{\substack{\uparrow \\ \text{Länge}}}{2\pi\rho}\cdot\ \underset{\substack{\uparrow \\ \text{Breite}}}{\mathrm{d}\rho} = 2\pi\rho\,\mathrm{d}\rho$$

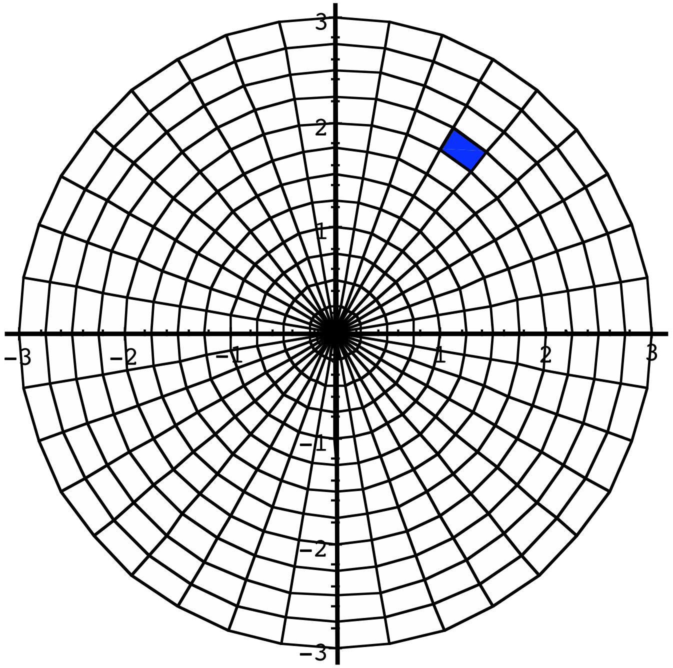
Kreisfläche:

$$\boxed{\begin{aligned} \int_0^r 2\pi\rho\,\mathrm{d}\rho &= 2\pi\int_0^r \rho\,\mathrm{d}\rho \\ &= 2\pi\frac{r^2}{2} = r^2\pi \end{aligned}}$$



“Viereck” (fast Rechteck)

$$\underset{\substack{\uparrow\\\text{Länge}}}{\rho\,\mathrm{d}\phi}\cdot\ \underset{\substack{\uparrow\\\text{Breite}}}{\mathrm{d}\rho}\ = \rho\,\mathrm{d}\phi\,\mathrm{d}\rho$$



“Viereck” (fast Rechteck)

$$\underset{\substack{\uparrow\\ \text{Länge}}}{\rho\, \mathrm{d}\phi}\cdot \underset{\substack{\uparrow\\ \text{Breite}}}{\mathrm{d}\rho} = \rho\, \mathrm{d}\phi\, \mathrm{d}\rho$$

Kreisfläche:

$$\boxed{\begin{aligned} \int_0^r \int_0^{2\pi} \rho\, \mathrm{d}\phi\, \mathrm{d}\rho &= \int_0^r \underbrace{\left[\int_0^{2\pi} \rho\, \mathrm{d}\phi \right]}_{2\pi\rho} \mathrm{d}\rho \\ &= 2\pi \frac{r^2}{2} = r^2 \pi \end{aligned}}$$

```
> Int( Int(rho, phi=0..2*Pi ), rho=0..r)
=  int( int(rho, phi=0..2*Pi ), rho=0..r);
```

$$\int_0^r \int_0^{2\,\pi} \rho \, d\phi \, d\rho = r^2 \, \pi$$

```
> Int( Int(rho, phi=0..2*Pi ), rho=0..r)
=  int( int(rho, phi=0..2*Pi ), rho=0..r);
```

$$\int_{0}^{r}\int_{0}^{2\,\pi}\rho\,d\phi\,d\rho = r^{2}\,\pi$$

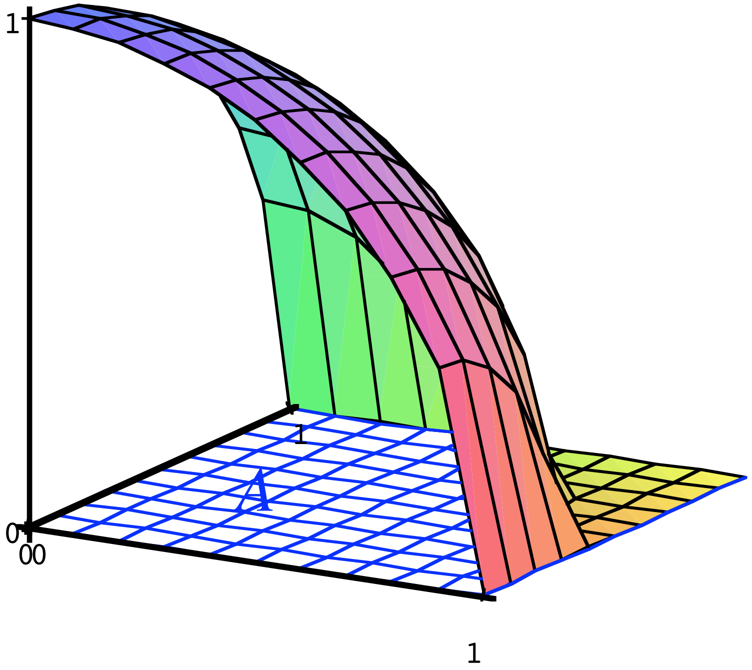
```
> Int( Int(rho, rho=0..r), phi=0..2*Pi )
=  int( int(rho, rho=0..r ), phi=0..2*Pi);
```

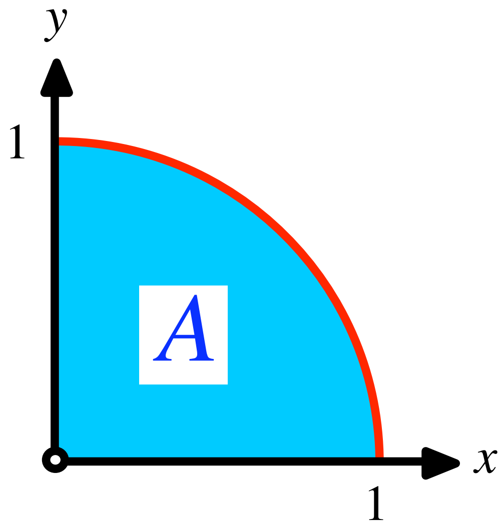
$$\int_{0}^{2\,\pi}\int_{0}^{r}\rho\,d\rho\,d\phi = r^{2}\,\pi$$

Kugelvolumen

$$V_{\text{Kugel}} = \frac{4}{3}\pi r^3$$

$$f(x,y) = \sqrt{1 - x^2 - y^2}$$





Achtelskugel

$$I = \iint\limits_{\substack{A \\ \uparrow \\ \text{Viertels-} \\ \text{kreis}}} \sqrt{1 - x^2 - y^2}\, \mathrm{d}x\, \mathrm{d}y = \int\limits_0^1 \left[\underbrace{\int\limits_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\, \mathrm{d}x}_{I_1} \right] \mathrm{d}y$$

Achtelskugel

$$I = \iint\limits_{\underset{\substack{\text{Viertels-}\\ \text{kreis}}}{\underset{\uparrow}{A}}} \sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\underbrace{\int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x}_{I_1} \right] \mathrm{d}y$$

Formelsammlung

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

Bei uns: $a^2 = 1 - y^2$

$$I_1 = \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x$$

Formelsammlung

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

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Formelsammlung

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

Bei uns: $a^2 = 1-y^2$

$$I_1 = \left(\frac{x}{2}\sqrt{1-y^2-x^2} + \frac{1-y^2}{2}\arcsin\left(\frac{x}{\sqrt{1-y^2}}\right)\right)\Bigg|_{0}^{\sqrt{1-y^2}}$$

$$I_1 = \int\limits_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x$$

Formelsammlung

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \tfrac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\tfrac{x}{a}\right) + C$$

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$$I_1 = \frac{\sqrt{1-y^2}}{2}\underbrace{\sqrt{1-y^2-\left(1-y^2\right)}}_{0} + \frac{1-y^2}{2}\underbrace{\arcsin\left(\frac{\sqrt{1-y^2}}{\sqrt{1-y^2}}\right)}_{\arcsin(1)=\frac{\pi}{2}}$$

$$I_1 = \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x$$

Formelsammlung

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + C$$

Bei uns: $a^2 = 1-y^2$

$$I_1 = \left(\frac{x}{2}\sqrt{1-y^2-x^2} + \frac{1-y^2}{2}\arcsin\left(\frac{x}{\sqrt{1-y^2}}\right)\right)\Bigg|_0^{\sqrt{1-y^2}}$$

$$I_1 = \frac{\sqrt{1-y^2}}{2}\underbrace{\sqrt{1-y^2-\left(1-y^2\right)}}_{0} + \frac{1-y^2}{2}\underbrace{\arcsin\left(\frac{\sqrt{1-y^2}}{\sqrt{1-y^2}}\right)}_{\arcsin(1)=\frac{\pi}{2}} = \frac{\pi}{4}\left(1-y^2\right)$$

Achtelskugel

$$I = \iint\limits_{\substack{A \\ \uparrow \\ \text{Viertels-} \\ \text{kreis}}} \sqrt{1 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\underbrace{\int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\,\mathrm{d}x}_{I_1 = \frac{\pi}{4}\left(1-y^2\right)} \right] \mathrm{d}y$$

Achtelskugel

$$I = \iint\limits_{\substack{A \\ \uparrow \\ \text{Viertels-} \\ \text{kreis}}} \sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_0^1 \left[\underbrace{\int\limits_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x}_{I_1=\frac{\pi}{4}\left(1-y^2\right)} \right] \mathrm{d}y$$

$$= \frac{\pi}{4}\int\limits_0^1 \left(1-y^2\right)\mathrm{d}y =$$

Achtelskugel

$$I = \iint\limits_{\substack{A \\ \uparrow \\ \text{Viertels-} \\ \text{kreis}}} \sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\underbrace{\int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x}_{I_1=\frac{\pi}{4}\left(1-y^2\right)} \right] \mathrm{d}y$$

$$= \frac{\pi}{4}\int_0^1 \left(1-y^2\right)\mathrm{d}y = \frac{\pi}{4}\left(y-\frac{y^3}{3}\right)\Bigg|_0^1$$

Achtelskugel

$$I = \iint\limits_{\substack{A \\ \uparrow \\ \text{Viertels-} \\ \text{kreis}}} \sqrt{1 - x^2 - y^2}\, \mathrm{d}x\, \mathrm{d}y = \int\limits_0^1 \left[\underbrace{\int\limits_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\, \mathrm{d}x}_{I_1 = \frac{\pi}{4}\left(1 - y^2\right)} \right] \mathrm{d}y$$

$$= \frac{\pi}{4} \int\limits_0^1 \left(1 - y^2\right) \mathrm{d}y = \frac{\pi}{4} \left(y - \frac{y^3}{3} \right) \Bigg|_0^1 = \frac{\pi}{4} \underbrace{\left(1 - \frac{1}{3}\right)}_{\frac{2}{3}} = \frac{\pi}{6}$$

Achtelskugel

$$I = \iint\limits_{\substack{A \\ \uparrow \\ \text{Viertels-} \\ \text{kreis}}} \sqrt{1-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \left[\underbrace{\int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\,\mathrm{d}x}_{I_1=\frac{\pi}{4}\left(1-y^2\right)} \right] \mathrm{d}y$$

$$= \frac{\pi}{4}\int_0^1 \left(1-y^2\right)\mathrm{d}y = \frac{\pi}{4}\left(y-\frac{y^3}{3}\right)\Bigg|_0^1 = \frac{\pi}{4}\underbrace{\left(1-\frac{1}{3}\right)}_{\frac{2}{3}} = \frac{\pi}{6}$$

$$V_{\text{Vollkugel}} = 8\cdot\frac{\pi}{6} = \frac{4}{3}\pi$$

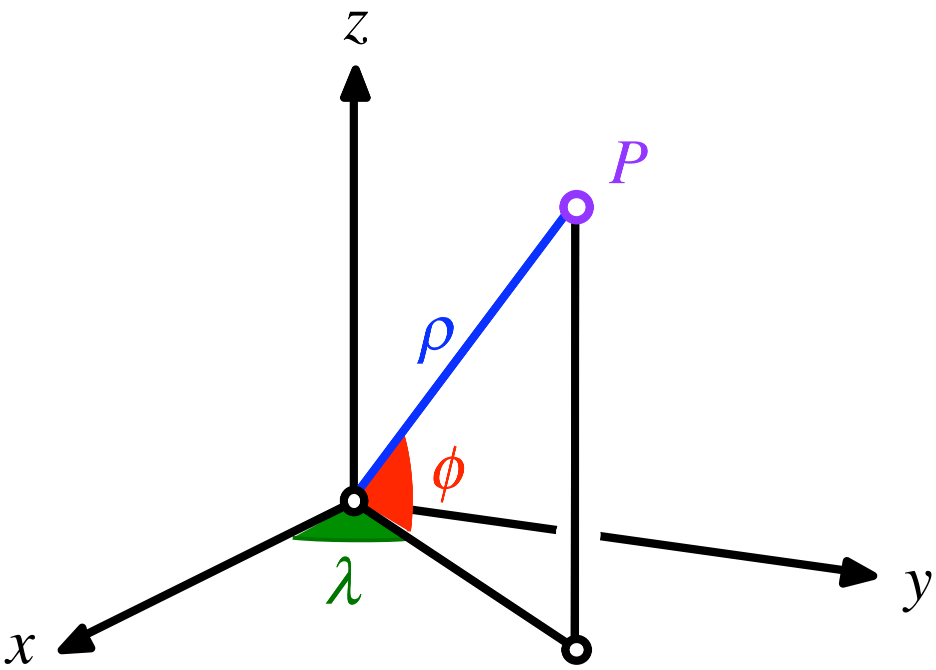
Achtelskugel

```
> Int( Int(sqrt(1-x^2-y^2), x=0..sqrt(1-y^2)), y=0..1)
= int( int(sqrt(1-x^2-y^2), x=0..sqrt(1-y^2)), y=0..1);
```

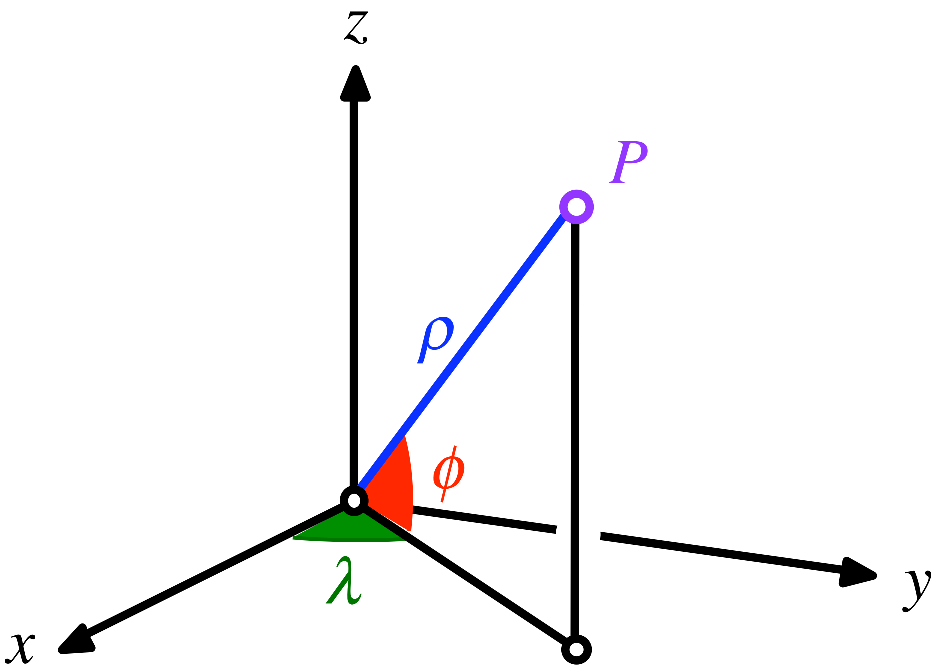
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-x^2-y^2}\, dx\, dy = \frac{1}{6}\pi$$

Nochmals Kugelvolumen

Kugelkoordinaten

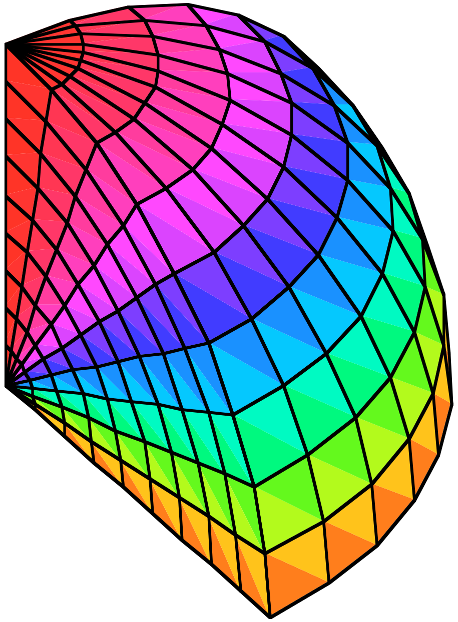


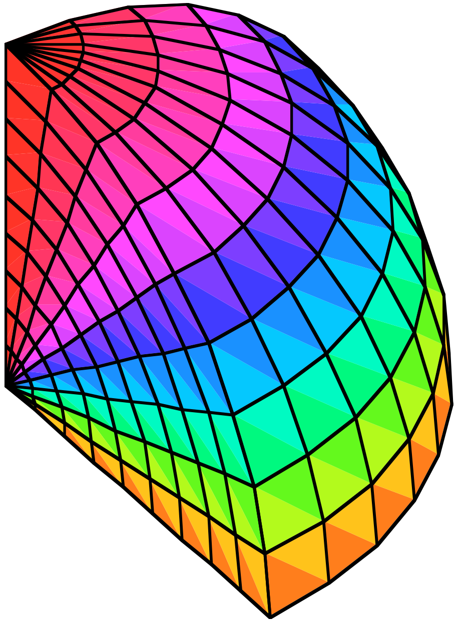
Kugelkoordinaten

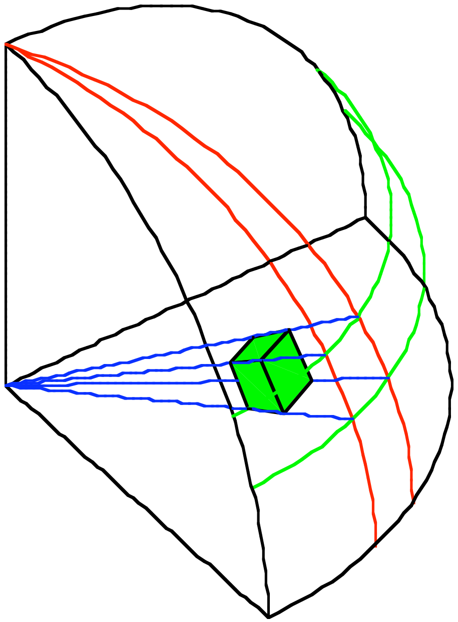


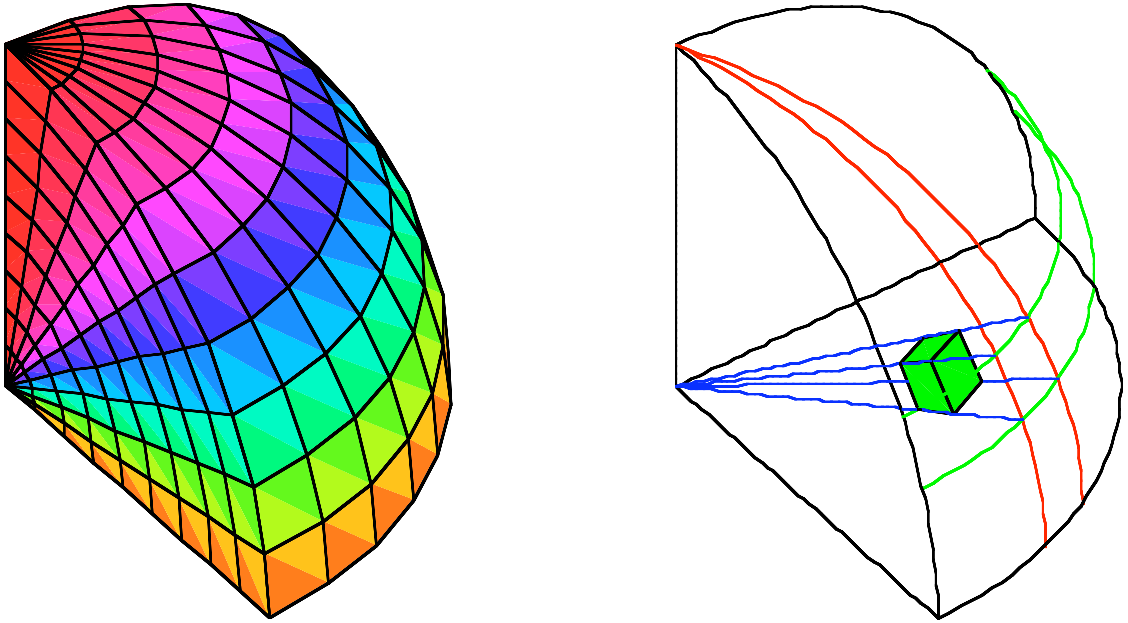
Kugel mit Radius r :

$$A = \left\{ (\rho, \phi, \lambda) \middle| \; \rho \in [0, r], \; \phi \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2} \right], \; \lambda \in [0, 2\pi] \right\}$$









"Quader"-Volumen:

$$\rho\,\mathrm{d}\phi\,\rho\,\mathrm{d}\lambda\cos(\phi)\,\mathrm{d}\rho = \rho^2\cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda$$

"Quader"-Volumen:

$$\rho\,\mathrm{d}\phi\,\rho\,\mathrm{d}\lambda\cos(\phi)\,\mathrm{d}\rho=\rho^2\cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda$$

"Quader"-Volumen:

$$\rho\,\mathrm{d}\phi\,\rho\,\mathrm{d}\lambda\cos(\phi)\,\mathrm{d}\rho = \rho^2\cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda$$

Kugelvolumen:

$$\iiint\limits_A \rho^2\cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda = \int\limits_0^{2\pi}\left[\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int\limits_0^{r}\rho^2\cos(\phi)\,\mathrm{d}\rho\right]\mathrm{d}\phi\right]\mathrm{d}\lambda$$

"Quader"-Volumen:

$$\rho \, \mathrm{d}\phi \, \rho \, \mathrm{d}\lambda \cos(\phi) \, \mathrm{d}\rho = \rho^2 \cos(\phi) \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\lambda$$

Kugelvolumen:

$$\iiint\limits_A \rho^2 \cos(\phi) \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\lambda = \int\limits_0^{2\pi} \left[\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int\limits_0^r \rho^2 \cos(\phi) \, \mathrm{d}\rho \right] \mathrm{d}\phi \right] \mathrm{d}\lambda$$

$$\iiint\limits_A \rho^2 \cos(\phi) \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\lambda = \int\limits_0^{2\pi} \left[\underbrace{\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\underbrace{\int\limits_0^r \rho^2 \cos(\phi) \, \mathrm{d}\rho}_{I_1} \right] \mathrm{d}\phi}_{I_2} \right] \mathrm{d}\lambda$$

$$I_1 = \int_0^r \rho^2 \cos(\phi)\,\mathrm{d}\rho = \frac{r^3}{3}\cos(\phi)$$

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$$\iiint_A \rho^2 \cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda = \int_0^{2\pi} \left[\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\underbrace{\int_0^r \rho^2 \cos(\phi)\,\mathrm{d}\rho}_{I_1 = \frac{r^3}{3}\cos(\phi)} \right] \mathrm{d}\phi}_{I_2} \right] \mathrm{d}\lambda$$

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$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{3}\cos(\phi)\,\mathrm{d}\phi$$

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$$\iiint_A \rho^2 \cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda = \int_0^{2\pi}\left[\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\underbrace{\int_0^r \rho^2 \cos(\phi)\,\mathrm{d}\rho}_{I_1=\frac{r^3}{3}\cos(\phi)}\right]\mathrm{d}\phi}_{I_2}\right]\mathrm{d}\lambda$$

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r^3}{3}\cos(\phi)\,\mathrm{d}\phi = \frac{r^3}{3}\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\phi)\,\mathrm{d}\phi}_{2} = \frac{2}{3}r^3$$

$$I_2 = \frac{2}{3} r^3$$

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$$\iiint\limits_A \rho^2 \cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda = \int_0^{2\pi} \left[\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\underbrace{\int_0^r \rho^2 \cos(\phi)\,\mathrm{d}\rho}_{I_1 = \frac{r^3}{3}\cos(\phi)} \right] \mathrm{d}\phi}_{I_2 = \frac{2}{3} r^3} \right] \mathrm{d}\lambda$$

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$$\iiint\limits_A \rho^2 \cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda = \int_0^{2\pi} \left[\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\underbrace{\int_0^r \rho^2 \cos(\phi)\,\mathrm{d}\rho}_{I_1 = \frac{r^3}{3}\cos(\phi)} \right] \mathrm{d}\phi}_{I_2 = \frac{2}{3} r^3} \right] \mathrm{d}\lambda$$

$$\iiint\limits_A \rho^2 \cos(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\lambda = \int_0^{2\pi} \frac{2}{3} r^3\,\mathrm{d}\lambda = \frac{2}{3} r^3 \underbrace{\int_0^{2\pi} \mathrm{d}\lambda}_{2\pi} = \frac{4}{3}\pi r^3$$

```
> Int(Int( Int(rho^2*cos(phi), rho=0..r),phi=-
Pi/2..Pi/2), lambda=0..2*Pi)
= int(int( int(rho^2*cos(phi), rho=0..r),phi=-
Pi/2..Pi/2), lambda=0..2*Pi);
```

$$\int_{0}^{2\pi}\int_{-1/2\pi}^{1/2\pi}\int_{0}^{r}\rho^{2}\cos(\phi)\,d\rho\,d\phi\,d\lambda=\frac{4}{3}r^{3}\pi$$