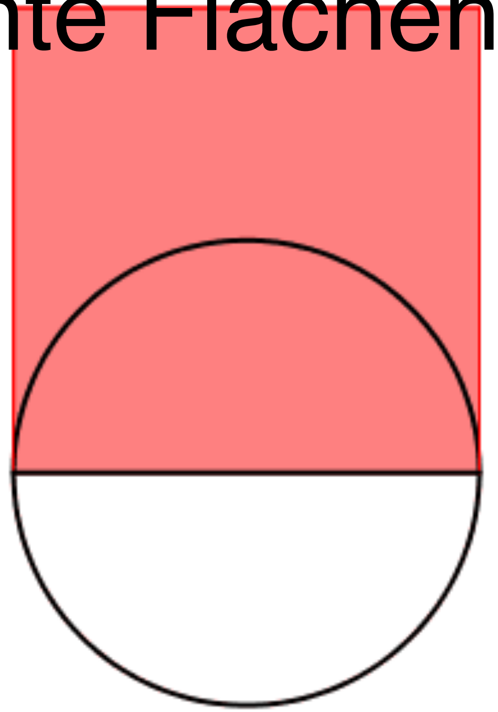
Hans Walser

Invariante Flächensummen

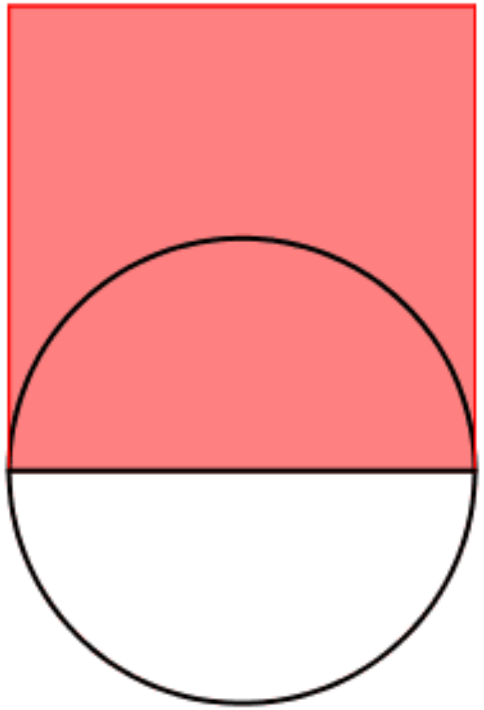


Quadratfläche $a^2 = 0$

Quadratfläche $b^2 = 4$

Flächensumme = 4

www.walser-h-m.ch/hans/Vortraege/20221126



Quadratfläche $a^2 = 0$
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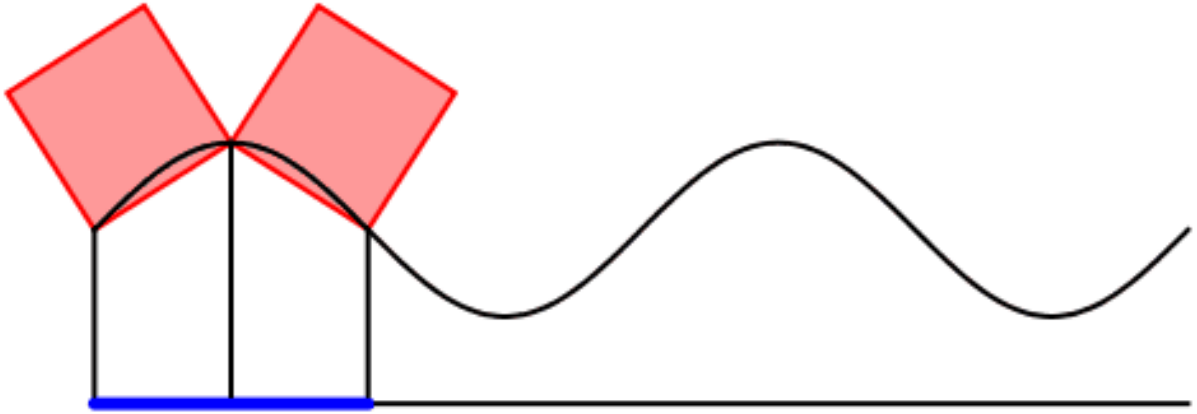
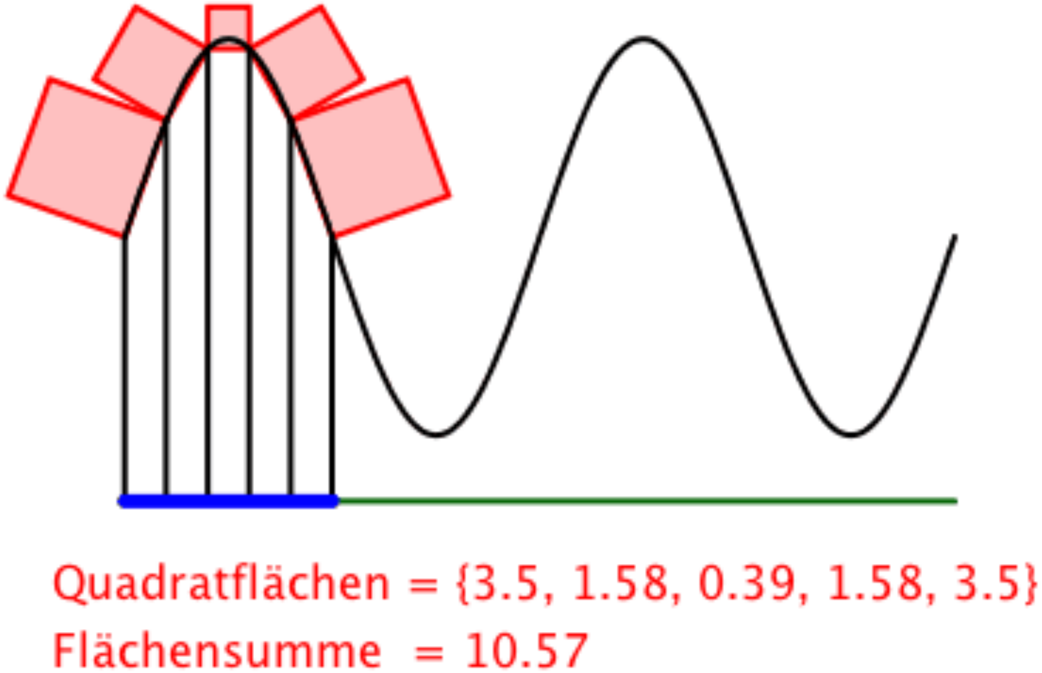
Invariante Flächensumme



www.walser-h-m.ch/hans/Vortraege/20221126

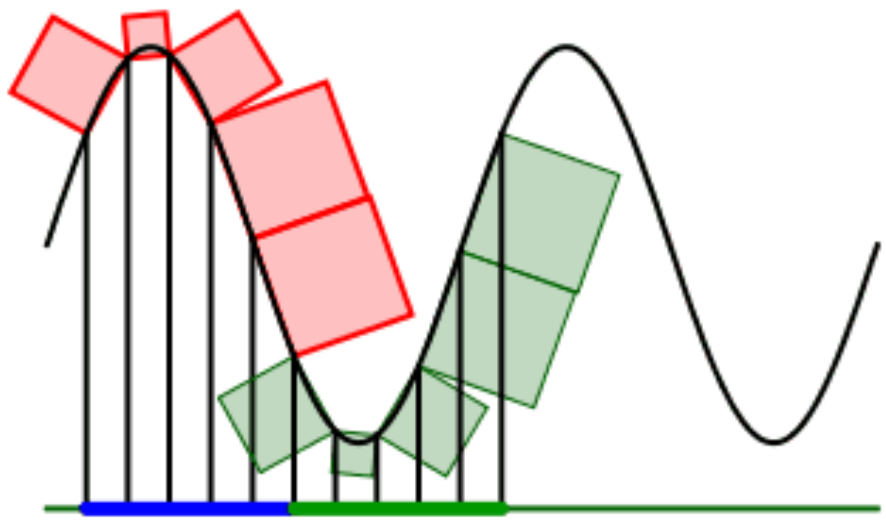
Als ich das erste Mal auf dem Dampfwagen saß



www.walser-h-m.ch/hans/Vortraege/20221126

Als ich das erste Mal auf dem Dampfwagen saß

Ganze Periodenlänge. Schubspiegelsymmetrie

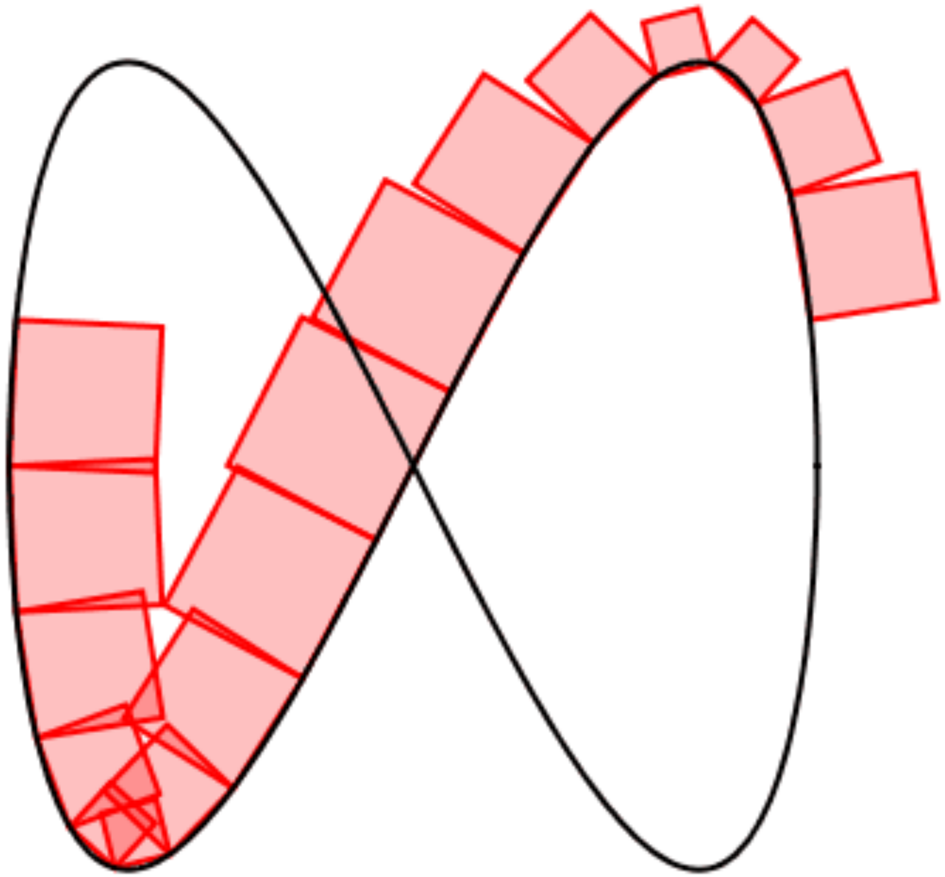


Quadratflächen = {1.69, 0.4, 1.48, 3.44, 3.56}

Flächensumme = 10.57

www.walser-h-m.ch/hans/Vortraege/20221126

Achterbahn (Lissajous-Kurve)

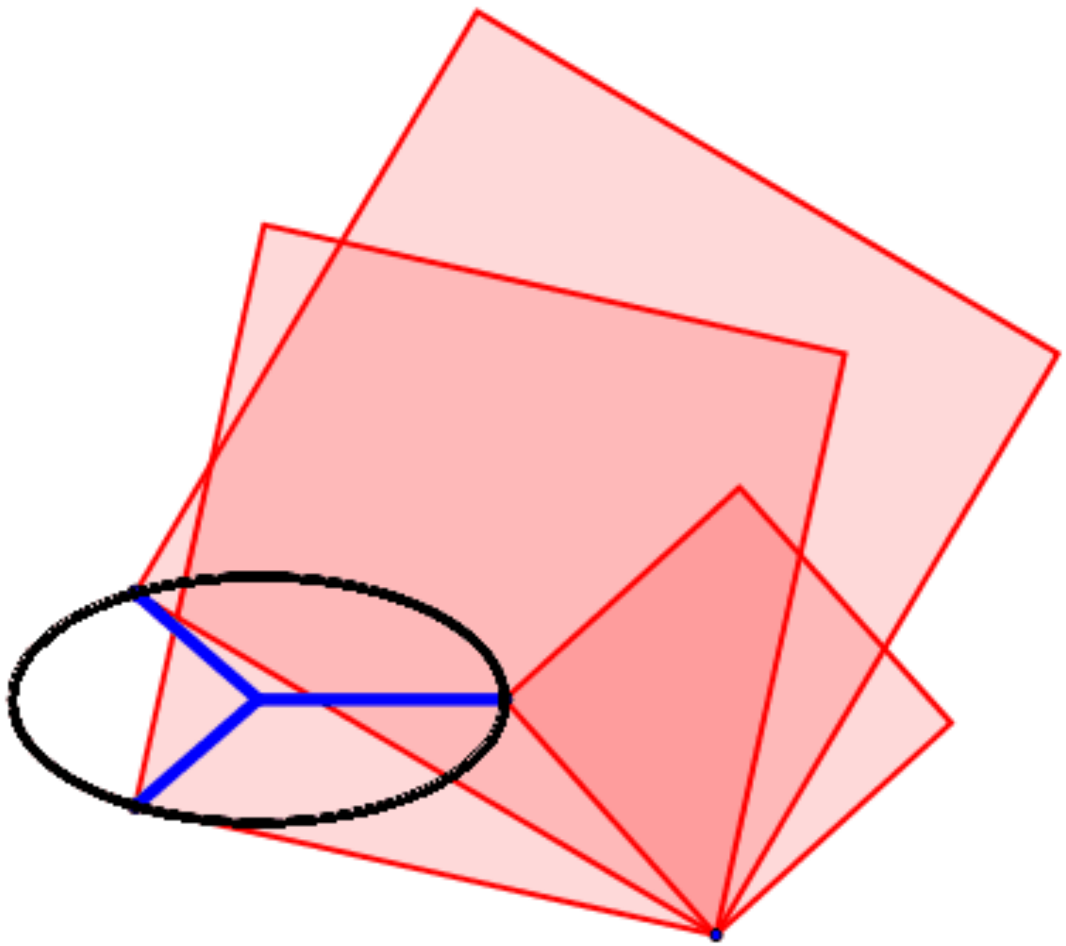


Quadratflächen = {0.1, 0.06, 0.02, 0.02, 0.05, 0.1, 0.15, 0.17, 0.15, 0.1, 0.05, 0.02, 0.02, 0.06, 0.1, 0.13, 0.13}

Flächensumme = 1.44

www.walser-h-m.ch/hans/Vortraege/20221126

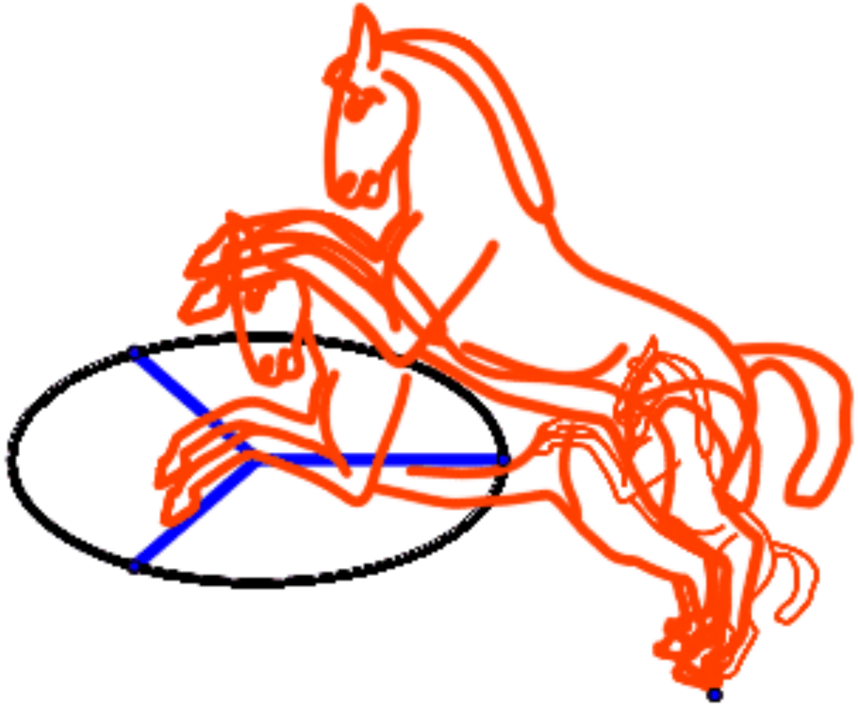
Externer Pivot



Quadratflächen = {23.39, 6.62, 30.01}

Flächensumme = 60.02

Externer Pivot
Тройка



Beweise?

Schlüsselformeln

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) = 0$$

$$\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = 0$$

$$\sum_{k=1}^{n} \cos^2\left(t + k\frac{2\pi}{n}\right) = \frac{n}{2}$$

$$\sum_{k=1}^{n} \sin^2\left(t + k\frac{2\pi}{n}\right) = \frac{n}{2}$$

Beweise?

Schlüsselformeln

$$\sum_{k=1}^{n} \cos\left(t + k\,\frac{2\pi}{n}\right) = 0$$

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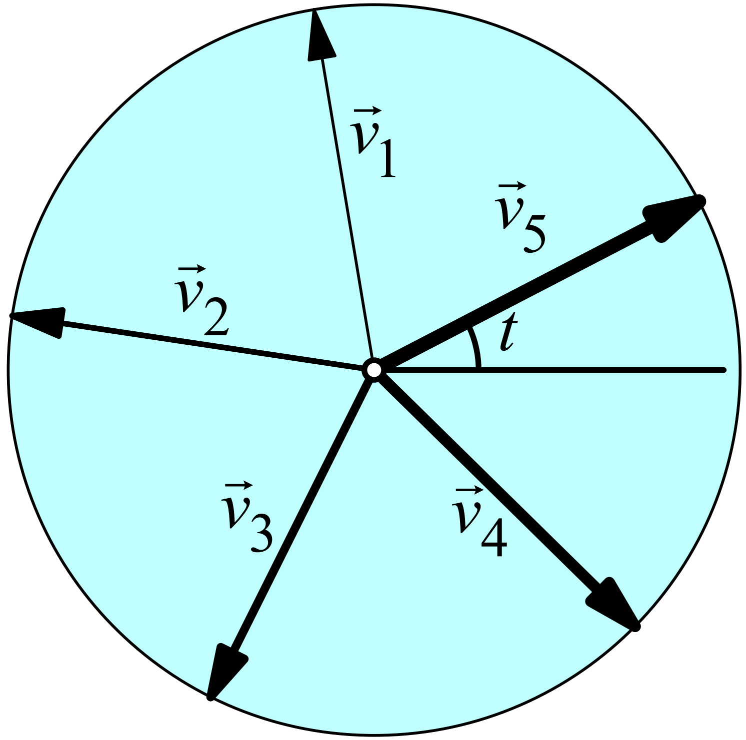
Beweis 1: regelmäßiges n-Eck

Schlüsselformeln

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) = 0$$

$$\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = 0$$

$$\vec{v}_k = \begin{bmatrix} \cos\left(t + k\frac{2\pi}{n}\right) \\ \sin\left(t + k\frac{2\pi}{n}\right) \end{bmatrix}$$



Beweis 1: regelmäßiges n-Eck

Schlüsselformeln

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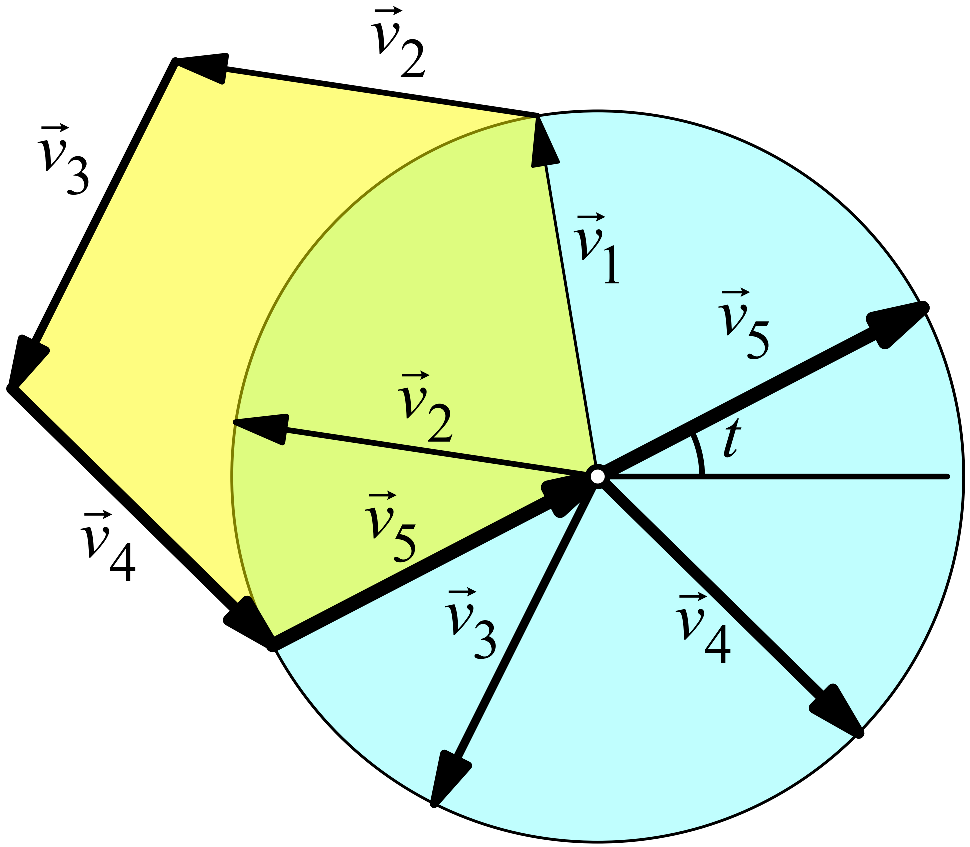


Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right)$$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)}$$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)} = \mathrm{e}^{\mathrm{i}t} \underbrace{\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}}_{?}$$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)} = \mathrm{e}^{\mathrm{i}t} \underbrace{\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}}_{?}$$

$$\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}} = \sum_{k=1}^{n} \left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{k}$$

Geometrische Reihe: $q = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n}\cos\left(t+k\frac{2\pi}{n}\right)+\mathrm{i}\sum_{k=1}^{n}\sin\left(t+k\frac{2\pi}{n}\right)=\sum_{k=1}^{n}\mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)}=\mathrm{e}^{\mathrm{i}t}\underbrace{\sum_{k=1}^{n}\mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}}_{?}$$

$$\sum_{k=1}^{n}\mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}=\sum_{k=1}^{n}\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{k}=\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\frac{1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n}}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}}$$

Geometrische Reihe: $q=\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)} = \mathrm{e}^{\mathrm{i}t} \underbrace{\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}}_{?}$$

$$\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}} = \sum_{k=1}^{n}\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{k} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}} \frac{1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n}}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}}$$

Zähler: $1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n}$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)} = \mathrm{e}^{\mathrm{i}t} \underbrace{\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}}_{?}$$

$$\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}} = \sum_{k=1}^{n} \left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{k} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}} \frac{1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n}}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}}$$

Zähler: $1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n} = 1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}n}$

Beweis 2: komplexe Zahlen

$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) + \mathrm{i}\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}\left(t+k\frac{2\pi}{n}\right)} = \mathrm{e}^{\mathrm{i}t} \underbrace{\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}}}_{?}$$

$$\sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}k\frac{2\pi}{n}} = \sum_{k=1}^{n} \left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{k} = \mathrm{e}^{\mathrm{i}\frac{2\pi}{n}} \frac{1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n}}{1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}}$$

Zähler: $$1-\left(\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}}\right)^{n} = 1-\mathrm{e}^{\mathrm{i}\frac{2\pi}{n}n} = 1-\mathrm{e}^{2\pi\mathrm{i}} = 1-1 = 0$$

Eulersche Formel

Beweise?

Schlüsselformeln

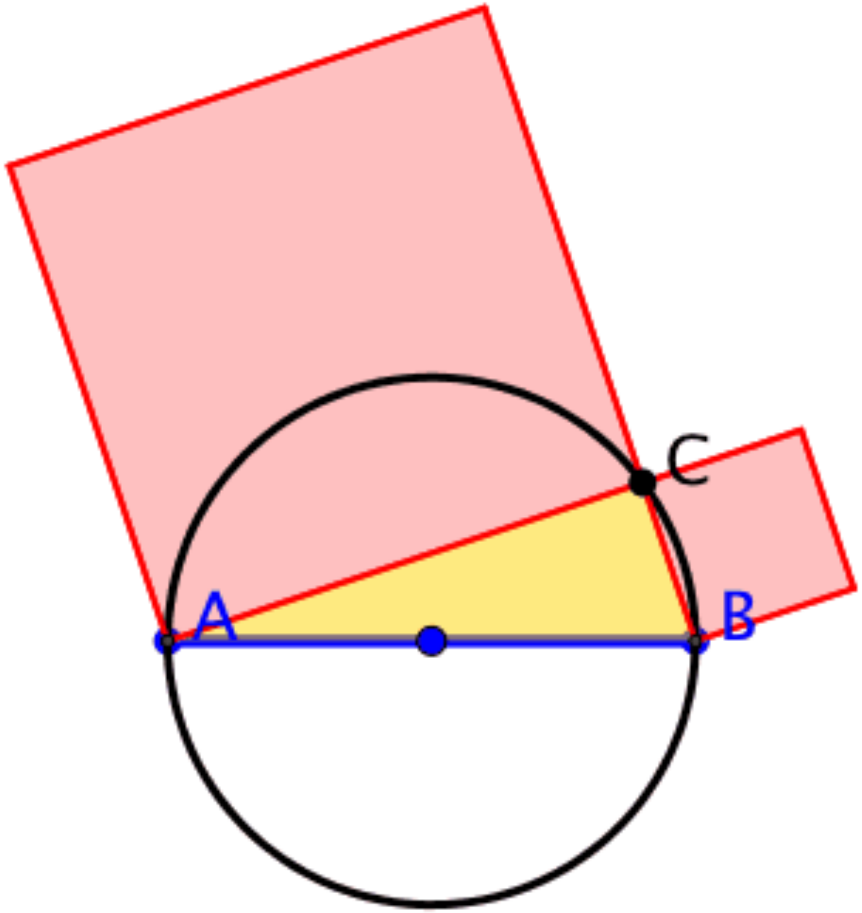
$$\sum_{k=1}^{n} \cos\left(t + k\frac{2\pi}{n}\right) = 0$$

$$\sum_{k=1}^{n} \sin\left(t + k\frac{2\pi}{n}\right) = 0$$

$$\Leftrightarrow \qquad e^{2\pi i} = 1$$

Eulersche Formel

Pythagoras

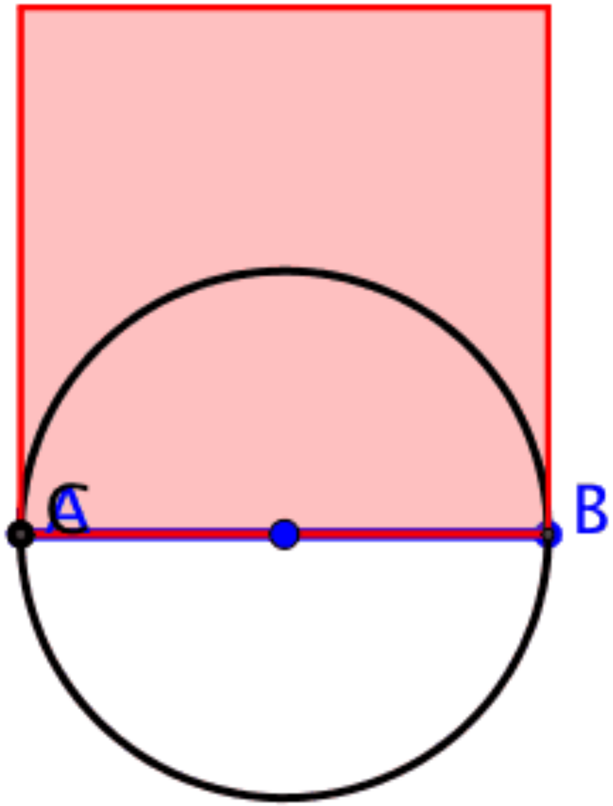


Quadratflächen = {0.4, 3.6}

Flächensumme = 4

Pythagoras

Morgen



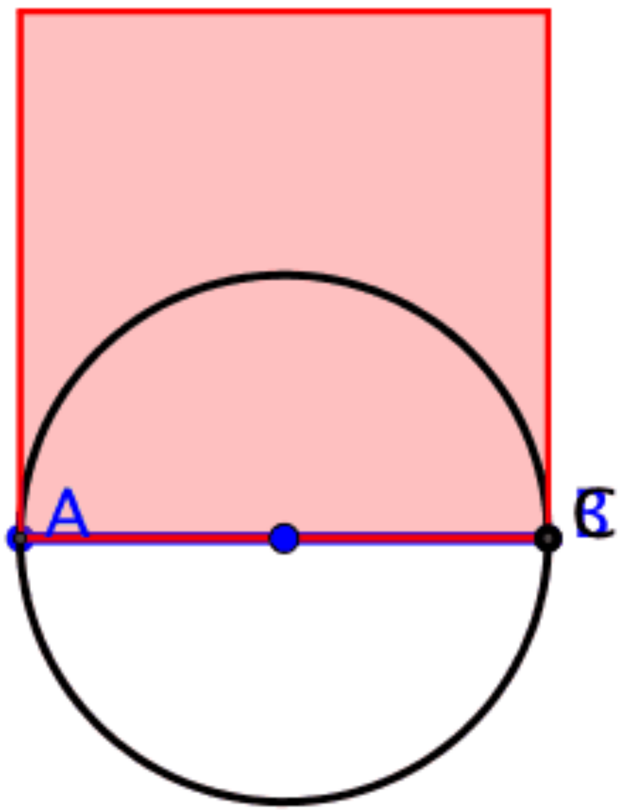
Abend

Quadratflächen = {4, 0}

Flächensumme = 4

Pythagoras bei Nacht

Morgen

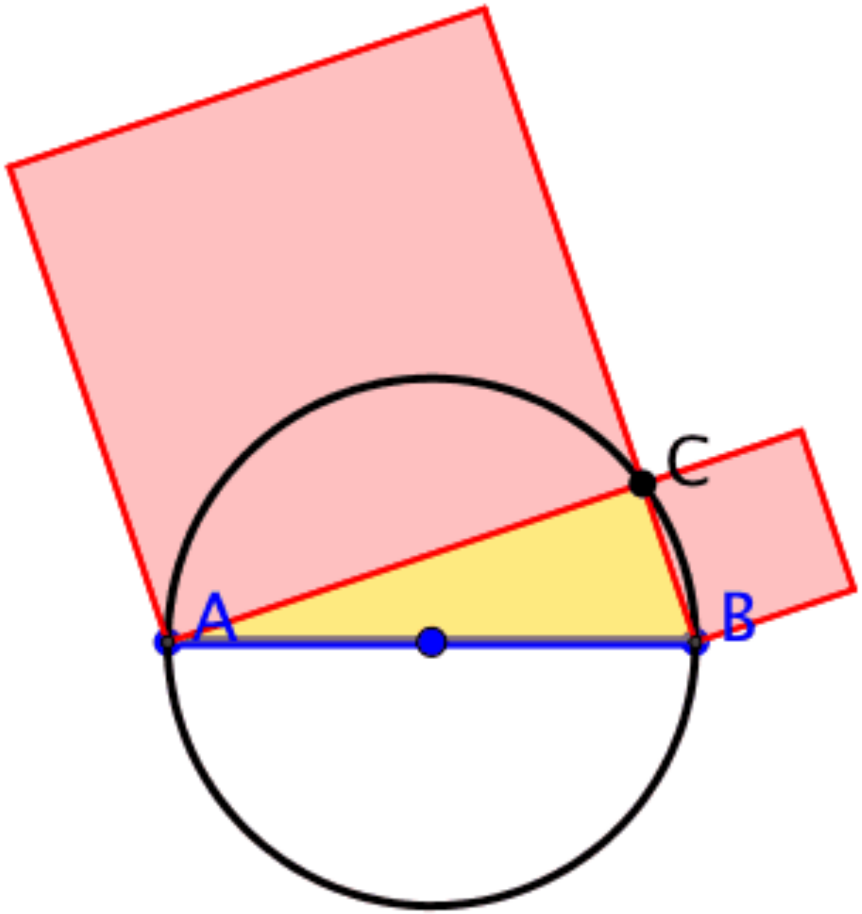


Abend

Quadratflächen = {0, 4}

Flächensumme = 4

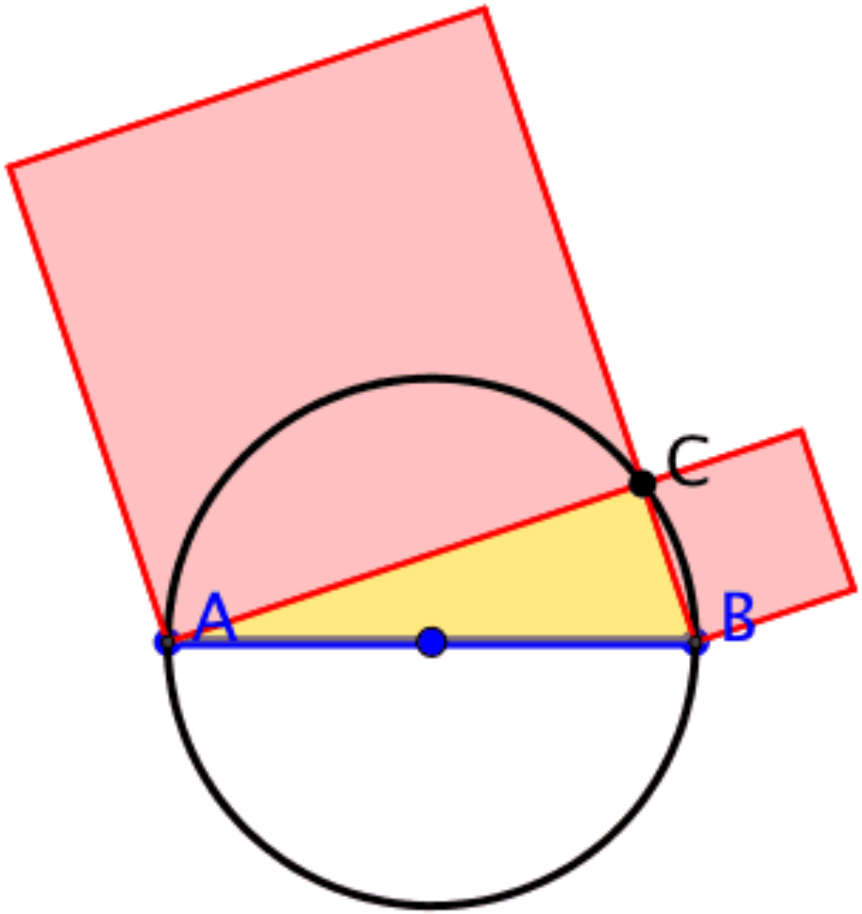
Kopernikus



Quadratflächen = {0.4, 3.6}

Flächensumme = 4

Kopernikus



Quadratflächen = {0.4, 3.6}

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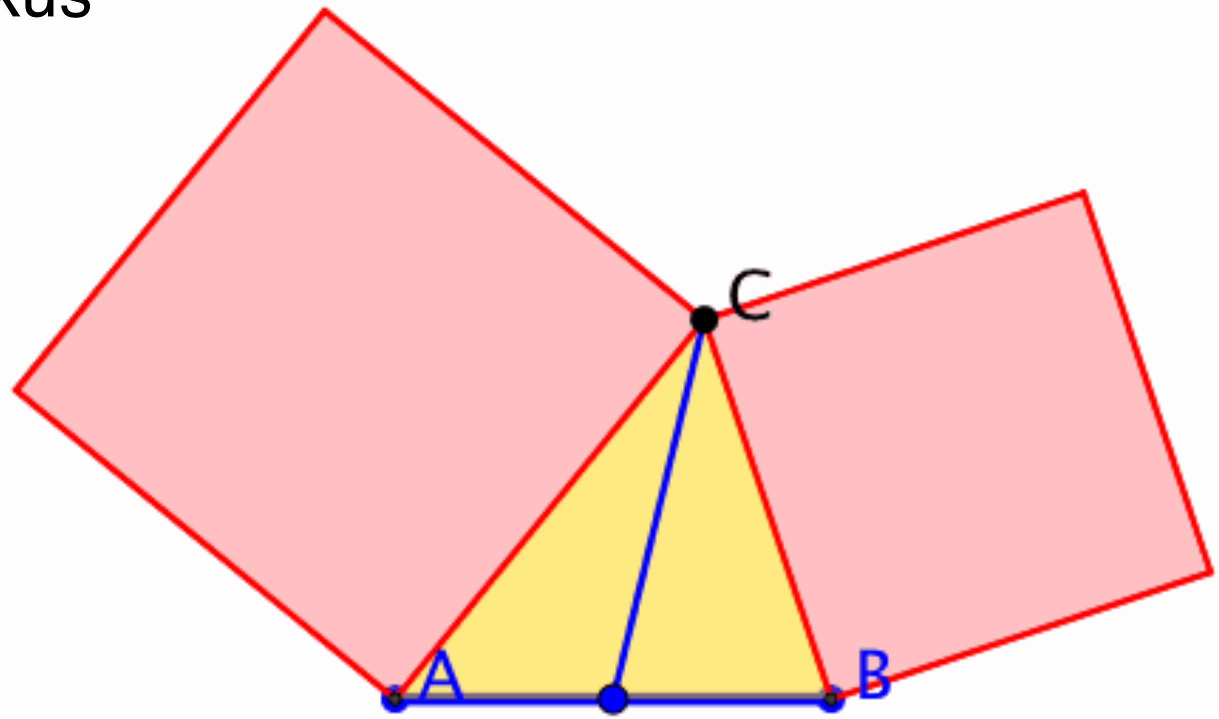
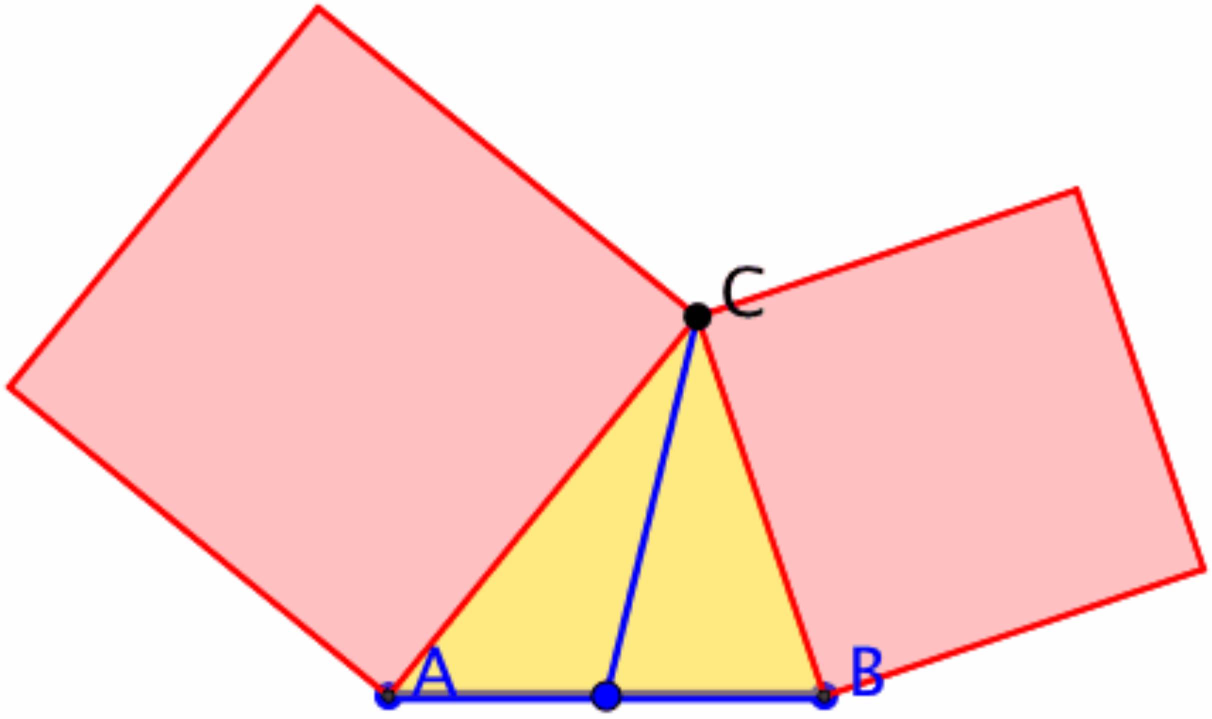
Kopernikus



Quadratflächen = {3.37, 5.04}
Flächensumme = 8.41

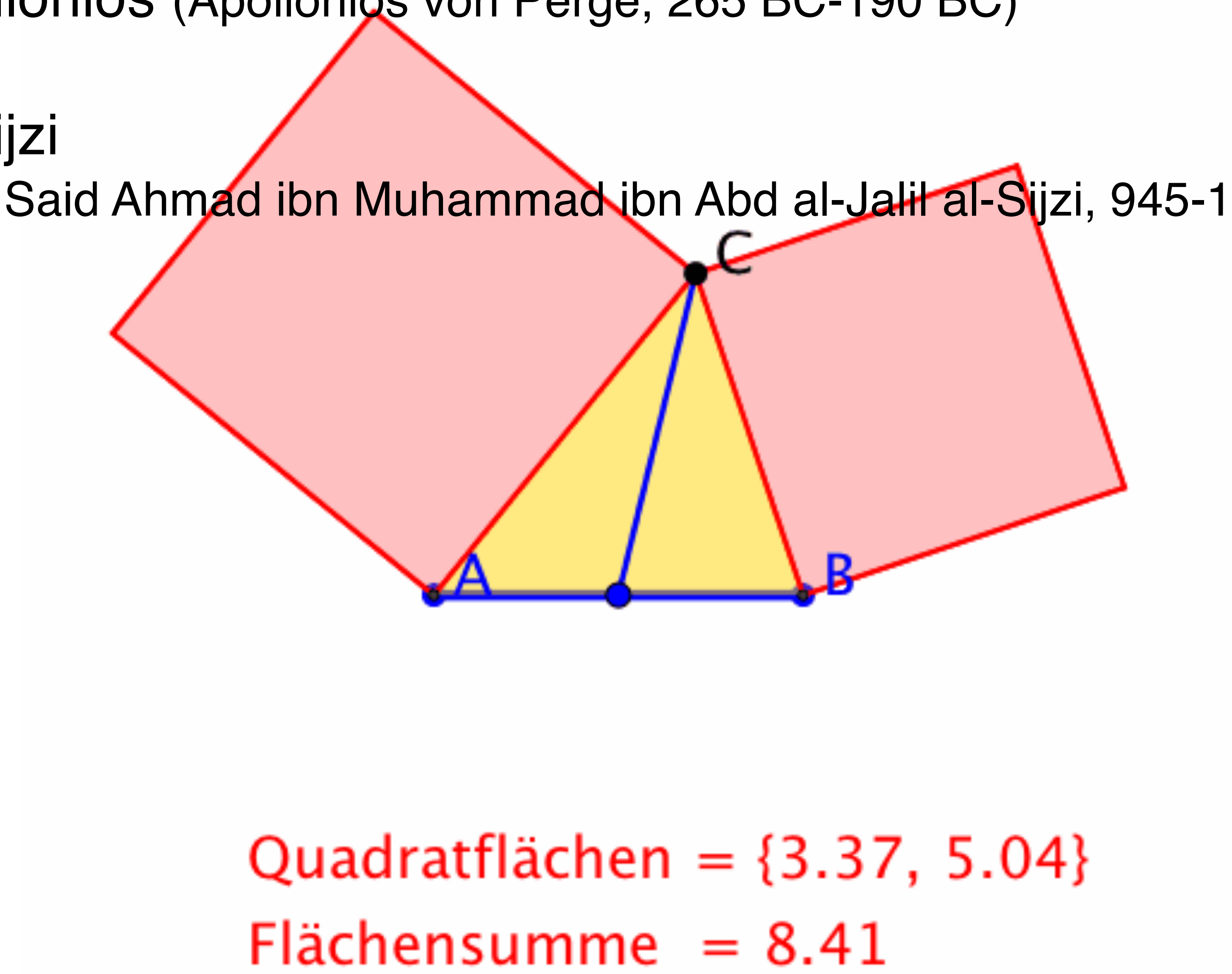


Quadratflächen = {3.37, 5.04}

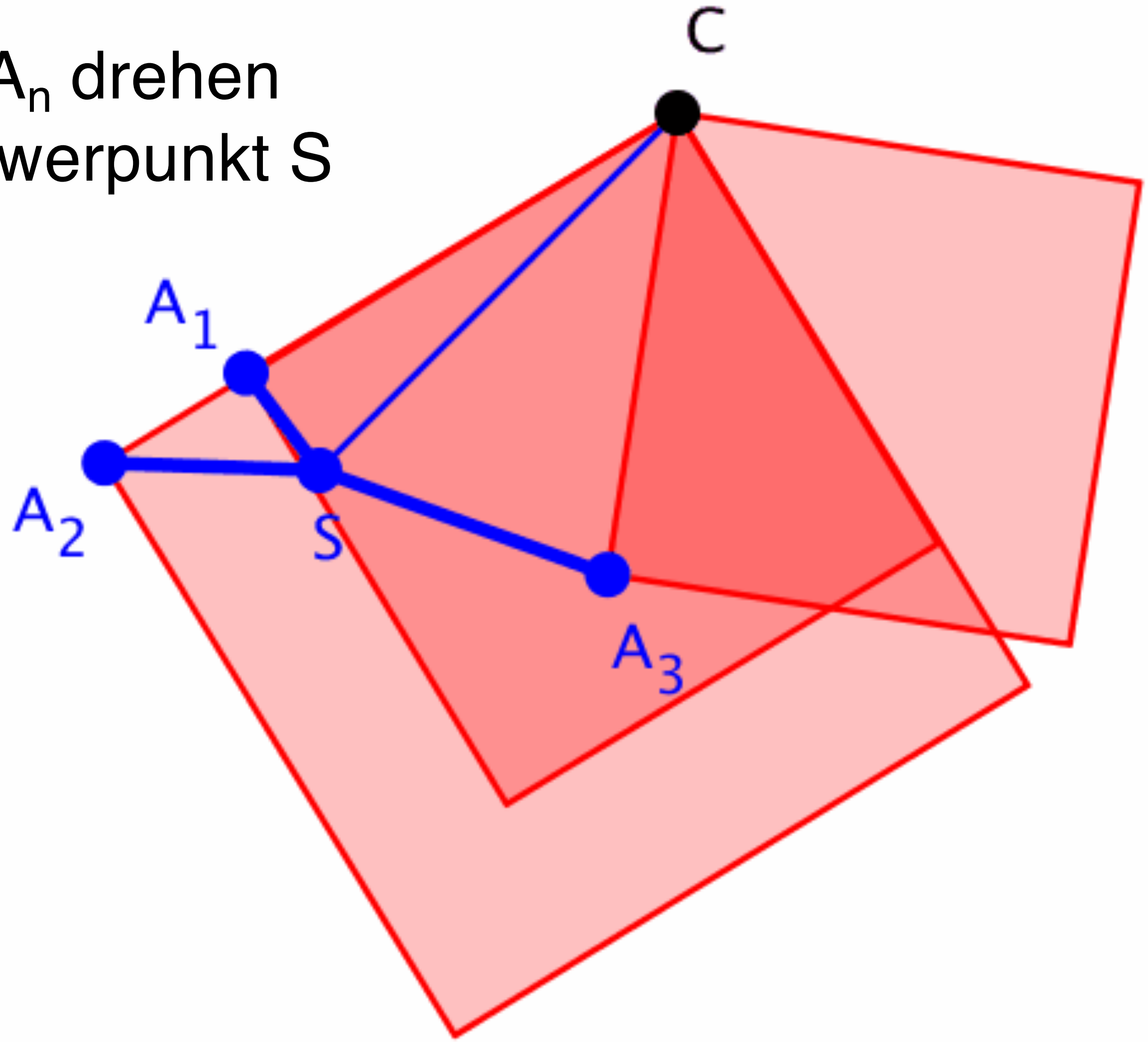
Flächensumme = 8.41

Apollonios (Apollonios von Perge, 265 BC-190 BC)

al-Sijzi
(Abu Said Ahmad ibn Muhammad ibn Abd al-Jalil al-Sijzi, 945-1020)



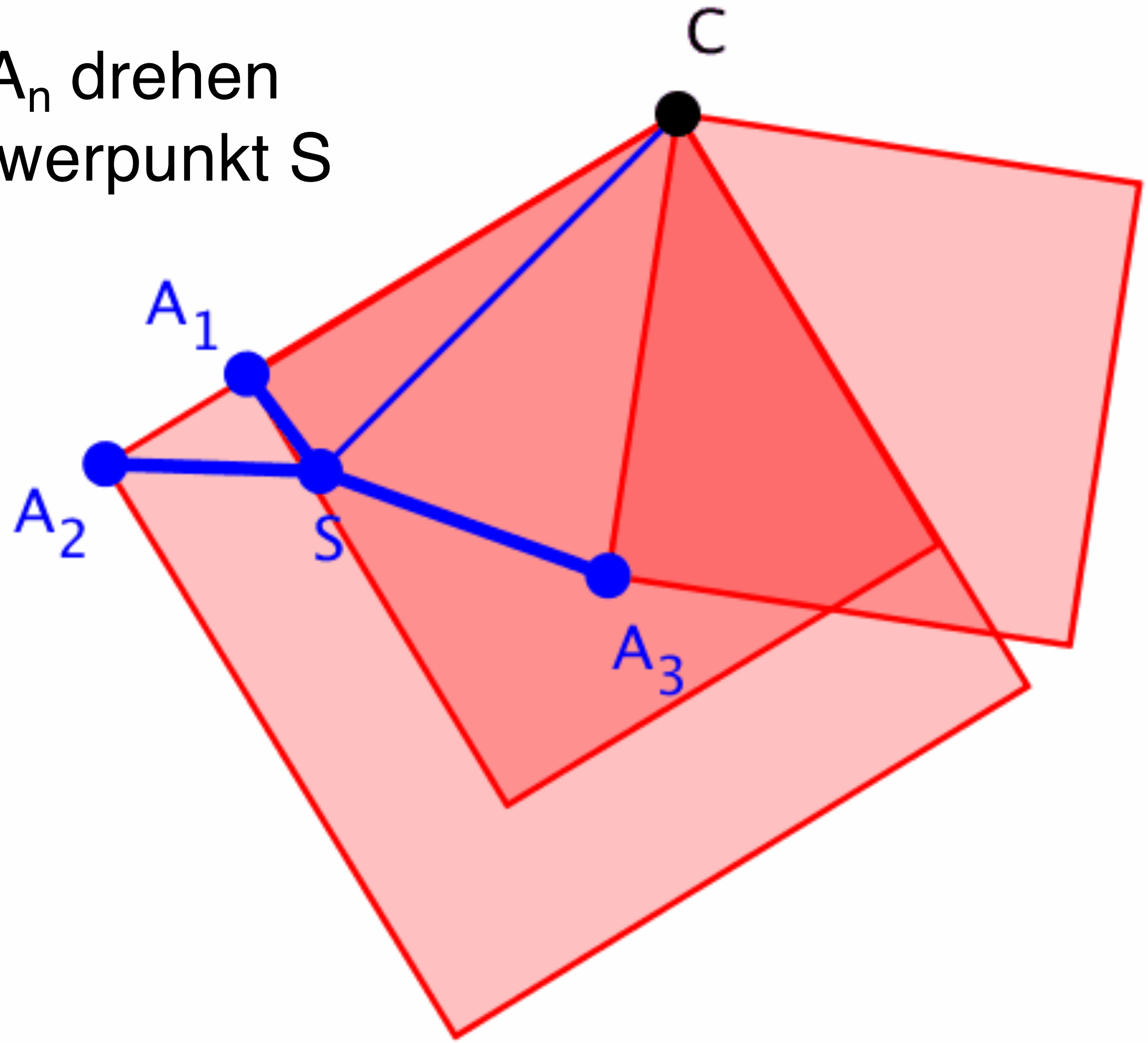
$A_1, \ldots, A_n$ drehen um Schwerpunkt S



Quadratflächen = {522.64, 253.12, 294.23}

Flächensumme = 1069.99

$A_1, \dots, A_n$ drehen um Schwerpunkt S



Quadratflächen = {522.64, 253.12, 294.23}

Flächensumme = 1069.99

$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

Summe der Quadrate der Abstände von C zu den Punkten $A_1, \ldots, A_n$ invariant.

$A_1, \dots, A_n$ drehen um Schwerpunkt S

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Summe der Quadrate der Abstände von C zu den Punkten $A_1, \dots, A_n$ invariant.

Beweis:

Ursprung in den Schwerpunkt S

$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

Summe der Quadrate der Abstände von C zu den Punkten $A_1, \ldots, A_n$ invariant.

Beweis:

Ursprung in den Schwerpunkt S

$$A_k\left(x_k, y_k\right), \quad k = 1, \ldots, n \qquad \sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} y_k = 0$$

$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

Summe der Quadrate der Abstände von C zu den Punkten $A_1, \ldots, A_n$ invariant.

Beweis:

Ursprung in den Schwerpunkt S

$$A_k\left(x_k, y_k\right), \quad k = 1, \ldots, n \qquad \sum_{k=1}^{n} x_k = 0, \quad \sum_{k=1}^{n} y_k = 0$$

$$C\left(x_C, y_C\right)$$

$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

$$\sum_{k=1}^{n} d\left(C, A_k\right)^2 = \sum_{k=1}^{n}\left(\left(x_k - x_C\right)^2 + \left(y_k - y_C\right)^2\right)$$

$A_1, \dots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

$$\sum_{k=1}^{n} d\left(C, A_k\right)^2 = \sum_{k=1}^{n}\left(\left(x_k - x_C\right)^2 + \left(y_k - y_C\right)^2\right)$$

$$= \sum_{k=1}^{n}\left(x_k^2 + y_k^2\right) - 2x_C \sum_{k=1}^{n} x_k - 2y_C \sum_{k=1}^{n} y_k + n\left(x_C^2 + y_C^2\right)$$

$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

$$\sum_{k=1}^{n} d\left(C, A_k\right)^2 = \sum_{k=1}^{n}\left(\left(x_k - x_C\right)^2 + \left(y_k - y_C\right)^2\right)$$

$$= \sum_{k=1}^{n}\left(x_k^2 + y_k^2\right) - 2x_C \underbrace{\sum_{k=1}^{n} x_k}_{=0} - 2y_C \underbrace{\sum_{k=1}^{n} y_k}_{=0} + n\left(x_C^2 + y_C^2\right)$$

$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt

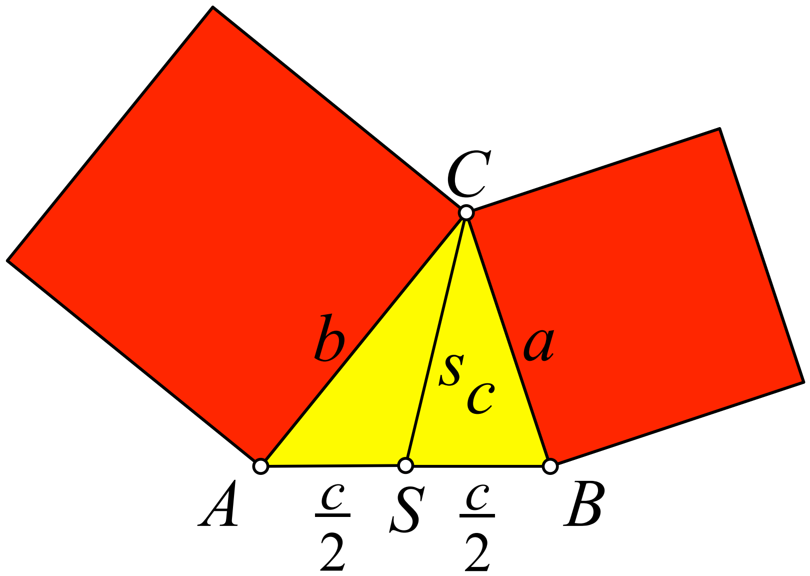
$$\sum_{k=1}^{n} d(C, A_k)^2 = \sum_{k=1}^{n} \left((x_k - x_C)^2 + (y_k - y_C)^2 \right)$$

$$= \sum_{k=1}^{n} \left(x_k^2 + y_k^2 \right) - 2x_C \underbrace{\sum_{k=1}^{n} x_k}_{=0} - 2y_C \underbrace{\sum_{k=1}^{n} y_k}_{=0} + n\left(x_C^2 + y_C^2 \right)$$

$$= \underbrace{\sum_{k=1}^{n} d(S, A_k)^2}_{\text{konstant}} + n \underbrace{d(SC)^2}_{\text{konstant}}$$

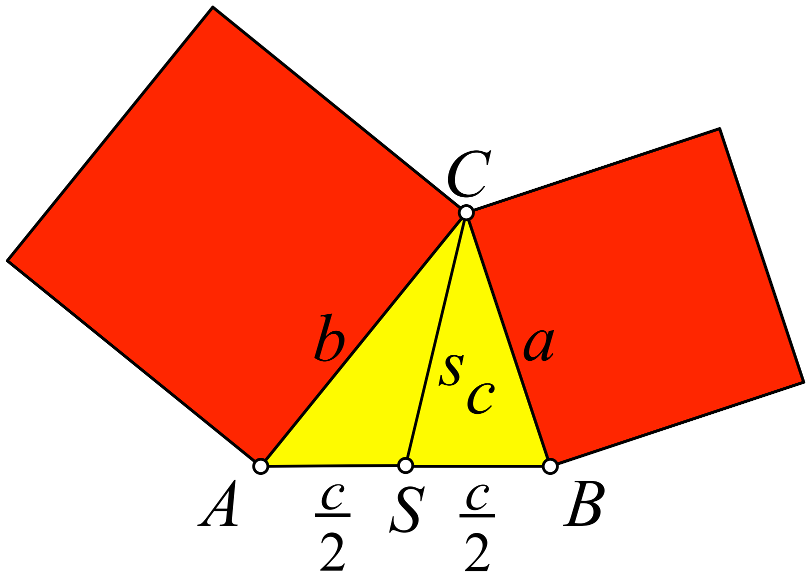
$A_1, \ldots, A_n$ drehen um Schwerpunkt S

C ein externer Punkt



$A_1, \ldots, A_n$ drehen um Schwerpunkt S

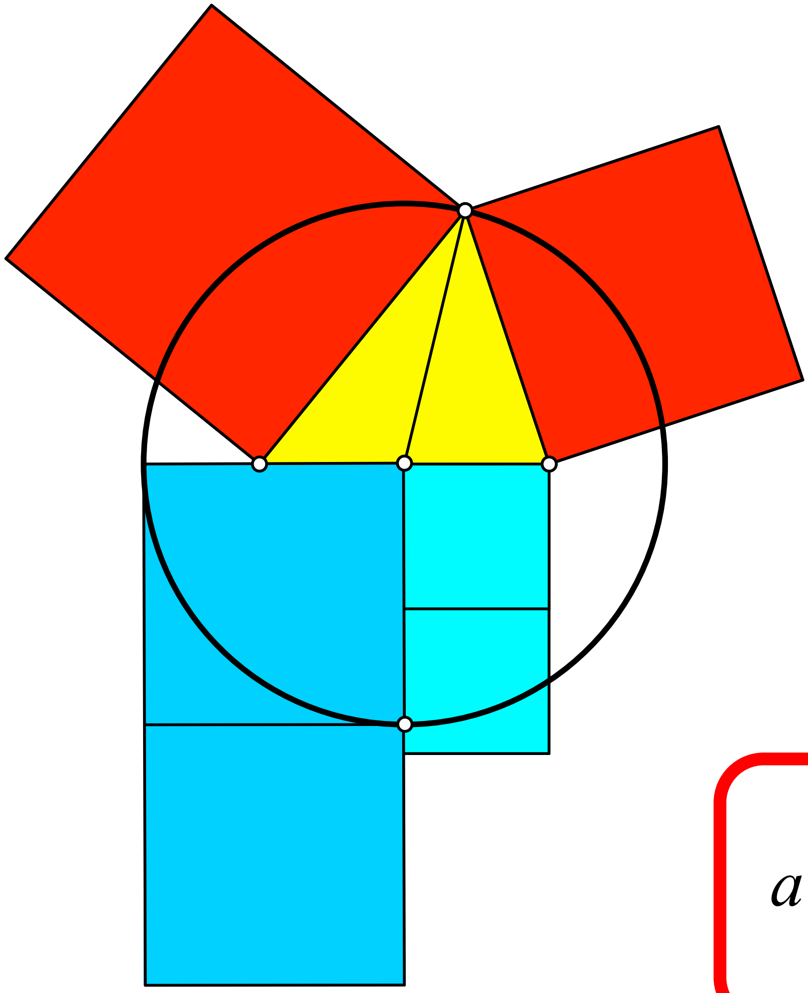
C ein externer Punkt



$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

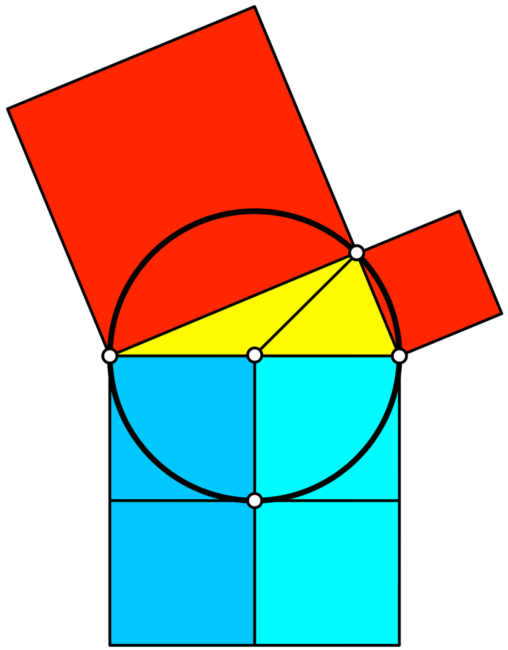
Apollonios

al-Sijzi



$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

Sonderfall Pythagoras



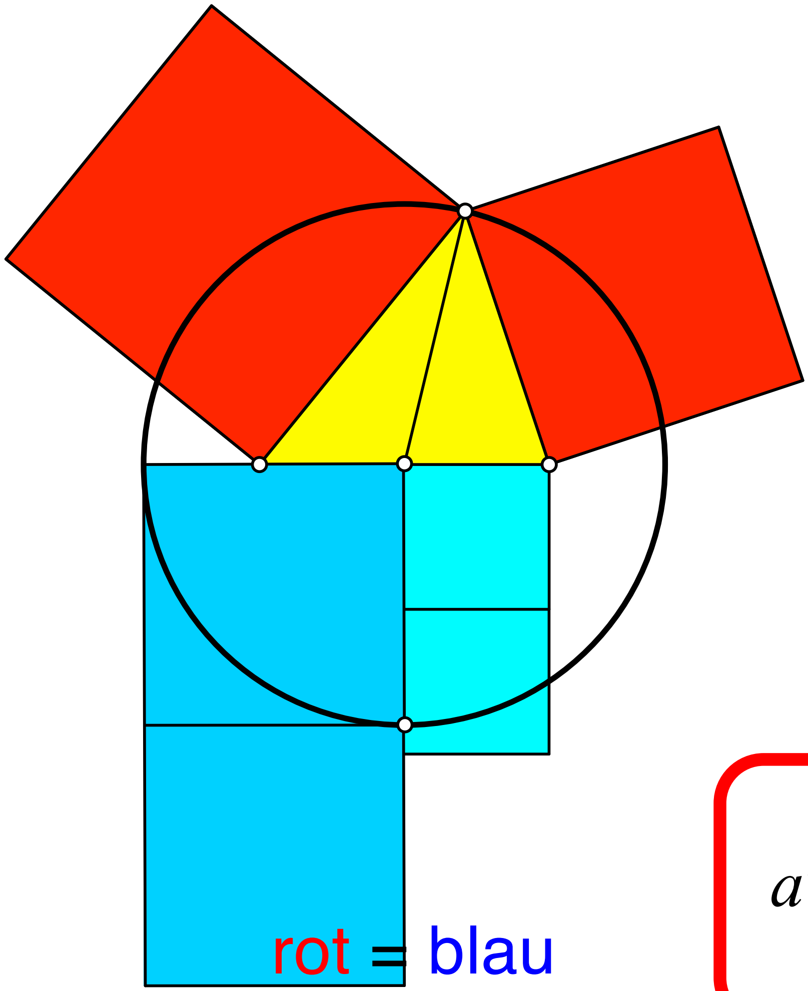
rot = blau

Sonderfall

$$s_c = \frac{1}{2}c$$

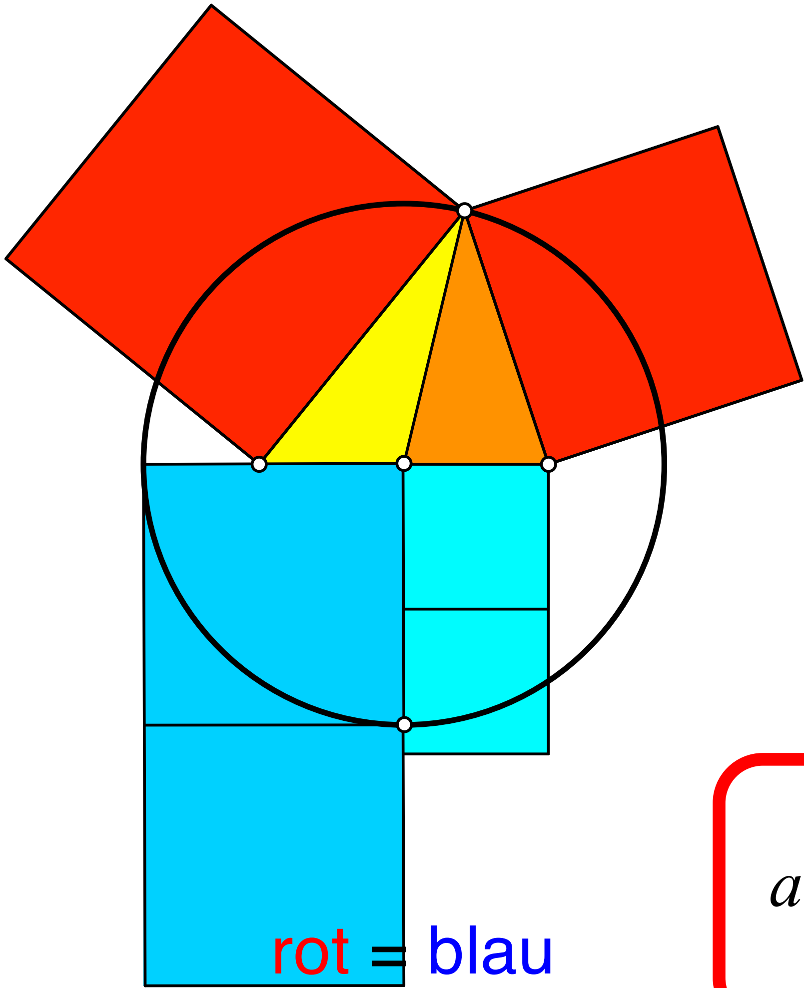
$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

Apollonios / al-Sijzi



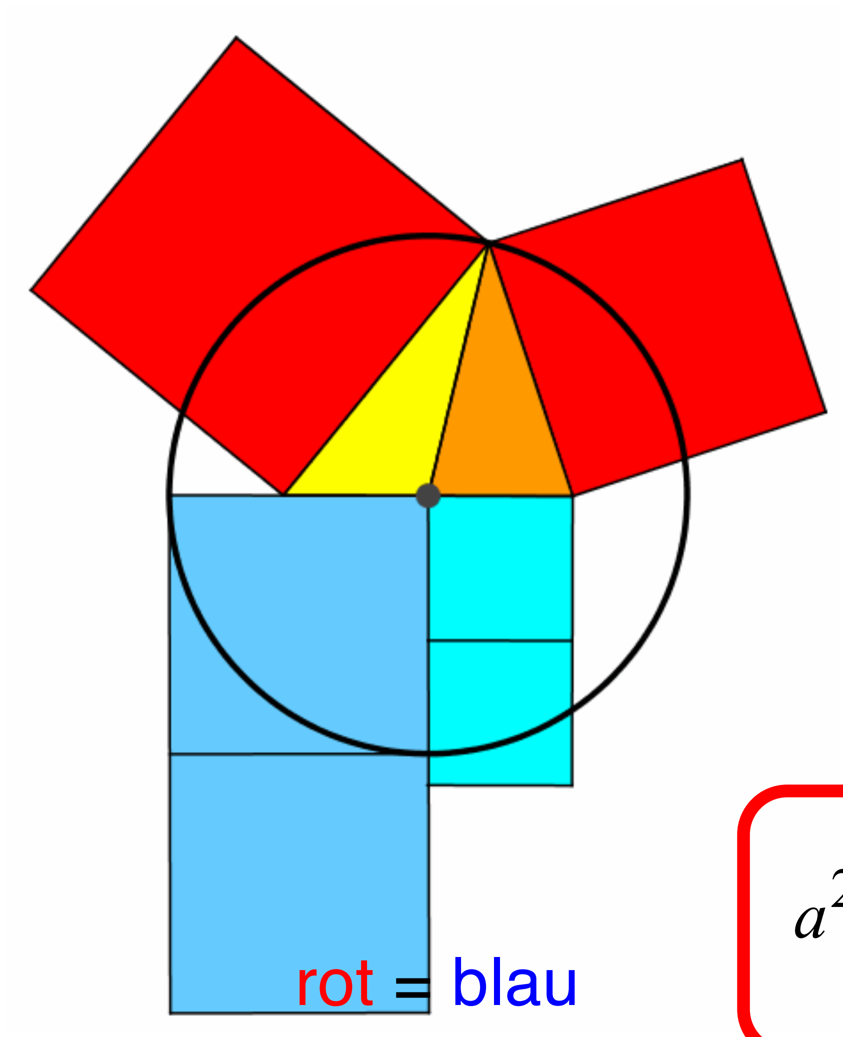
$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

Apollonios / al-Sijzi



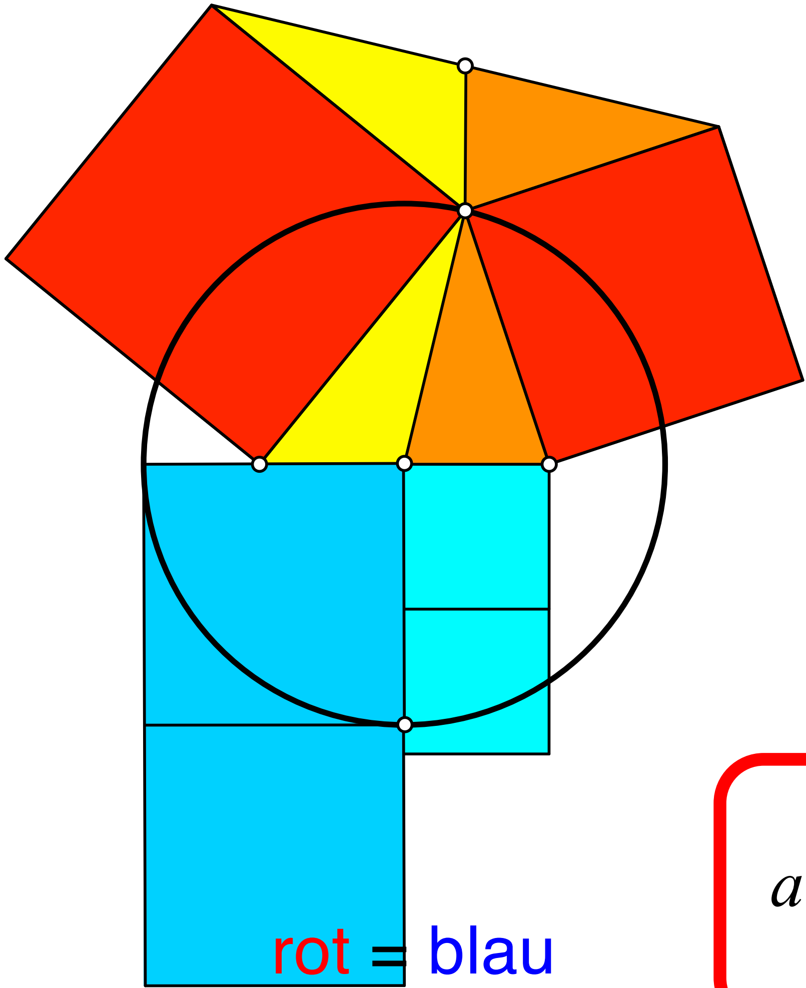
$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

Apollonios / al-Sijzi

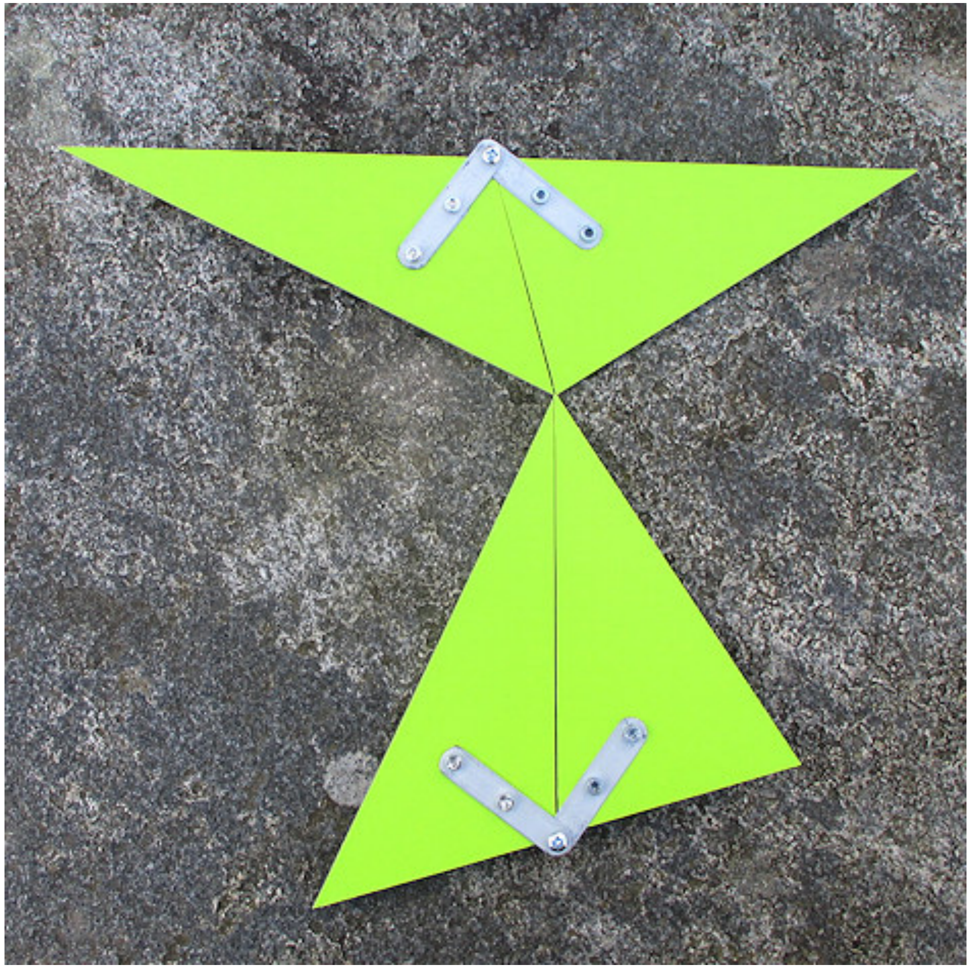


$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

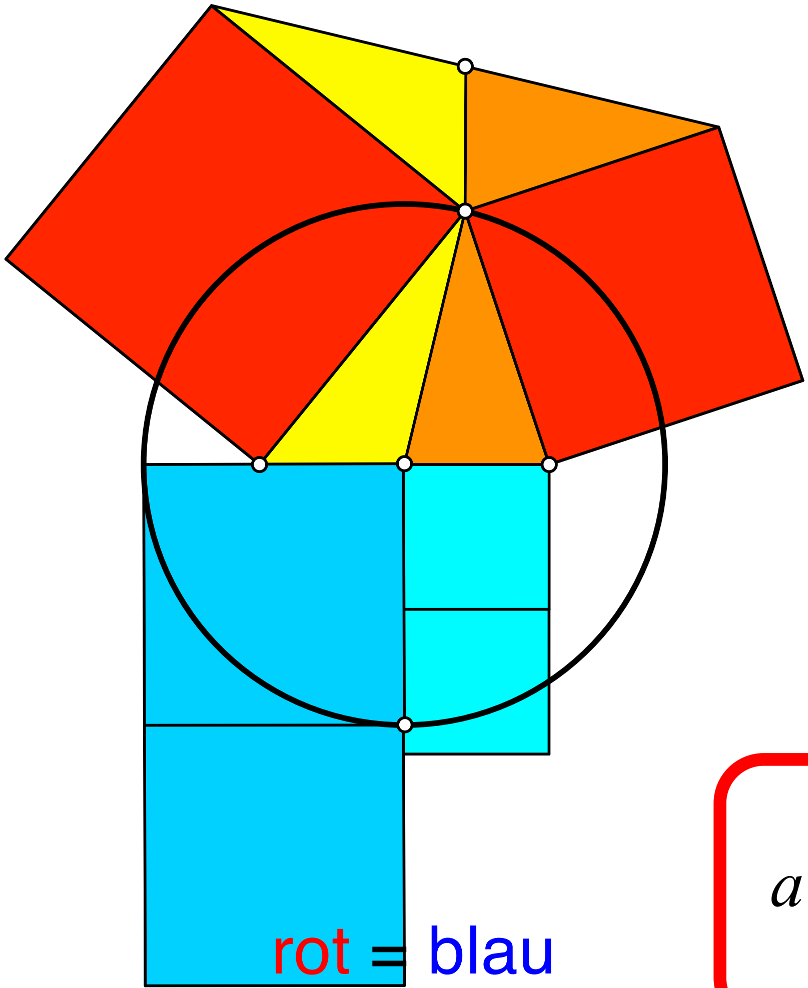
Apollonios / al-Sijzi



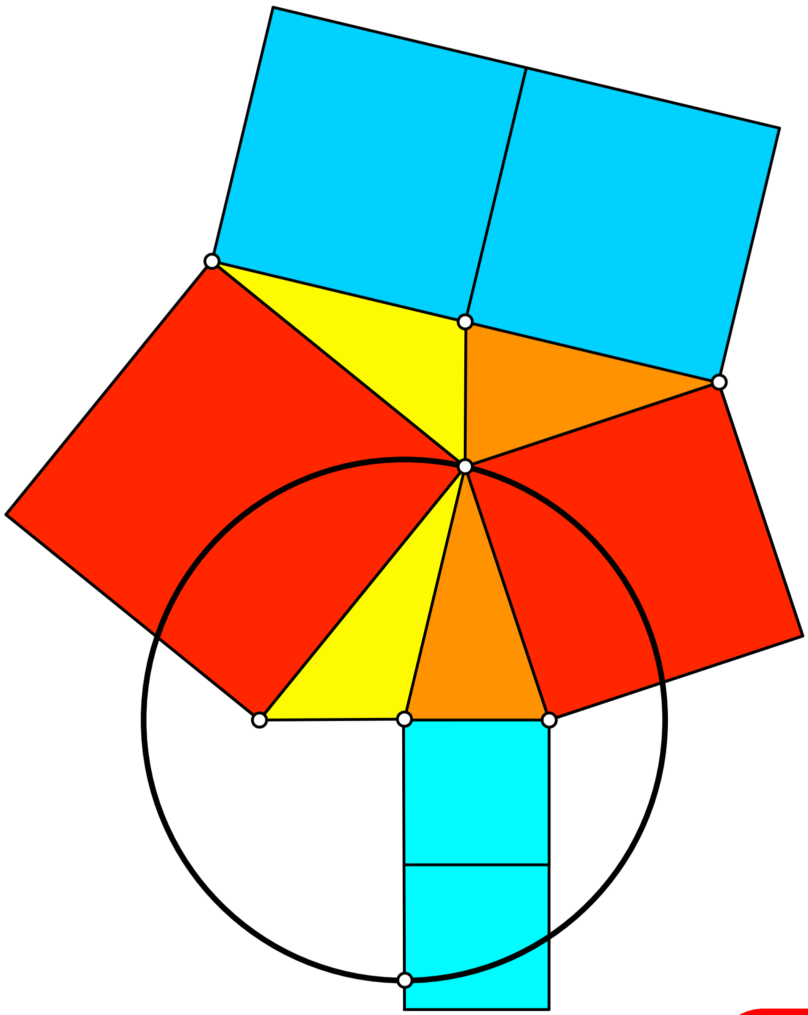
$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$



Modell

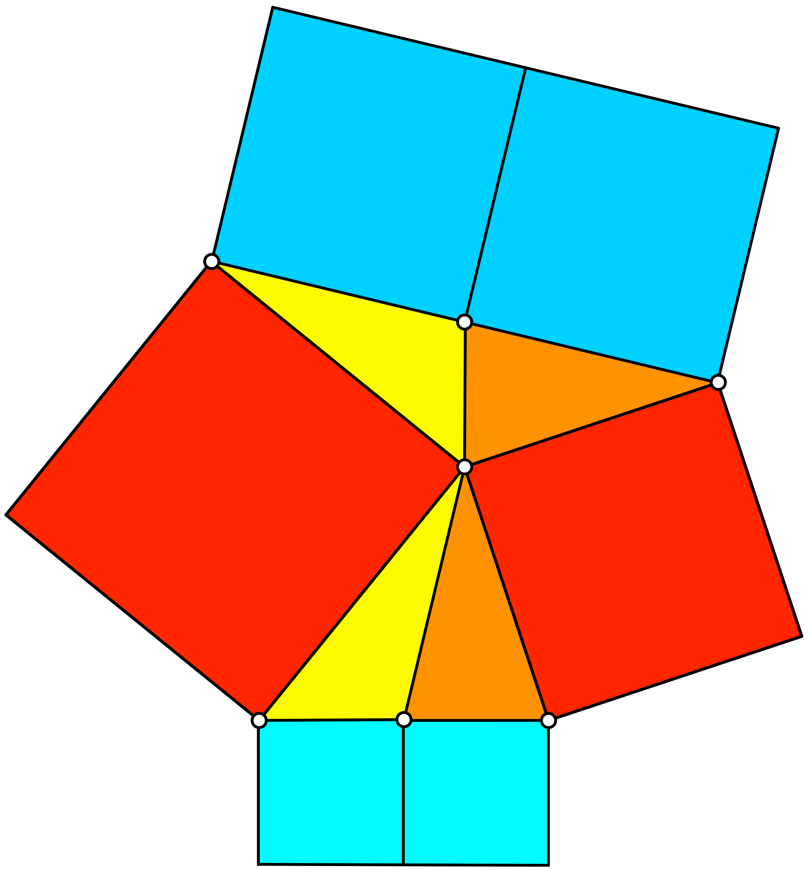


$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$



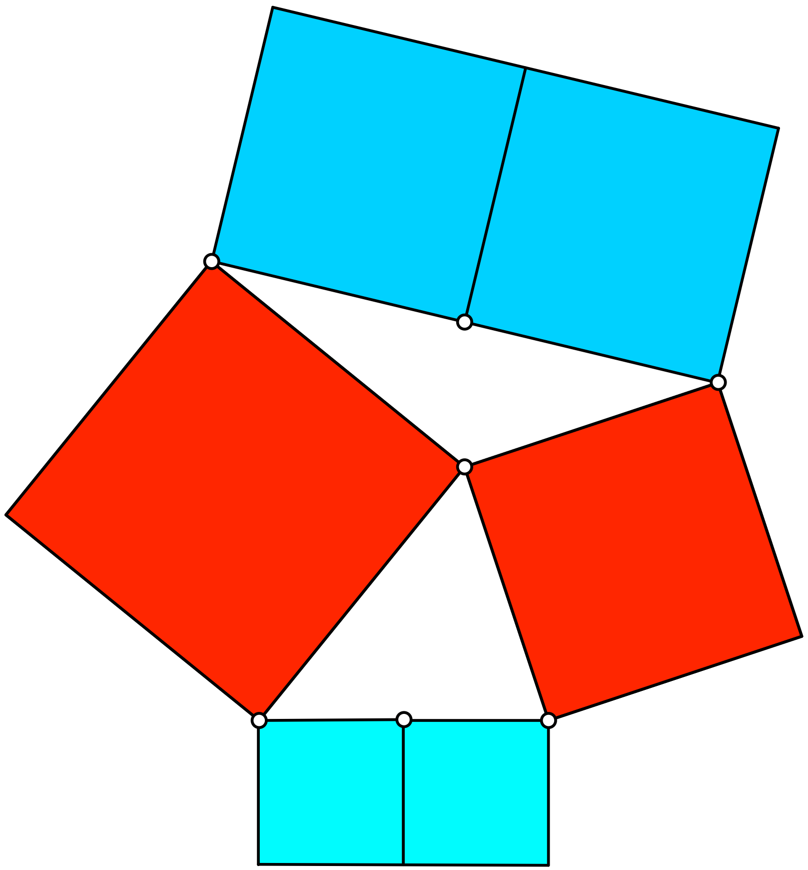
rot = blau

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$

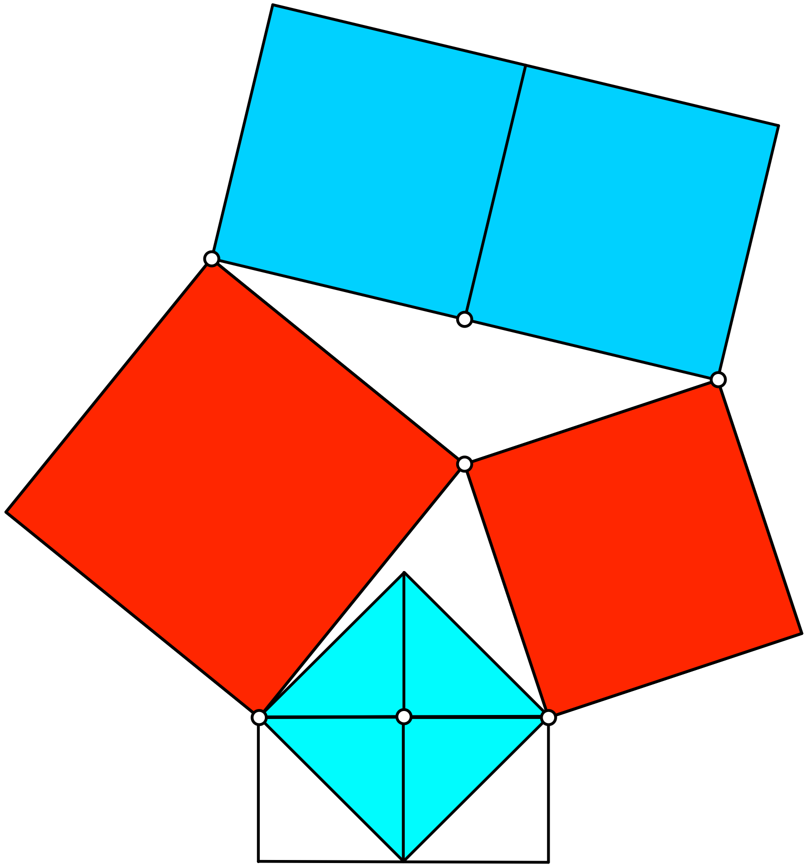


rot = blau

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2s_c^2$$



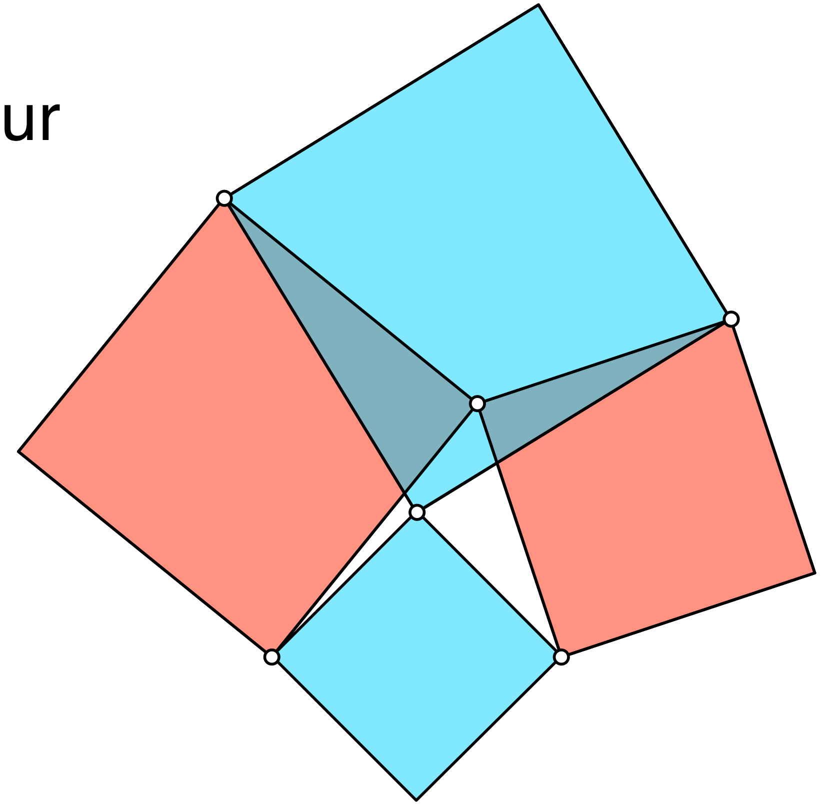
rot = blau



rot = blau

Papillon

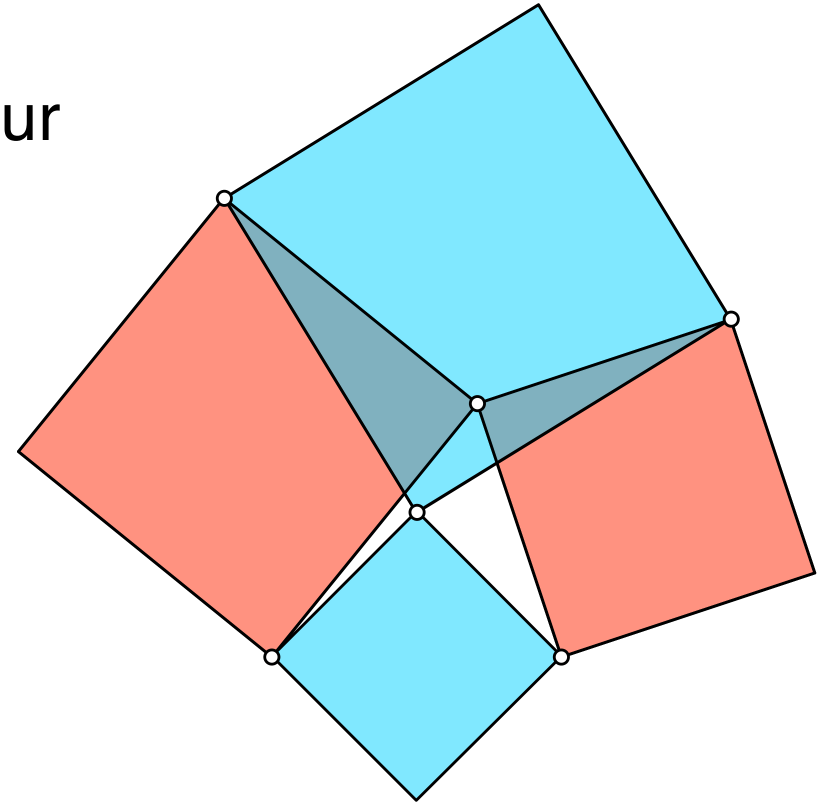
Schließungsfigur



rot = blau

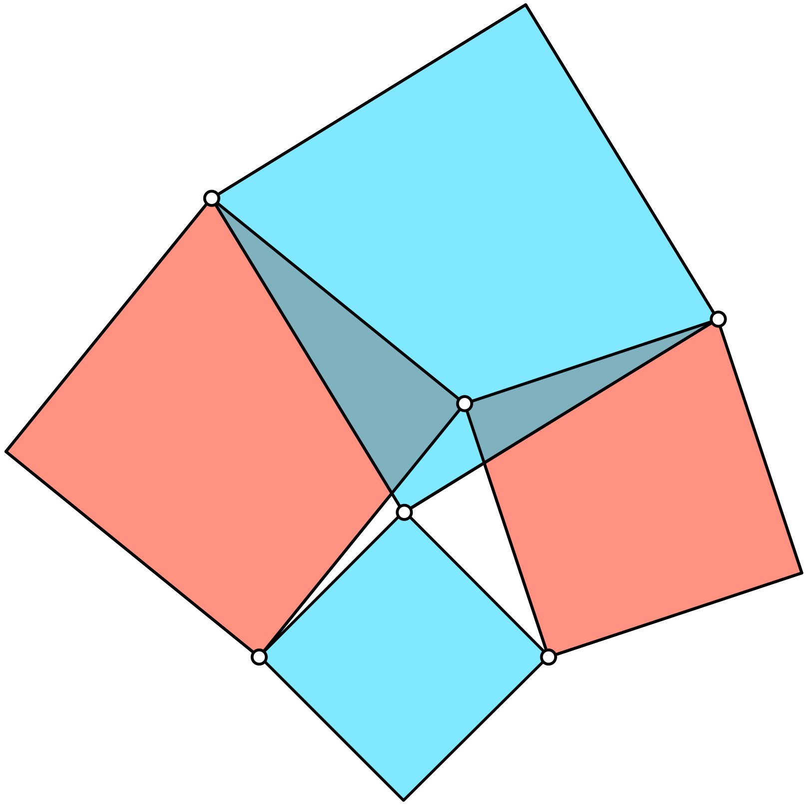
Papillon

Schließungsfigur



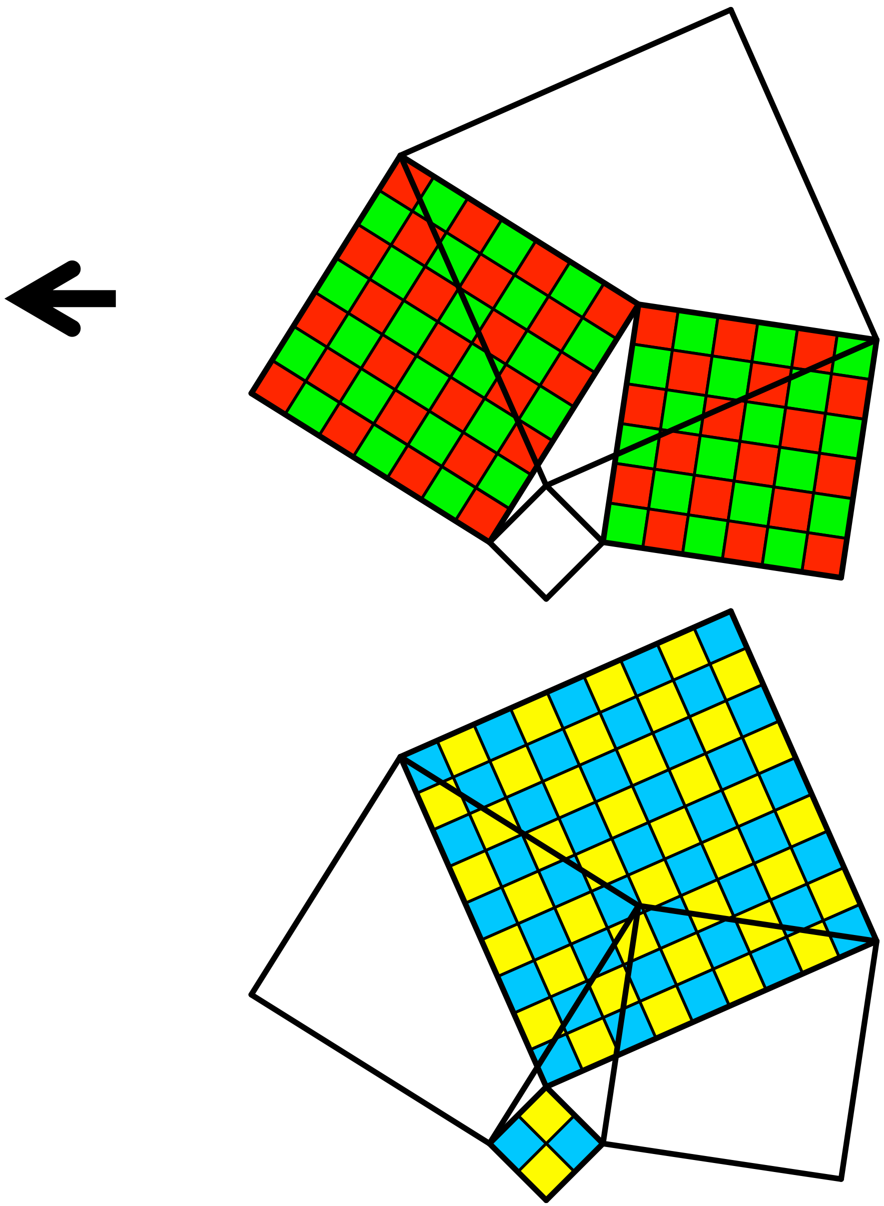
Beweis der Schließungseigenschaft:
Walser, H. (2021): Spiel mit Quadraten
MU Der Mathematikunterricht
Jahrgang 67. Heft 3. August 2021. 17-27

Papillon



rot = blau

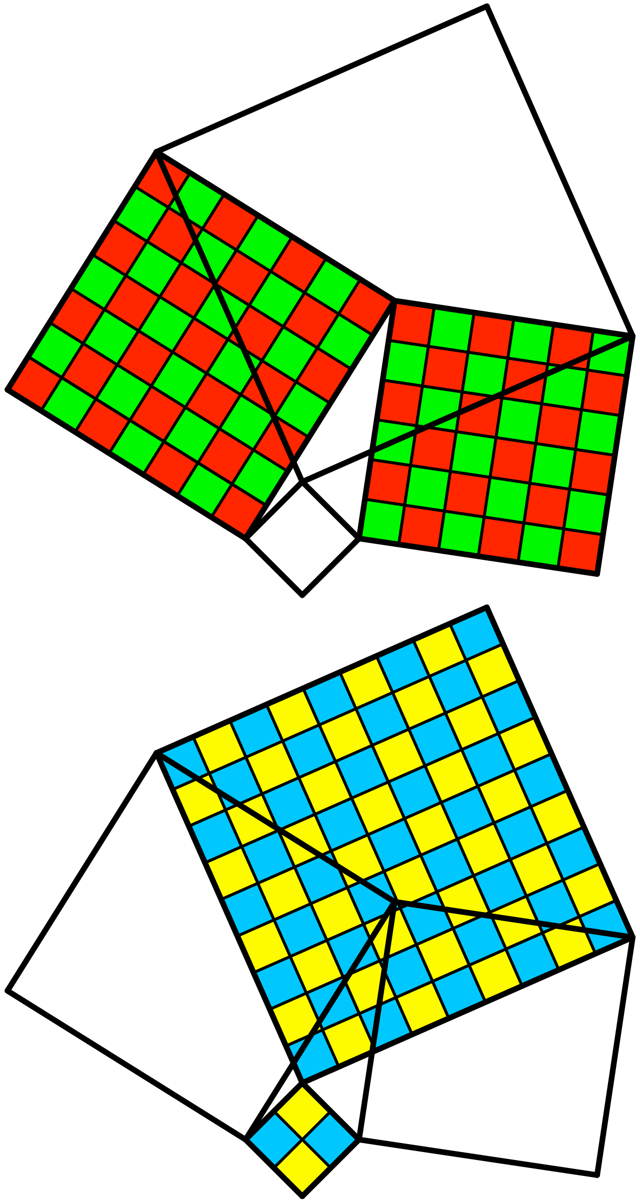
4	7	1	8
5	5	1	7
5	10	2	11
6	7	2	9
6	13	3	14
7	9	3	11
7	11	1	13
8	9	1	12
8	11	4	13
9	13	5	15
10	11	5	14
11	12	3	16
11	13	1	17
13	13	7	17
13	14	2	19



rot/grün = blau/gelb

ganzzahlig

4	7	1	8
5	5	1	7
5	10	2	11
6	7	2	9
6	13	3	14
7	9	3	11
7	11	1	13
8	9	1	12
8	11	4	13
9	13	5	15
10	11	5	14
11	12	3	16
11	13	1	17
13	13	7	17
13	14	2	19



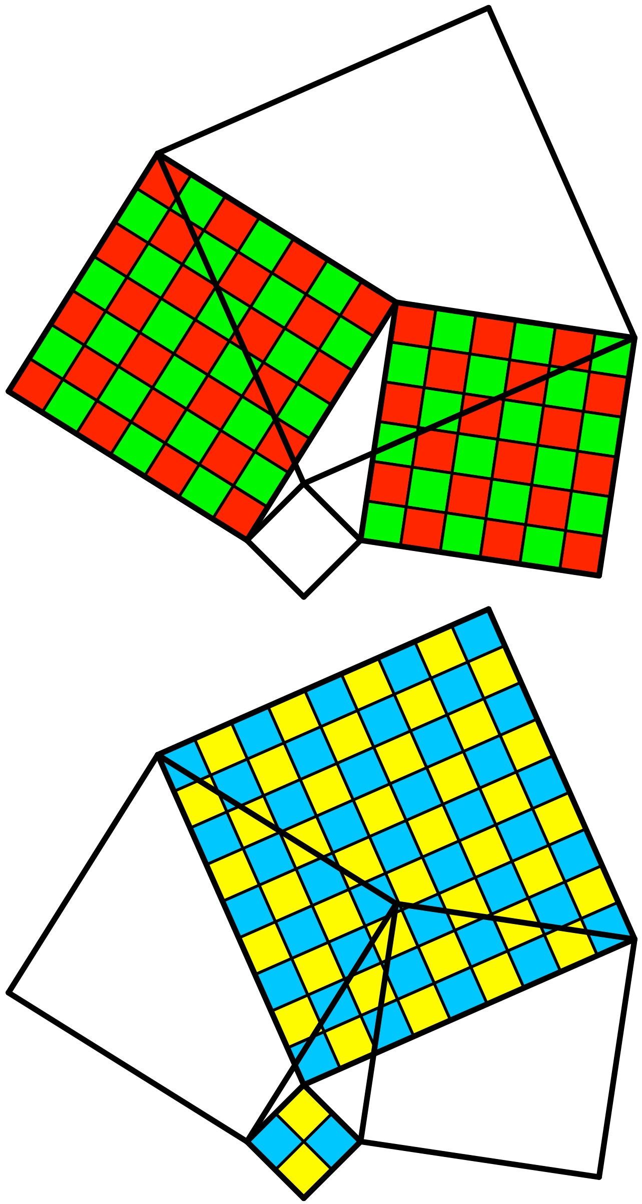
ganzzahlig

$$6^2 + 7^2 = 2^2 + 9^2$$

$$36 + 49 = 4 + 81 = 85$$

rot/grün = blau/gelb

4	7	1	8
5	5	1	7
5	10	2	11
6	7	2	9
6	13	3	14
7	9	3	11
7	11	1	13
8	9	1	12
8	11	4	13
9	13	5	15
10	11	5	14
11	12	3	16
11	13	1	17
13	13	7	17
13	14	2	19



rot/grün = blau/gelb

ganzzahlig

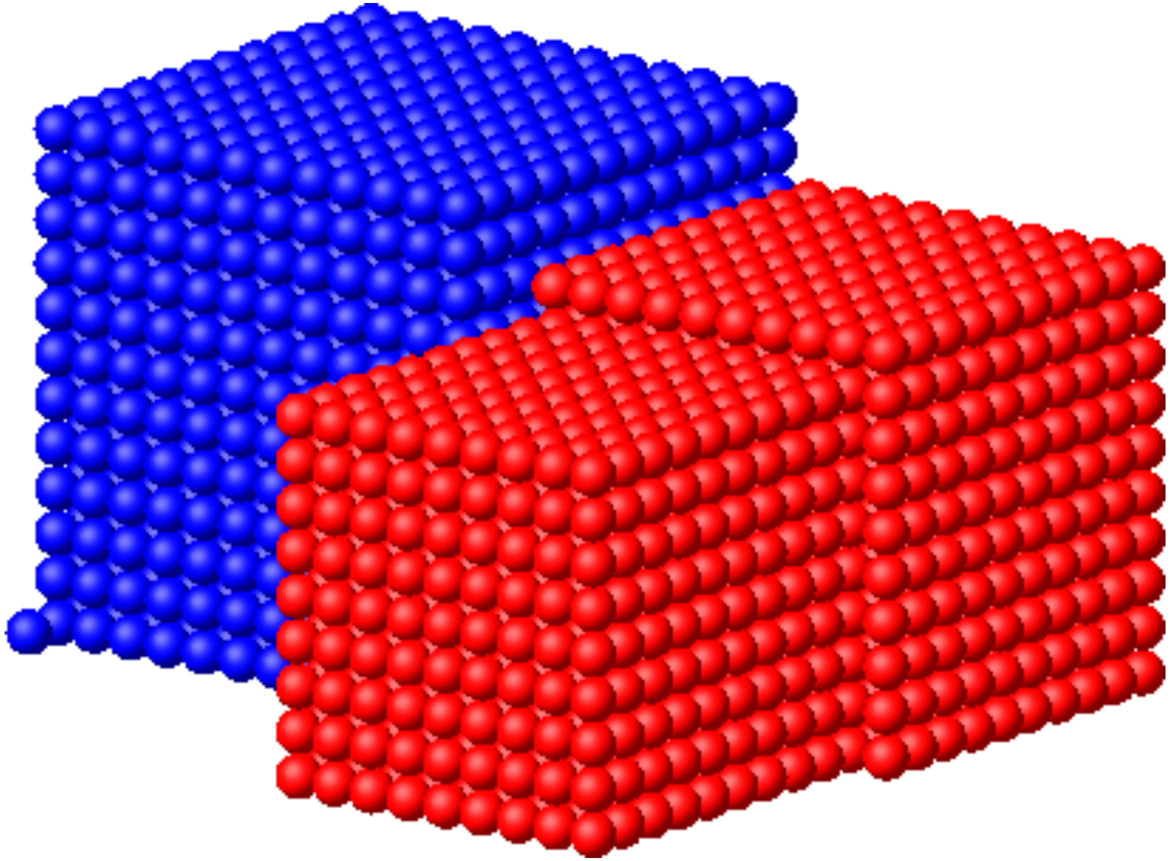
$$6^2 + 7^2 = 2^2 + 9^2$$

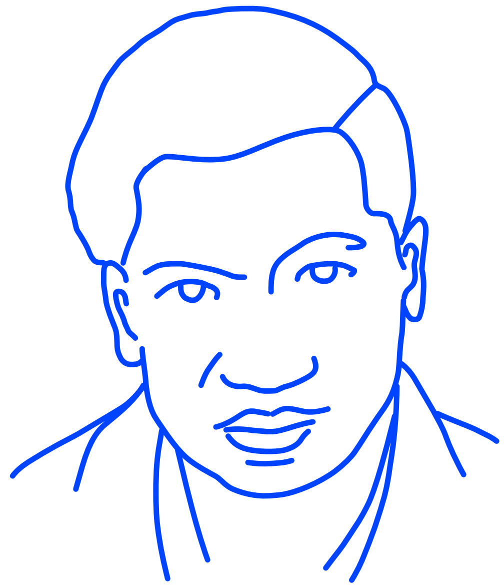
$$36 + 49 = 4 + 81 = 85$$

S. Ramanujan
1887-1920

$$9^3 + 10^3 = 1^3 + 12^3$$

$$729 + 1000 = 1 + 1728$$





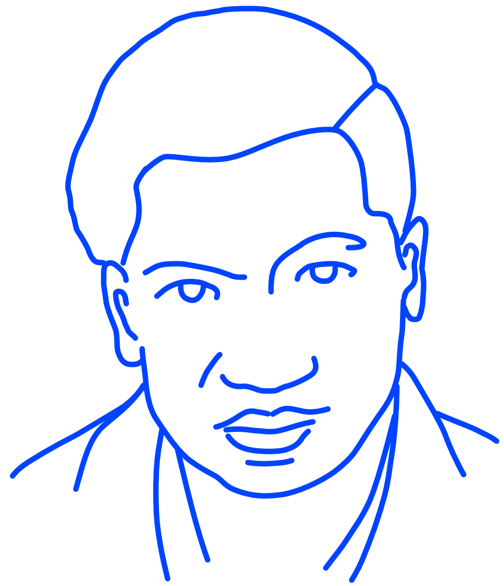
S. Ramanujan
1887-1920

$$9^3 + 10^3 = 1^3 + 12^3$$

$$729 + 1000 = 1 + 1728$$

rot = blau

9	10	1	12	1729
9	15	2	16	4104
18	20	2	24	13832
15	33	2	34	39312
27	30	3	36	46683
18	30	4	32	32832
16	33	9	34	40033
19	24	10	27	20683
31	33	12	40	65728
26	36	17	39	64232



S. Ramanujan
1887-1920

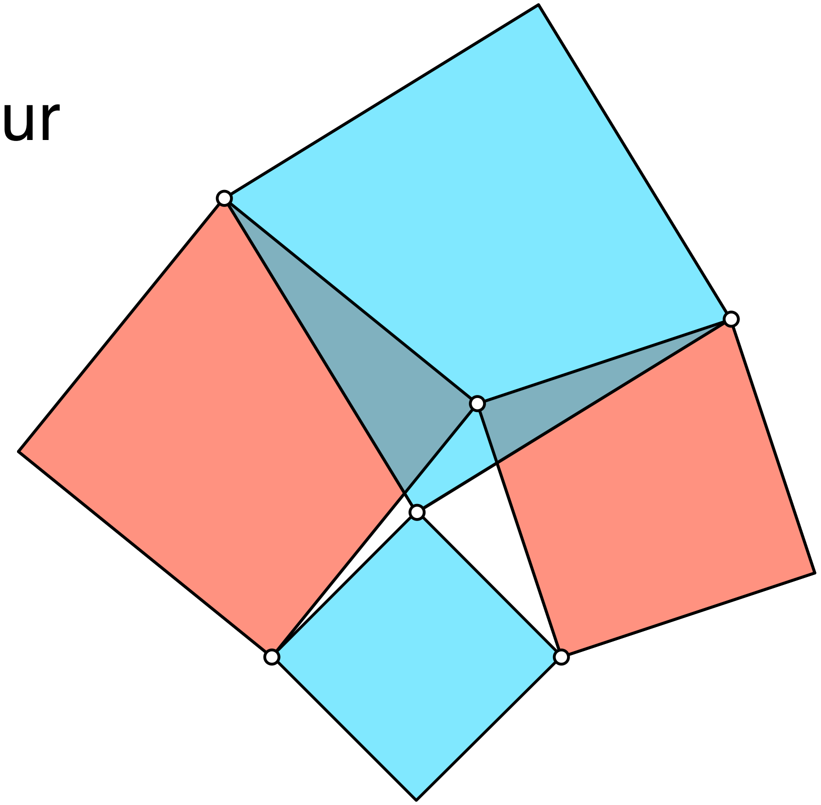
$$9^3 + 10^3 = 1^3 + 12^3$$

$$729 + 1000 = 1 + 1728$$

$$59^4 + 158^4 = 133^4 + 134^4 = 635'318'657$$

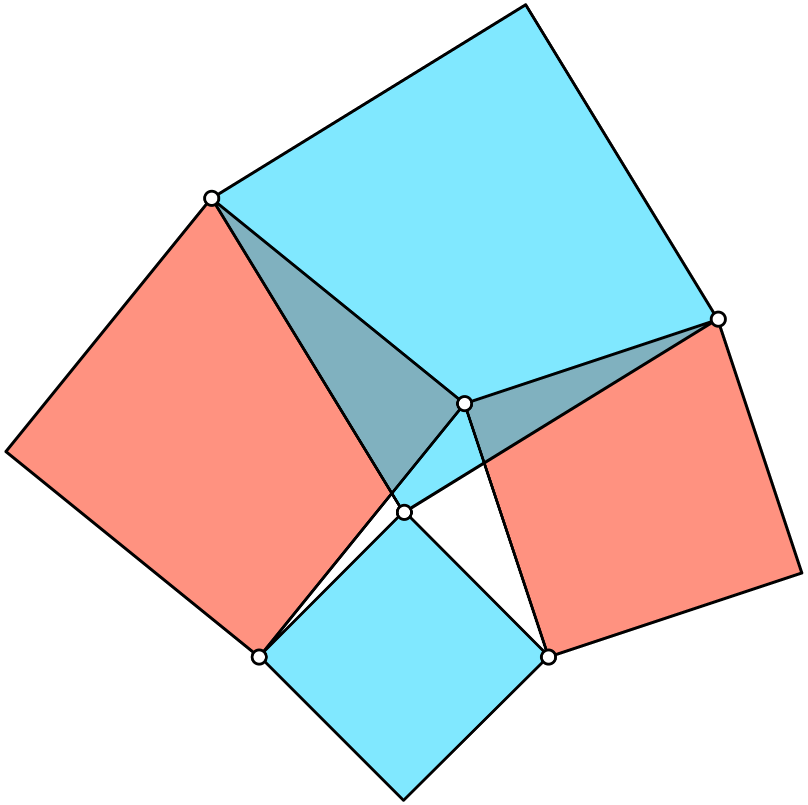
Papillon

Schließungsfigur



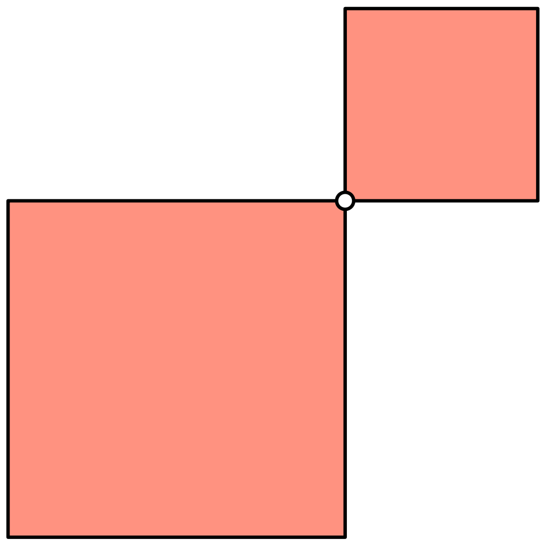
rot = blau

Zurück zu Pythagoras

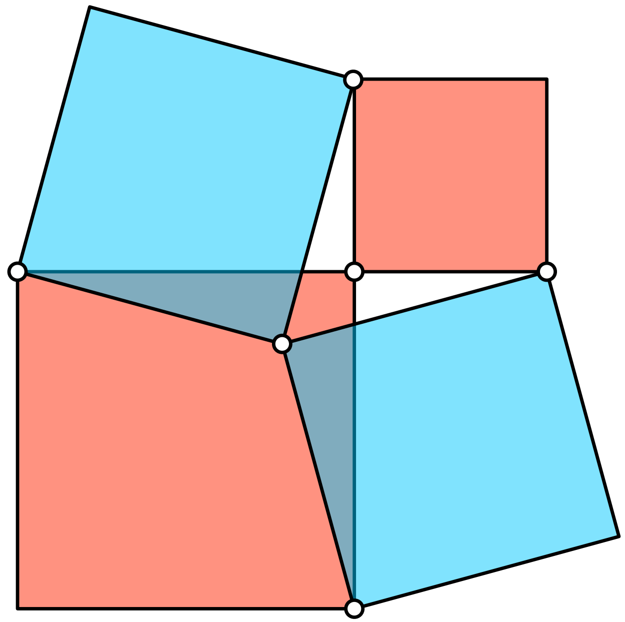


rot = blau

Zurück zu Pythagoras

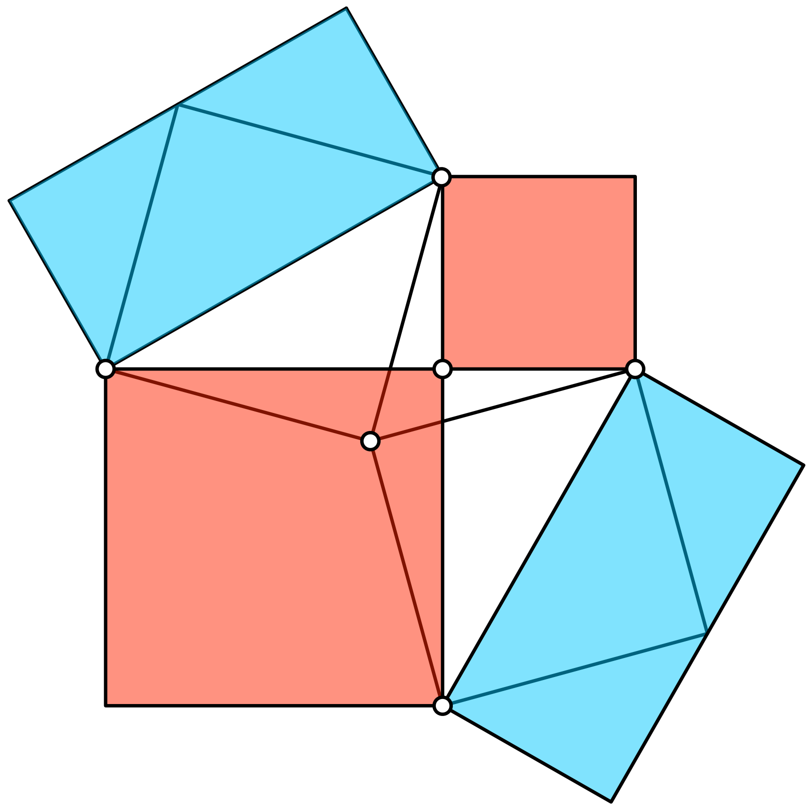


Zurück zu Pythagoras



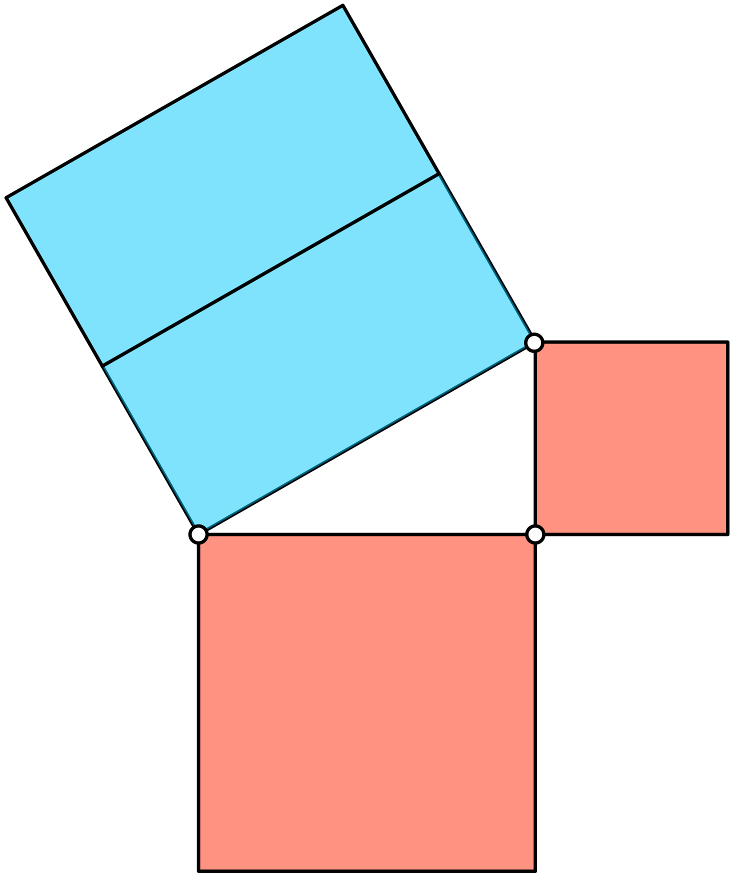
rot = blau

Zurück zu Pythagoras



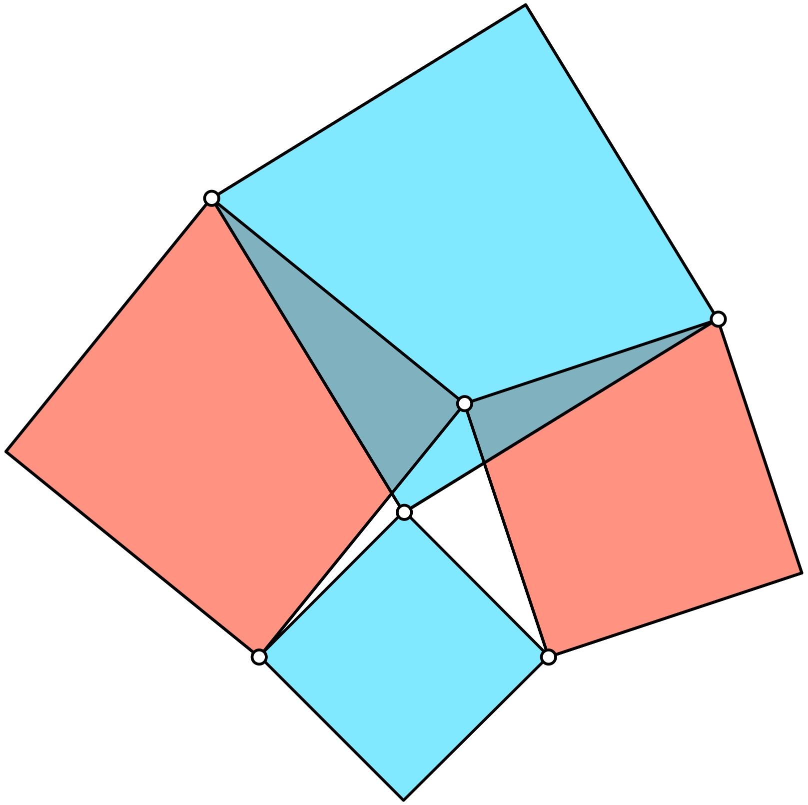
rot = blau

Zurück zu Pythagoras



rot = blau

Papillon



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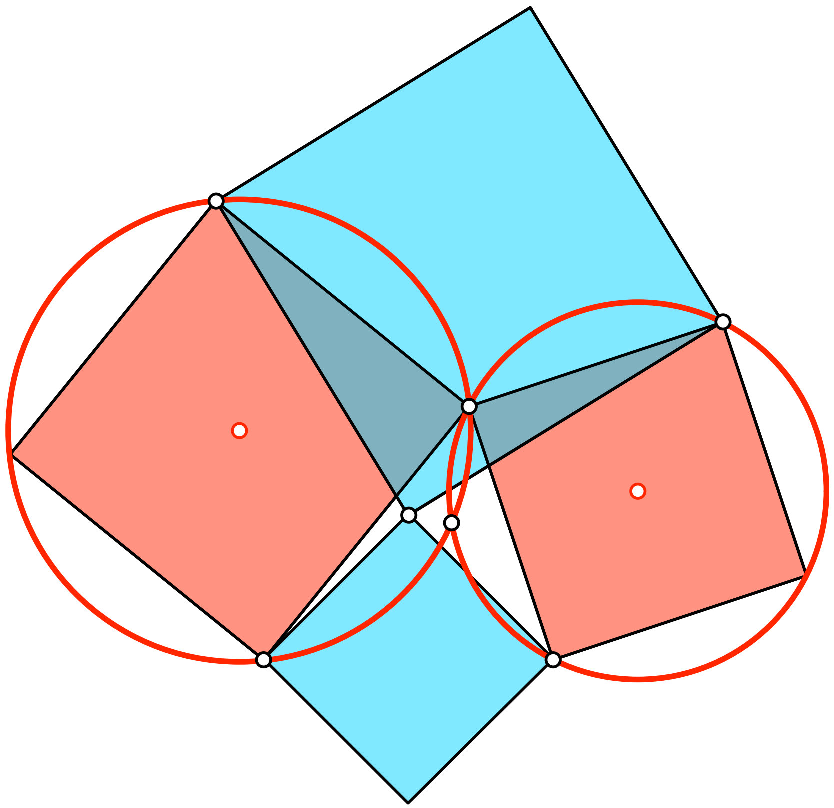
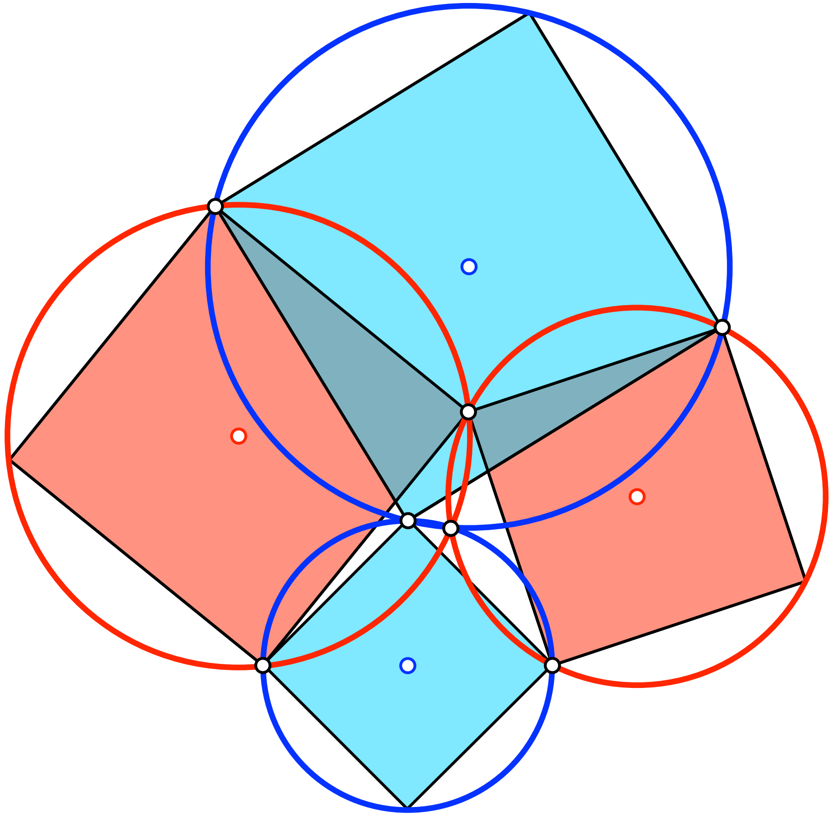


Umkreise

Papillon

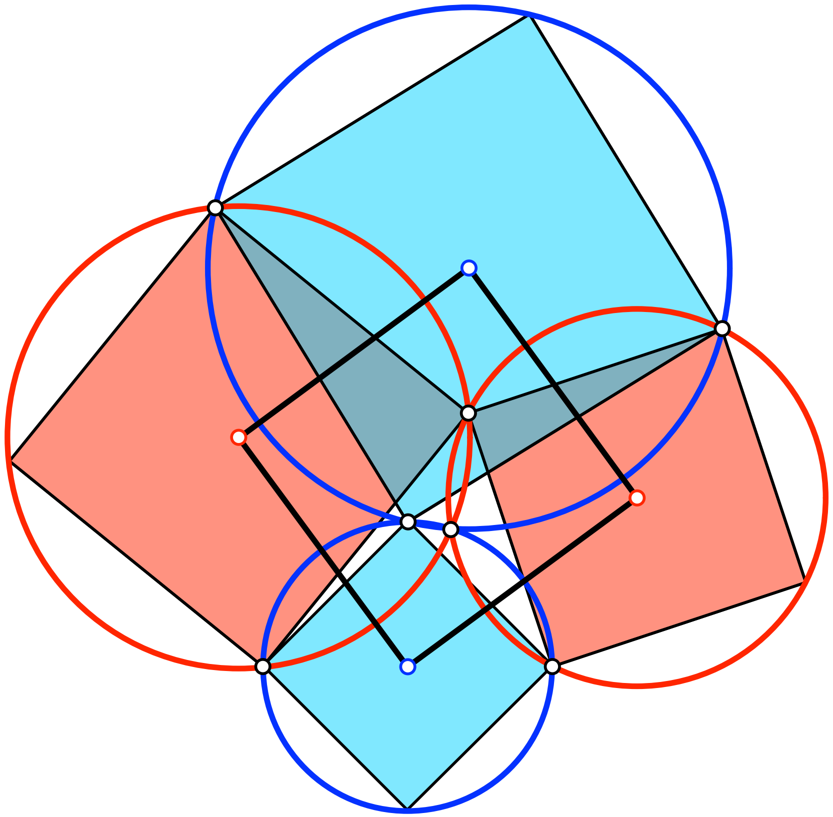
Gemeinsamer
Schnittpunkt



Umkreise

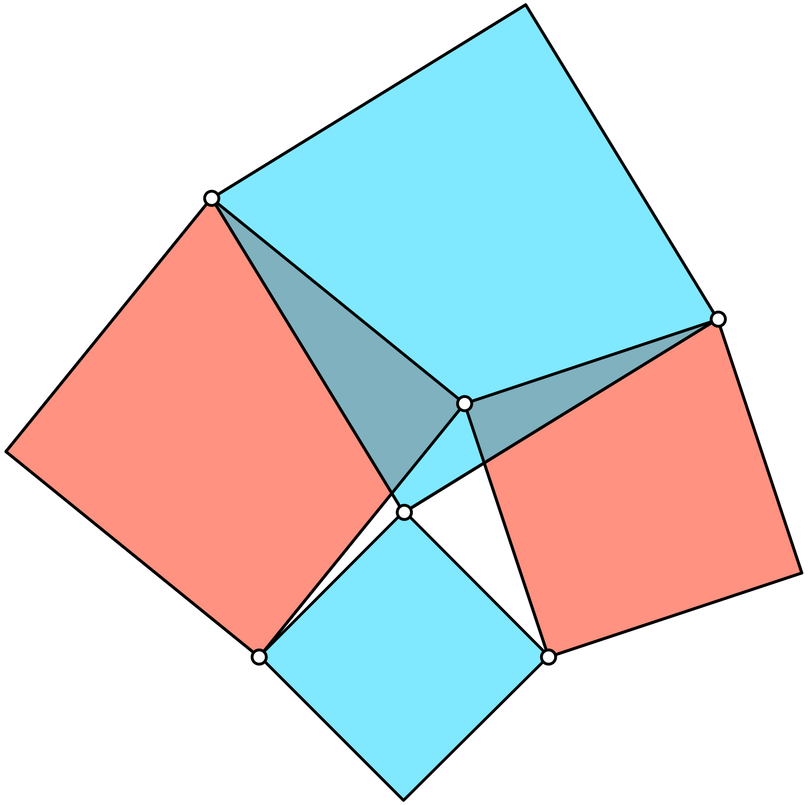
Papillon

Quadrat



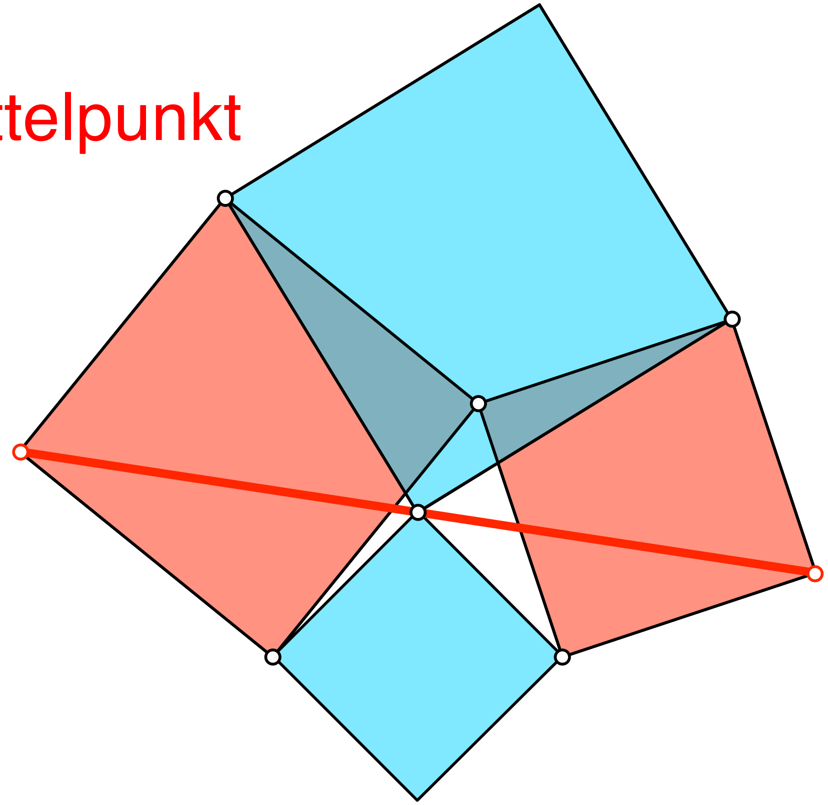
Umkreise

Papillon



Papillon

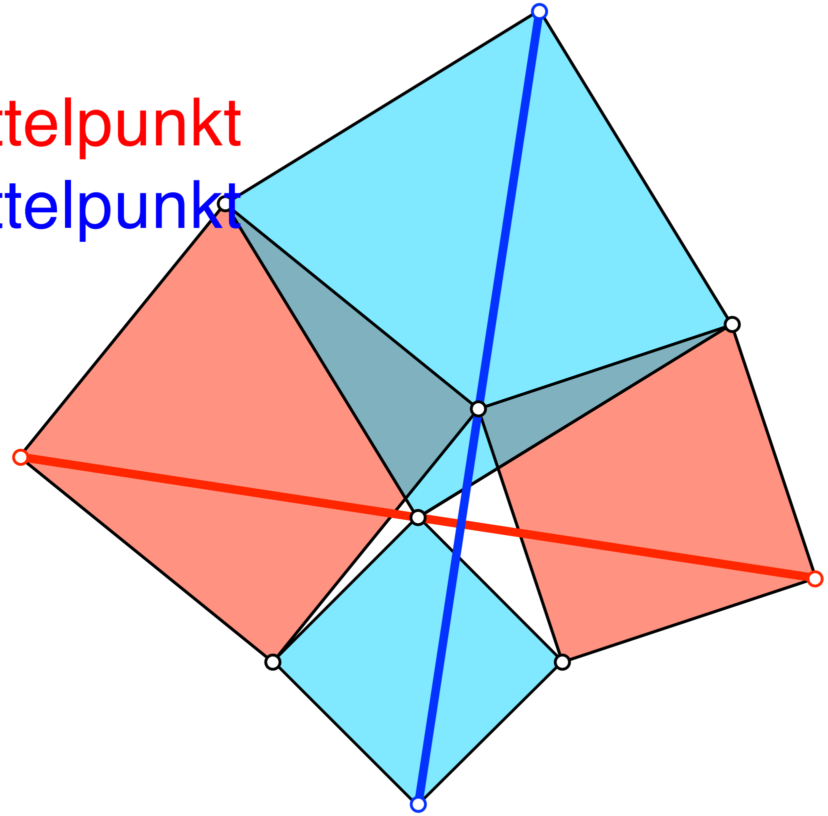
Strecke mit Mittelpunkt



Papillon

Strecke mit Mittelpunkt

Strecke mit Mittelpunkt



rot = blau

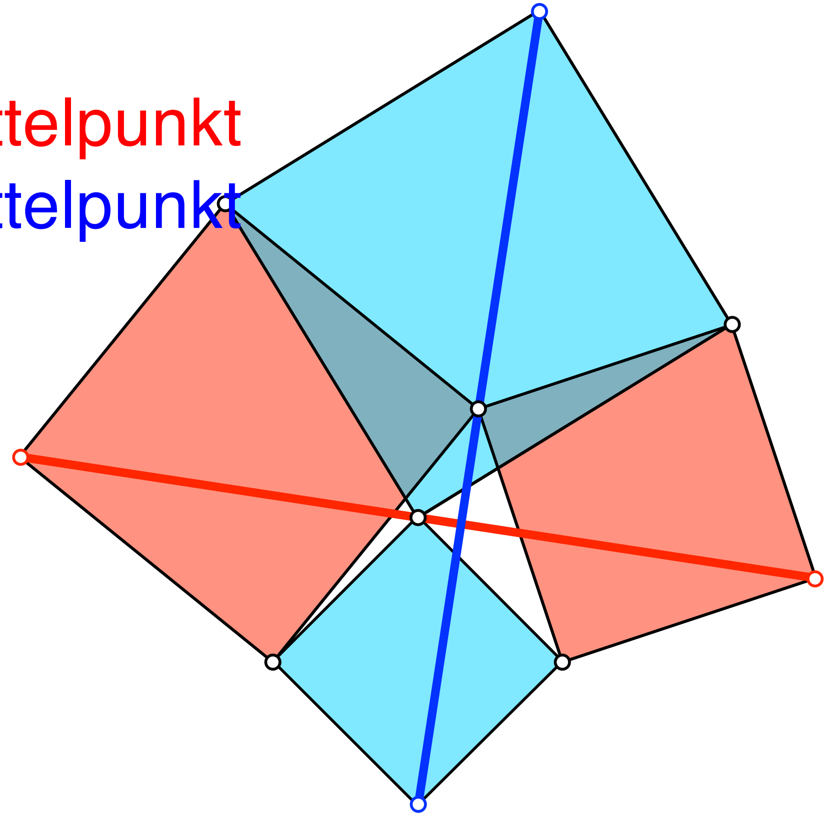
Papillon

Strecke mit Mittelpunkt

Strecke mit Mittelpunkt

orthogonal

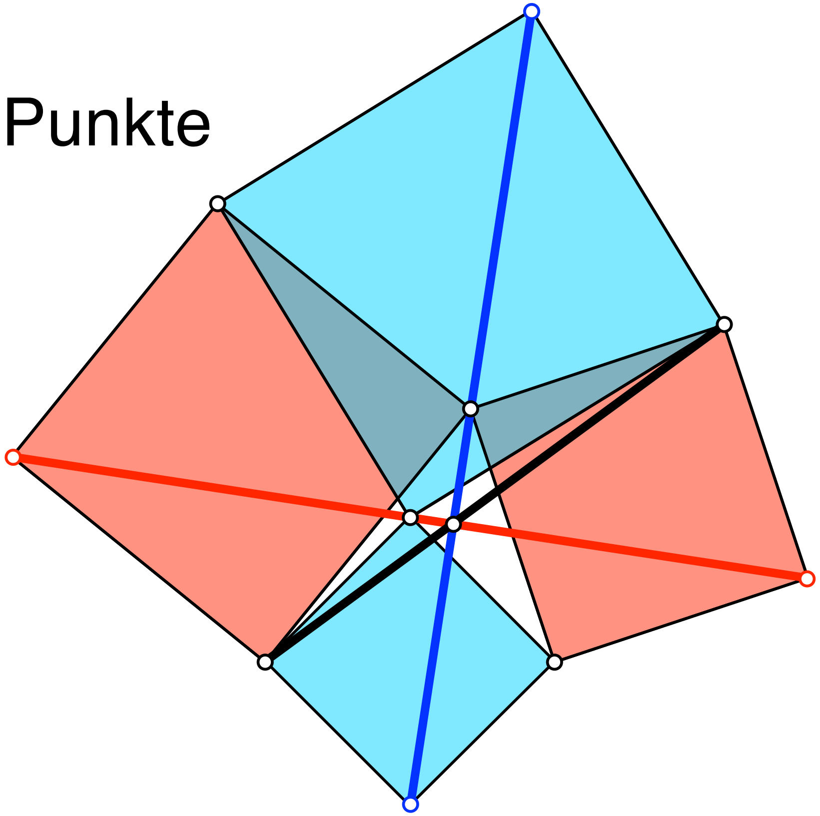
gleich lang



rot = blau

Papillon

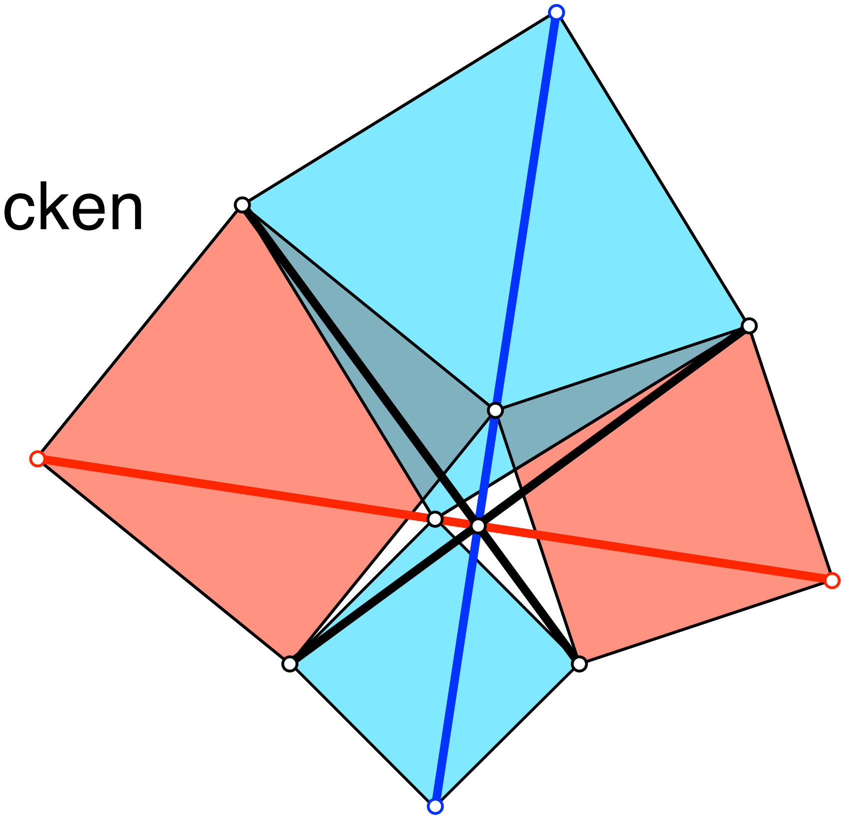
Drei kollineare Punkte



rot = blau

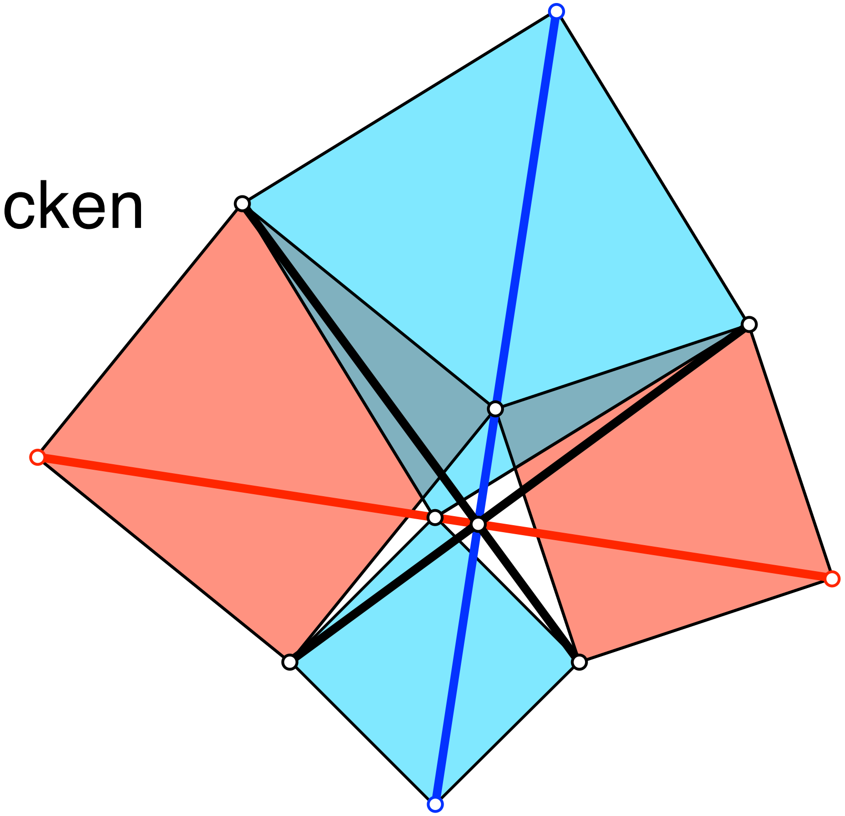
Papillon

gleich lange
schwarze Strecken
45°-Winkel



Papillon

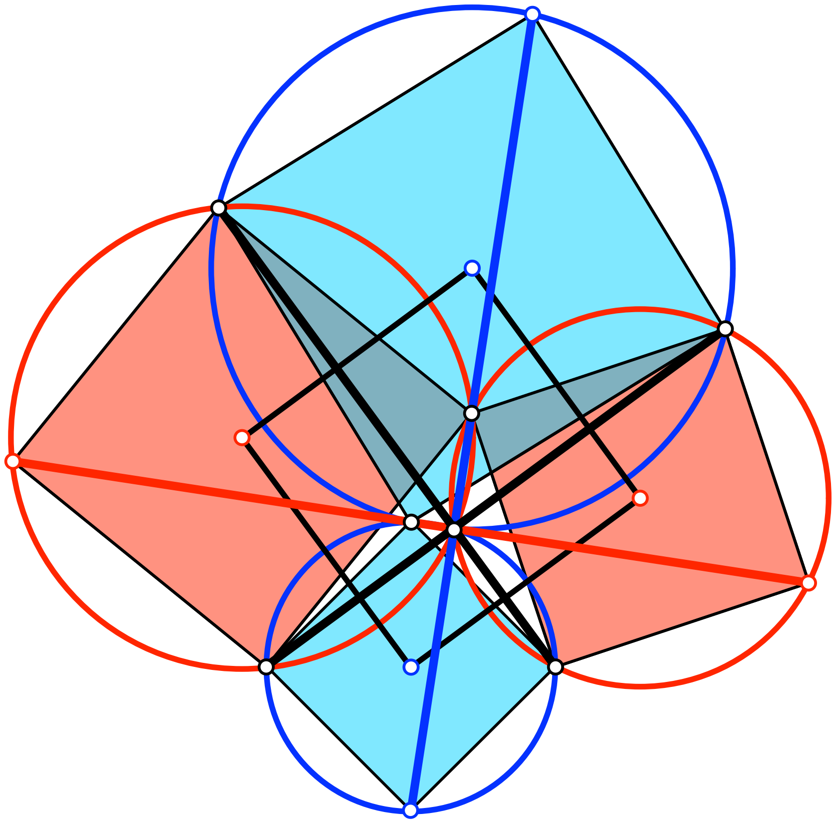
gleich lange
schwarze Strecken
45°-Winkel



$\text{rot} = \sqrt{2}\ \text{schwarz}$

$\text{blau} = \sqrt{2}\ \text{schwarz}$

Papillon



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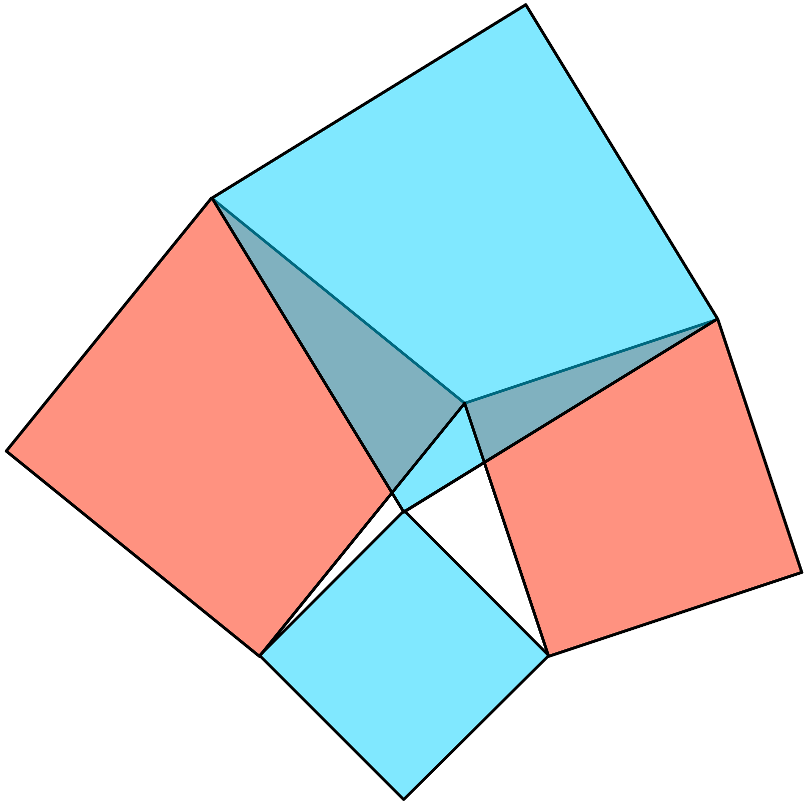
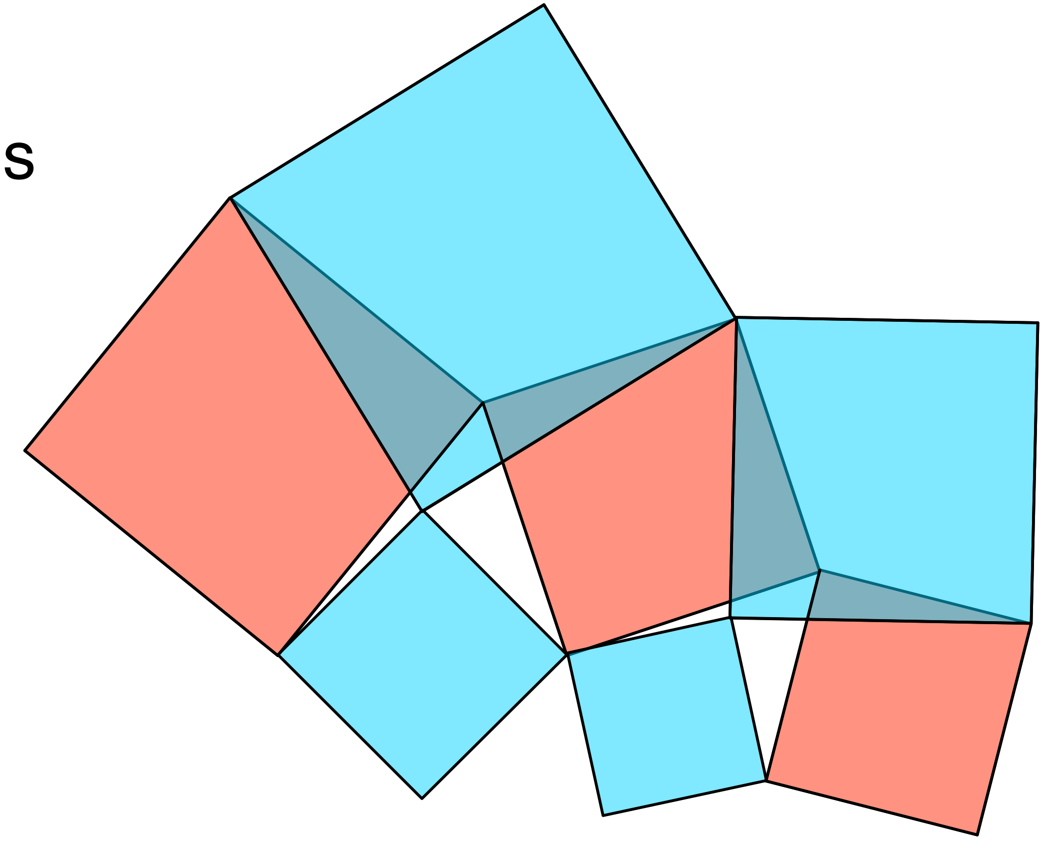
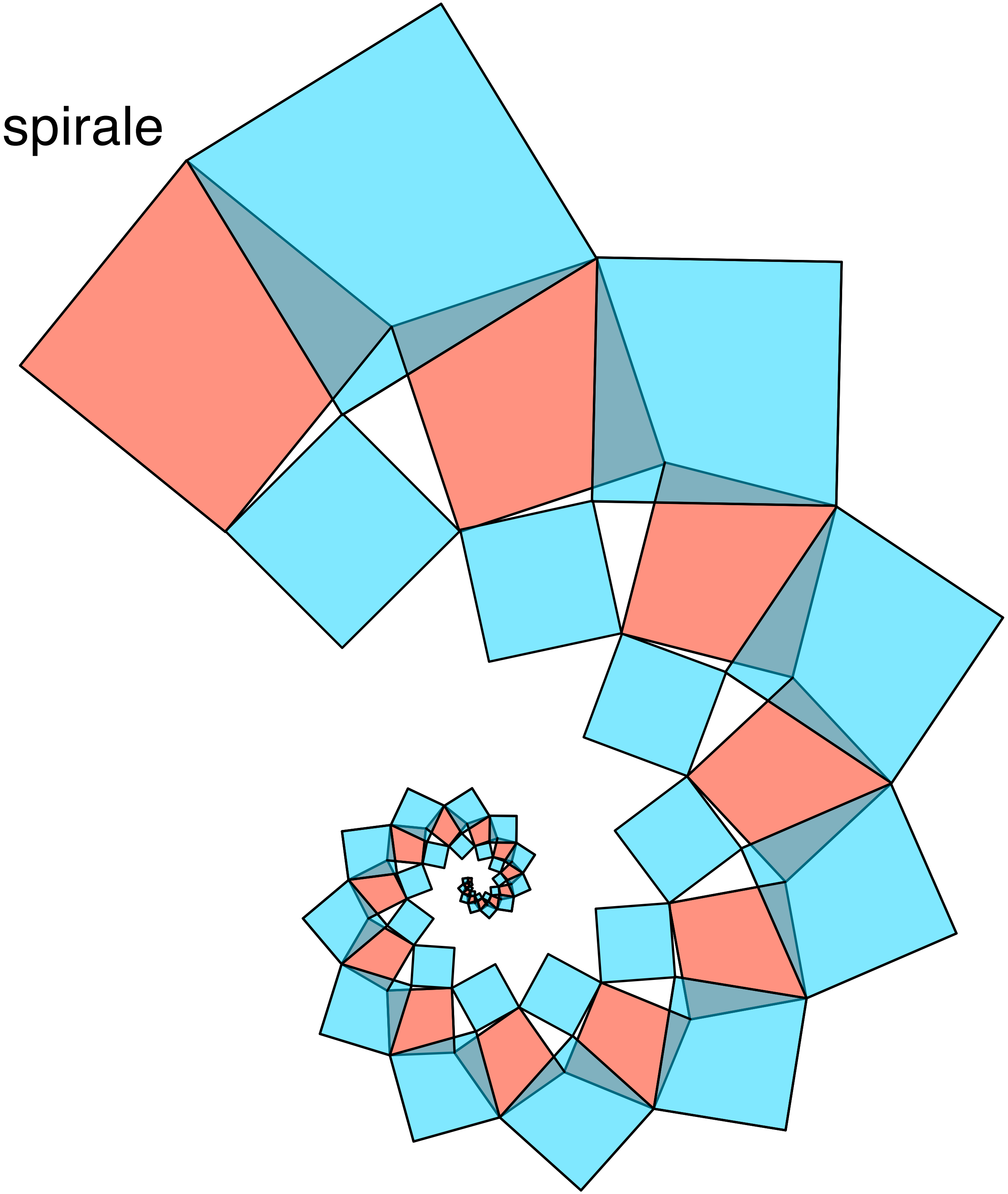


Papillons

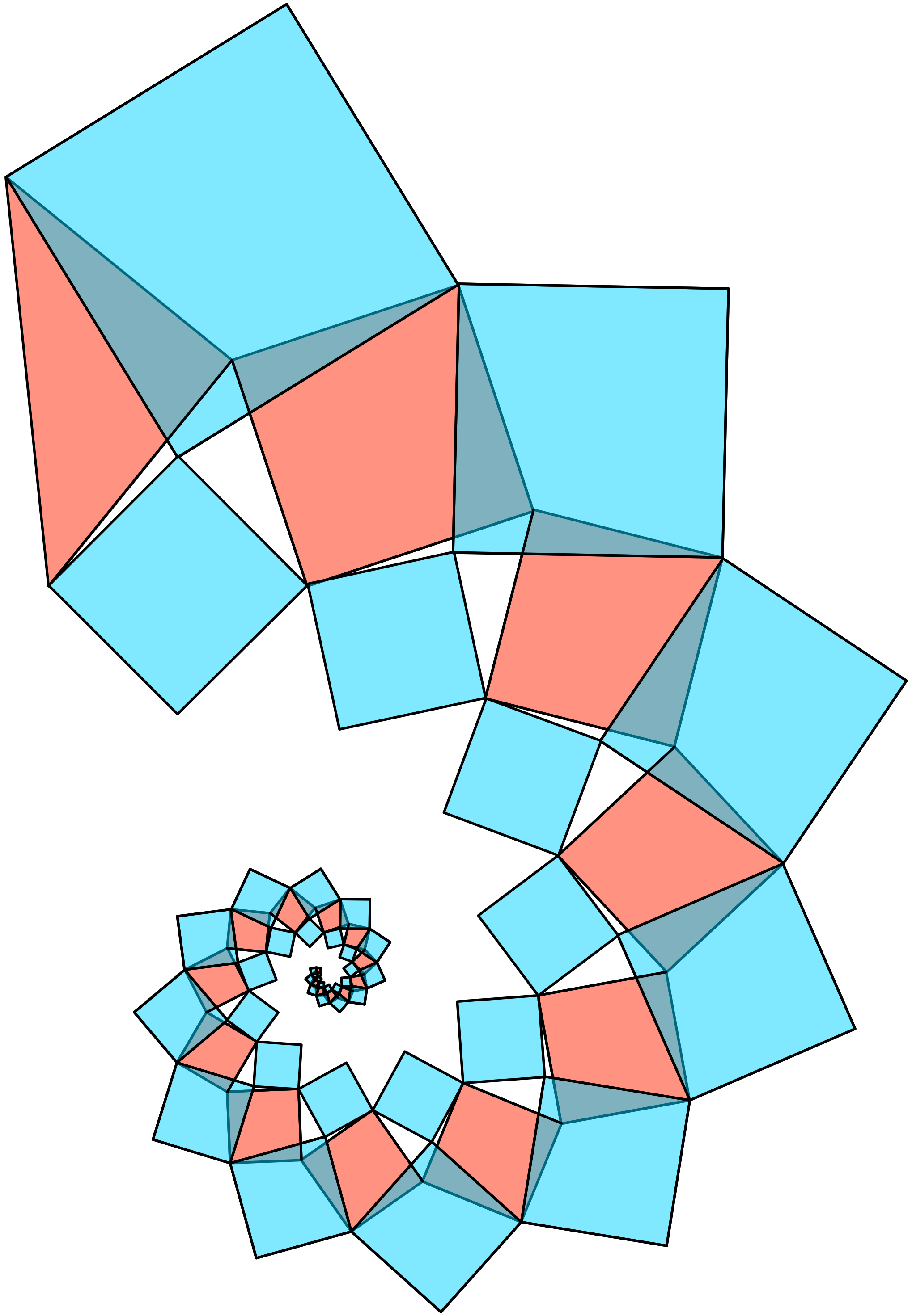


Papillonspirale

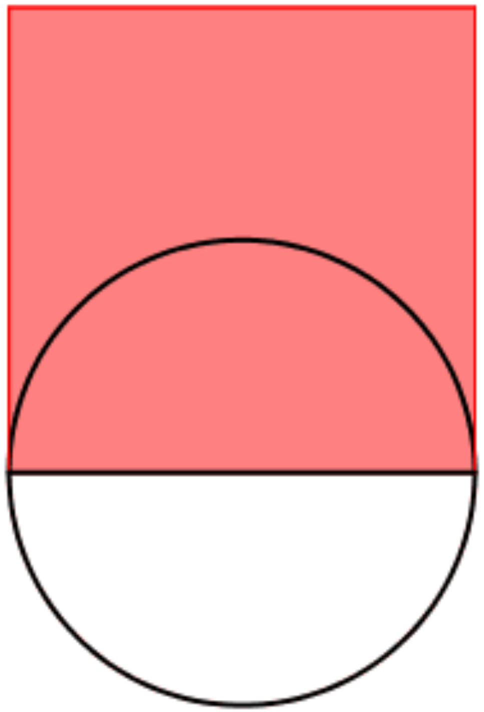


Papillonspirale

rot = ½ blau



Danke



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